

Preparation Course

Mathematics

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1 Units and Numbers

1.1 International System of Units

The International System of Units, also referred to as the **SI system** after the abbreviation from the French name *Le Système International d'Unités*, is the officially and internationally recognized system of measurement. The SI system identifies seven **base units**, which are defined in terms of the exact value of a corresponding **defining constant** (see Table 1.1 further below):

1. **Second:** The second is defined by taking the fixed numerical value of the caesium frequency $\Delta\nu_{\text{Cs}}$, the unperturbed ground-state hyperfine transition frequency of the caesium-133 atom, to be 9 192 631 770 when expressed in the unit Hz, which is equal to s^{-1} .
2. **Metre:** The metre is defined by taking the fixed numerical value of the speed of light in vacuum c to be 299 792 458 when expressed in the unit $\text{m} \cdot \text{s}^{-1}$, where the second is defined in terms of the caesium frequency $\Delta\nu_{\text{Cs}}$.
3. **Kilogram:** The kilogram is defined by taking the fixed numerical value of the Planck constant h to be $6.626\,070\,15 \times 10^{-34}$ when expressed in the unit $\text{J} \cdot \text{s}$, which is equal to $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$, where the metre and the second are defined in terms of c and $\Delta\nu_{\text{Cs}}$.
4. **Ampere:** The ampere is defined by taking the fixed numerical value of the elementary charge e to be $1.602\,176\,634 \times 10^{-19}$ when expressed in the unit C, which is equal to $\text{A} \cdot \text{s}$, where the second is defined in terms of $\Delta\nu_{\text{Cs}}$.
5. **Kelvin:** The Kelvin is defined by taking the fixed numerical value of the Boltzmann constant k to be $1.380\,649 \times 10^{-23}$ when expressed in the unit $\text{J} \cdot \text{K}^{-1}$, which is equal to $\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$, where the kilogram, metre and second are defined in terms of h , c and $\Delta\nu_{\text{Cs}}$.
6. **Mole:** One mole contains exactly $6.022\,140\,76 \times 10^{23}$ elementary entities. This number is the fixed numerical value of the Avogadro constant, N_A , when expressed in the unit mol^{-1} and is called the Avogadro number.
7. **Candela:** The candela is defined by taking the fixed numerical value of the luminous efficacy of monochromatic radiation of frequency 540×10^{12} Hz, K_{cd} , to be 683 when expressed in the unit $\text{lm} \cdot \text{W}^{-1}$, which is equal to $\text{cd} \cdot \text{sr} \cdot \text{W}^{-1}$, or $\text{cd} \cdot \text{sr} \cdot \text{kg}^{-1} \cdot \text{m}^{-2} \cdot \text{s}^3$, where the kilogram, metre and second are defined in terms of h , c and $\Delta\nu_{\text{Cs}}$.

By combining several SI base units in a unique way, we can formulate so-called **derived units**. Examples of derived physical quantities, which are expressed in SI derived units, include area (m^2), energy density ($\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}$), molality ($\text{mol} \cdot \text{kg}^{-1}$), resistivity ($\text{kg} \cdot \text{m}^3 \cdot \text{s}^{-3} \cdot \text{A}^{-2}$), luminous energy ($\text{s} \cdot \text{cd}$), and thermal conductivity ($\text{kg} \cdot \text{m} \cdot \text{s}^{-3} \cdot \text{K}^{-1}$).

The SI system furthermore designates special names to the units of measurement of twenty-two derived physical quantities, which are listed below in Table 1.2.

Table 1.1: The seven SI base units with their corresponding SI defining constants

Physical Quantity	Unit	Symbol	Constant	Symbol	Value	Unit
Time	second	s	caesium hyperfine frequency	$\Delta\nu_{\text{Cs}}$	9 192 631 770	Hz
Length	metre	m	speed of light in vacuum	c	299 792 458	$\text{m} \cdot \text{s}^{-1}$
Mass	kilogram	kg	Planck constant	h	$6.626\,070\,15 \times 10^{-34}$	$\text{J} \cdot \text{s}$
Electric current	ampere	A	elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	C
Thermodynamic temperature	kelvin	K	Boltzmann constant	k	$1.380\,649 \times 10^{-23}$	$\text{J} \cdot \text{K}^{-1}$
Amount of substance	mole	mol	Avogadro constant	N_{A}	$6.022\,140\,76 \times 10^{23}$	mol^{-1}
Luminous intensity	candela	cd	luminous efficacy of a defined visible radiation	K_{cd}	683	$\text{lm} \cdot \text{W}^{-1}$

Table 1.2: The twenty-two SI derived units with special names and equivalent SI base units

Physical Quantity	Unit	Symbol	Base units
Plane angle	radian	rad	1
Solid angle	steradian	sr	1
Frequency	hertz	Hz	s^{-1}
Force	newton	N	$\text{kg} \cdot \text{m} \cdot \text{s}^{-2}$
Pressure/Stress	pascal	Pa	$\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}$
Energy/Work/Amount of heat	joule	J	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$
Power/Radiant flux	watt	W	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3}$
Electric charge	coulomb	C	$\text{A} \cdot \text{s}$
Electric potential difference	volt	V	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3} \cdot \text{A}^{-1}$
Capacitance	farad	F	$\text{kg}^{-1} \cdot \text{m}^{-2} \cdot \text{s}^4 \cdot \text{A}^2$
Electric resistance	ohm	Ω	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-3} \cdot \text{A}^{-2}$
Electric conductance	siemens	S	$\text{kg}^{-1} \cdot \text{m}^{-2} \cdot \text{s}^3 \cdot \text{A}^2$
Magnetic flux	weber	Wb	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{A}^{-1}$
Magnetic flux density	tesla	T	$\text{kg} \cdot \text{s}^{-2} \cdot \text{A}^{-1}$
Inductance	henry	H	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{A}^{-2}$
Celsius temperature	degree Celsius	$^{\circ}\text{C}$	K
Luminous flux	lumen	lm	$\text{cd} \cdot \text{sr}$
Illuminance	lux	lx	$\text{cd} \cdot \text{sr} \cdot \text{m}^{-2}$
Activity referred to a radionuclide	becquerel	Bq	s^{-1}
Absorbed dose/Kerma	gray	Gy	$\text{m}^2 \cdot \text{s}^{-2}$
Dose equivalent	sievert	Sv	$\text{m}^2 \cdot \text{s}^{-2}$
Catalytic activity	katal	kat	$\text{mol} \cdot \text{s}^{-1}$

1.2 Dimensional Analysis

The **dimensions** of a physical quantity can be described as a generalization of the unit of measurement. That is, the dimensions remain invariable for a particular quantity whereas the unit in which it is expressed may vary.

For example, the dimension of the distance d between the Andromeda Galaxy and the Milky Way Galaxy is equal to length, which we write as $[L]$. Given that 1 light-year is measured as 9.46×10^{15} m and 1 parsec (pc) as 3.26 light-years, we can write the distance between both galaxies in terms of the unit metre, light-year, or parsec: $d = 2.32 \times 10^{22}$ m, $d = 2.45 \times 10^6$ light-years, or $d = 7.52 \times 10^5$ pc.

The dimensions of a physical quantity are useful for a number of reasons. In the first place, they let us **verify the validity of a formula**. Take for instance the Reynolds number Re , which is a dimensionless number that reflects the nature of the dynamical pattern of a fluid. Suppose that we have written down the following formula:

$$Re = \frac{\rho u}{\mu}$$

whereby ρ represents the fluid density (in $\text{kg} \cdot \text{m}^{-3}$), u the flow speed (in $\text{m} \cdot \text{s}^{-1}$), and μ the dynamic viscosity (in $\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}$). To check whether the above formula is correct, we perform a dimensional analysis. With $[M]$ and $[T]$ the dimension of the physical quantity mass and time, respectively, the dimensions of the right-hand side of the above formula are determined as follows:

$$\frac{[\rho] \cdot [u]}{[\mu]} = \frac{([M][L]^{-3})([L][T]^{-1})}{([M][L]^{-1}[T]^{-1})} = [L]^{-1}$$

Since the Reynolds number is a dimensionless quantity, we know that we made a mistake. The correct formula requires an additional dimension of length in the numerator, so that the final formula is the following:

$$Re = \frac{\rho u L}{\mu}$$

Secondly, the dimensions can be used to **formulate a physical law**. Suppose we would like to find an expression for Kepler's third law, which describes the relationship between, on the one hand, the orbital period T of an object—the period is the time it takes that object to complete one revolution—and, on the other hand, the distance r from that object to the massive object around which it orbits and the mass M_m of that massive object. If a_0 represents a certain numerical constant and given that at such large scales the universal gravitational constant G , which has a unit $\text{kg}^{-1} \cdot \text{m}^3 \cdot \text{s}^{-2}$, also plays a role, we can then algebraically write:

$$T = a_0 G^\alpha M_m^\beta r^\gamma$$

In terms of dimensions, this becomes:

$$[T] = ([M]^{-1}[L]^3[T]^{-2})^\alpha ([M])^\beta ([L])^\gamma$$

The above expression is equivalent to solving the following system of three equations:

$$\begin{cases} -\alpha + \beta = 0 \\ 3\alpha + \gamma = 0 \\ -2\alpha = 1 \end{cases}$$

The solution is equal to $\alpha = \beta = -\frac{1}{2}$ and $\gamma = \frac{3}{2}$. This means that Kepler's third law takes on the following form:

$$T = a_0 G^{-\frac{1}{2}} M_m^{-\frac{1}{2}} r^{\frac{3}{2}} = \frac{a_0}{\sqrt{G M_m}} r^{\frac{3}{2}}$$

Finally, the dimensions also allow us to **determine the scale of fundamental constants**. Let us for example estimate the strength of the electromagnetic force—one of the four fundamental forces of nature—which is reflected by the dimensionless coupling constant α , a.k.a. the fine-structure constant. With the relevant parameters equal to the Planck constant h , the speed of light c , the elementary charge e , and the permeability of free space μ_0 , which has a unit $\text{H} \cdot \text{m}^{-1}$, the expression for α becomes (with a_0 a certain constant):

$$\alpha = a_0 \mu_0^{p_1} c^{p_2} e^{p_3} h^{p_4}$$

The corresponding dimensions are:

$$1 = ([M][L]^2[T]^{-2}[I]^{-2}[L]^{-1})^{p_1} ([L][T]^{-1})^{p_2} ([I])^{p_3} ([M][L]^2[T]^{-1})^{p_4}$$

This corresponds to the following system of four equations:

$$\begin{cases} p_1 + p_4 = 0 \\ p_1 + p_2 + 2p_4 = 0 \\ -2p_1 - p_2 - p_4 = 0 \\ -2p_1 + p_3 = 0 \end{cases}$$

Since all the parameters are relevant, we must exclude the solution whereby they are all equal to zero. If we start with setting, for instance, $p_1 = 1$, we find that $p_2 = 1$, $p_4 = -1$, and $p_3 = 2$. If you know that $a_0 = \frac{1}{2}$, the value of the fine-structure constant α then becomes:

$$\begin{aligned} \alpha &= \frac{1}{2} \frac{\mu_0 c e^2}{h} = \frac{1}{2} \frac{(1.256637062 \times 10^{-6}) (299,792,458) [1.602176634 \times 10^{-19}]^2}{6.62607015 \times 10^{-34}} \\ &= \frac{1}{2} (0.0145947) \approx 7.30 \times 10^{-3} \end{aligned}$$

This corresponds with a scale of approximately 10^{-2} or an order of magnitude of -2 (see next section).

1.3 SI Prefixes and Orders of Magnitude

The SI units can be combined with **SI prefixes**, which represent decimal submultiples (with a power < 0) and multiples (with a power > 0) extending from 10^{-24} to 10^{24} .

For instance, one nanotesla (nT) is equal to 10^{-9} T, 5.45 exajoule to 5.45×10^{18} J, and 63 femtometer to 63×10^{-15} m. Similarly, a mass density of $2.20 \text{ g} \cdot \text{cm}^{-3}$ is equal to $2.20 (10^{-3} \text{ kg}) \cdot (10^{-2} \text{ m})^{-3}$ or $2,200 \text{ kg} \cdot \text{m}^{-3}$, and a molar ionic conductivity (for the chloride anion) of $7.642 \times 10^{-3} \text{ S} \cdot \text{m}^2 \cdot \text{mol}^{-1}$ to $7.642 \times 10^{-3} \text{ S} \cdot (10^2 \text{ cm})^2 \cdot \text{mol}^{-1}$ or $76.42 \text{ S} \cdot \text{cm}^2 \cdot \text{mol}^{-1}$.

Table 1.3: The SI prefixes with their corresponding powers of ten

Prefix	Symbol	Factor	Prefix	Symbol	Factor
yocto	y	10^{-24}	deca	da	10^1
zepto	z	10^{-21}	hecto	h	10^2
atto	a	10^{-18}	kilo	k	10^3
femto	f	10^{-15}	mega	M	10^6
pico	p	10^{-12}	giga	G	10^9
nano	n	10^{-9}	tera	T	10^{12}
micro	μ	10^{-6}	peta	P	10^{15}
milli	m	10^{-3}	exa	E	10^{18}
centi	c	10^{-2}	zetta	Z	10^{21}
deci	d	10^{-1}	yotta	Y	10^{24}

The **order of magnitude** of a number is a measurement of scale and can be found by taking the logarithm of base 10 of that number—for more details on logarithms see Section 2.2.3—and rounding it to the nearest integer.

For example, the order of magnitude of 10^9 is equal to $\log_{10}(10^9) = 9$. Going back to the example about the fine-structure constant in Section 1.2, the logarithm of 7.30×10^{-3} is equal to $\log_{10}(7.30 \times 10^{-3}) = -2.14$. Rounding it off to the nearest integer means that the order of magnitude is equal to -2 (and not -3). A final example is the number 46,993, which we could also write as 4.70×10^4 . Its logarithm is calculated as $\log_{10}(46,993) = 4.67$. This gives us an order of magnitude of 5 (and not 4).

1.4 Ratios and Proportions

A **ratio** describes the way in which a certain number relates to another number. The relationship between the numbers can be expressed in plain words, fractions, scales, or percentages. A **fraction** is a mathematical expression of a rational number with the general form $\frac{n_1}{n_2}$ whereby n_1 and n_2 are both integers and $n_2 \neq 0$. A **scale** expresses the numerical

relationship through a colon notation and a **percentage** reflects the number of parts per 100.

As an example suppose that a large glass bowl standing on your kitchen table contains small chocolate Easter eggs with a red, silver, and golden paper wrapping, and that for every 3 red-coloured eggs there are 15 golden eggs. This relationship can be expressed as 1 to 5 (words), $\frac{1}{5}$ (fractions), 1:5 (scales), or 20% (percentages).

Scales are commonly used when drawing maps of geographical areas. For example, if a map of the city of Adelaide in Australia shows a scale of 1:500,000 (cm) it means that every 1 cm on the map corresponds to a distance of 5 km within that city. The number 500,000 is called the **scale factor**. Say you measure a straight distance of 8.23 cm on your map between Aberfoyle Park and Golden Grove. Multiplying that distance with the scale factor leads you to understand that both locations are approximately 41.2 km apart in the real world.

In the above example of the eggs, the units of the two numbers that are being compared are the same (i.e., eggs), so that they cancel out in the fractional description of the ratio. If the units are different, we obtain a special kind of fractional ratio called **rate**.

For instance, the city of Portsmouth in the United Kingdom has a population density of 129,000 inhabitants per 25 square kilometers. The units of the numbers in the numerator and denominator of the fraction do not cancel out, so that the rate is written as $\frac{129,000 \text{ inhabitants}}{25 \text{ km}^2}$.

If the denominator of a rate is equal to 1, the rate is called a **unit rate**.

For example, a bathtub is filled at a constant rate of 6.5 L per 1.35 minutes. The corresponding unit rate is equal to $\frac{6.5 \times 10^3 \text{ mL}}{(60 + 0.35 \cdot 60) \text{ s}} = 80.2 \text{ mL} \cdot \text{s}^{-1}$ or $\frac{80.2 \times 10^{-3} \text{ L}}{\frac{1}{3.600} \text{ hour}} = 289 \text{ L per hour}$.

A **proportion** is defined as the equality of two ratios. In other words, a proportion implies that the ratios remain invariable.

Let us consider the law of Gay-Lussac, which states that the rate of the pressure P to the temperature T of a gas (for a given volume and a given amount of gas) remains constant: $\frac{P}{T} = k$. Suppose that the initial temperature T_1 of a certain gas is increased by a factor of 1.75, so that the final temperature measures $T_2 = 1.75 T_1$. The pressure of the gas in the final situation can then be expressed as $P_2 = k T_2 = k (1.75 T_1) = 1.75 (k T_1) = 1.75 P_1$. We can now see that the ratios remain constant:

$$\frac{P_2}{T_2} = \frac{1.75 P_1}{1.75 T_1} = \frac{P_1}{T_1}$$

In the above example of the law of Gay-Lussac, the pressure P is **directly proportional** to the temperature T , i.e., both physical quantities vary to the same extent: if the temperature T_1 is driven upwards by a factor of 1.75, then the pressure P_1 equally goes up by that same factor.

A counterexample whereby two quantities are not directly proportional to one another is the speed v of transverse waves traveling along a stretched cord:

$$v = \sqrt{\frac{T}{\mu}}$$

If the tension force T in the cord is doubled, then the speed v of the wave also increases but to a lesser extent, i.e., by a factor of $\sqrt{2}$ (< 2).

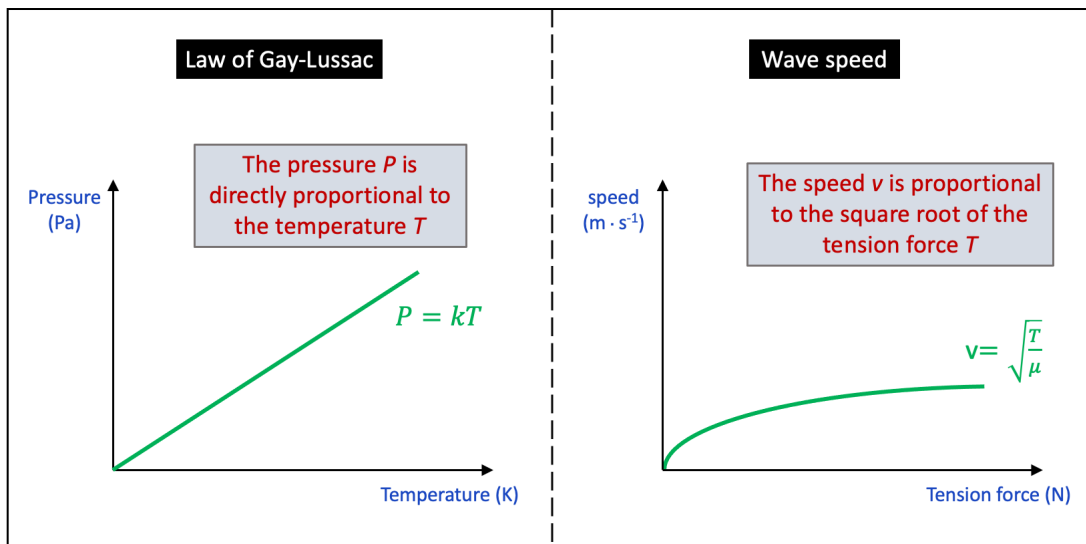


Figure 1.1: A graphical representation of the law of Gay-Lussac (left) and the speed of transverse waves on a stretched cord (right)

What the above two examples have in common is that the change of the two related quantities occurs in the same direction, i.e., if the value of one quantity decreases—or increases—so does the value of the other quantity (under the assumption that any other quantity remains constant). However, if the values of two related quantities change in the opposite direction, we say that these quantities are **inversely proportional** to one another.

For instance, the apparent brightness b of a star or galaxy is inversely related to the square of the distance d between the star or galaxy and the Earth. For a given luminosity L , the formula reads as follows:

$$b = \frac{L}{4\pi d^2}$$

If the distance d between the star or galaxy is cut by half, it follows that the apparent brightness b is enhanced by a factor of 4.

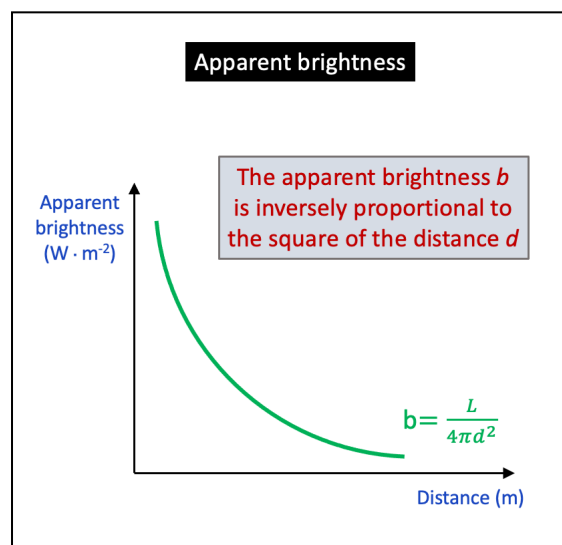


Figure 1.2: A graphical representation of the formula for the apparent brightness b of a star or galaxy

Proportionality is mathematically represented by the symbol \propto . For the examples with respect to the law of Gay-Lussac, the wave speed on a cord, and the apparent brightness, we can thus write: $P \propto T$, $v \propto T^{1/2}$, and $b \propto d^{-2}$.

Proportions can furthermore be used to **solve particular problems**. Let us again consider Kepler's third law, which we discussed in Section 1.2. Say we have the following problem: Given that Global Positioning System (GPS) satellites orbit Earth at a height of $h_1 = 20.2 \times 10^3$ km with a period of $T_1 = 12.0$ hours, what is the average period T_2 of the International Space Station (ISS) if you know that its average orbital height is equal to $h_2 = 370$ km?

From the formula derived in Section 1.2, we see that the term $\frac{a_0}{\sqrt{GM_m}}$ (with M_m the mass of the Earth) is constant. Therefore, we can write the following proportion:

$$\frac{T_1}{r_1^{3/2}} = \frac{T_2}{r_2^{3/2}}$$

In Kepler's third law, the distances r_1 and r_2 represent the orbital radii, measured from the center of the Earth. If we know that the radius of the Earth is equal to $r = 6.38 \times 10^6$ m, it follows the orbital radii are equal to $r_1 = h_1 + r = 2.02 \times 10^7 + 6.38 \times 10^6 = 2.66 \times 10^7$ m and $r_2 = h_2 + r = 3.70 \times 10^5 + 6.38 \times 10^6 = 6.75 \times 10^6$ m, respectively. The period T_2 of the ISS is then calculated as follows:

$$T_2 = T_1 \left(\frac{r_2}{r_1} \right)^{3/2} = 12.0 \left(\frac{6.75 \times 10^6}{2.66 \times 10^7} \right)^{3/2} = 1.54 \text{ hours or } 1 \text{ h } 32 \text{ min}$$

Finally, let us look a little closer to the concept of **percentages**. To illustrate the use of percentages, we would like to calculate the mass percent composition of 1 mole of the compound glucose ($\text{C}_6\text{H}_{12}\text{O}_6$), given that the molar mass of carbon, hydrogen, and oxygen is equal to $M_C = 12.01 \text{ g} \cdot \text{mol}^{-1}$, $M_H = 1.01 \text{ g} \cdot \text{mol}^{-1}$, and $M_O = 16.00 \text{ g} \cdot \text{mol}^{-1}$, respectively.

Since the molar mass of glucose is equal to $M_g = 6 \times 12.01 + 12 \times 1.01 + 6 \times 16.00 = 180.2 \text{ g} \cdot \text{mol}^{-1}$, the relative mass of the element carbon within 1 mole of the compound glucose is then calculated as $6 \times \frac{M_C}{M_g} = 6 \times \frac{12.01}{180.2} = 0.40$. To obtain the percentage, we multiply that amount by 100: 40%.

Similarly, the relative mass (in percentage) of the elements hydrogen and oxygen is equal to $12 \times \frac{M_H}{M_g} \times 100 = 12 \times \frac{1.01}{180.2} \times 100 = 6.73\%$ and $6 \times \frac{M_O}{M_g} \times 100 = 6 \times \frac{16.00}{180.2} \times 100 = 53.3\%$, respectively.

For smaller amounts, we can make use of the measures **parts per million (ppm)** and **parts per billion (ppb)**. Given that 1% is defined as 1 part per 100 and that 1 ppm is equal to the ratio $\frac{1}{1,000,000}$, it follows that 1 ppm = 0.0001%. To the same extent, 1 ppb = 0.0000001%.

For instance, an average Canadian adult of 70.0 kg is believed to take in about 2.6 μg of uranium on a daily basis. Taking into account that 1 kg is equal to $10^9 \mu\text{g}$, the equivalent amount in terms of ppb is equal to $\frac{2.6}{70} = 0.037$ ppb.

2 Algebra

In general terms, **algebra** is the subfield within mathematics that describes the relationship between variables by relying upon mathematical statements.

2.1 Definitions

A mathematical statement is usually written with the help of an **algebraic expression**, which consists of any number of terms connected through the mathematical operation of either addition or subtraction (see also polynomials in section 2.3).

A **term** is the main component of an algebraic expression and is made up of factors. A **factor** can be a numerical coefficient or a combination of coefficients and variables that are connected through the mathematical operation of either multiplication or division. A **numerical coefficient** is any real number with a fixed value and a **variable** is a letter of the Roman or Greek alphabet that represents a real number whereby its value can vary. A term that only contains a numerical coefficient is called a **constant**.

Let us consider the following example:

$$3.9 + s - 1.6\pi t + \frac{\alpha t^2}{2}$$

The above algebraic expression contains four terms. The first term "3.9" is a constant. The second term "s" is a product of two factors: the numerical coefficient 1 and the variable s. The third term "1.6πt" consists of the factor 1.6π, which is the numerical coefficient, and the factor t, which is the variable. The final term " $\frac{\alpha t^2}{2}$ " comprises the numerical coefficient 1/2 and the product of the variable α and two variables t (t²).

A mathematical statement that indicates the equality of two algebraic expressions is called an **equation**, whereas any other type of relationship between two expressions is referred to as an **inequality**, i.e., not equal to (≠), smaller than (<), greater than (>), smaller than or equal to (≤), and greater than or equal to (≥). In the below examples, only the second one qualifies as an equation:

$$\left\{ \begin{array}{l} 5x^2 - 2 \geq \frac{3}{4}y + z^3 \\ 344 + 385 = 3^6 \\ 88t < 2.3 + 5.5t^2 \end{array} \right.$$

A special kind of equation is the **formula**, which signals the relationship between two or more variables. For instance, the formula for the volume of a cone expresses the relation between the volume V on the one hand and the diameter d of the base and the height h of the cone at the other hand:

$$V = \frac{\pi}{12} d^2 h$$

Another special kind of equation is the **identity**, which is always true regardless of the input value of the variables. An example of an algebraic identity is the following:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

2.2 Laws of Indices, Roots, and Logarithms

2.2.1 Indices

An **index** or a **power** is a mathematical notation whereby a product of the same real number or variable, multiplied a certain number of times, is written in a succinct way. That real number is referred to as the **base** and the index refers to the number of times that the base is being multiplied.

For example, the expression $7 \times 7 \times 7 \times 7 \times 7$ can be written as 7^5 , whereby the base and the index are equal to 7 and 5, respectively. We read this notation as ‘seven to the power of five’ or ‘the fifth power of seven’.

The **Laws of Indices** include a set of rules that are used to manipulate algebraic expressions *with the same base*. Fig. 2.1 provides a list of these rules, whereby a is a non-zero real number and r and s are integers.

➤ $a^0 = 1$	➤ $\frac{a^r}{a^s} = a^{r-s}$
➤ $a^1 = a$	➤ $(a^r)^s = a^{rs}$
➤ $a^r \times a^s = a^{r+s}$	➤ $a^{-r} = \frac{1}{a^r}$

Figure 2.1: The Laws of Indices

With the help of the above rules, we can for instance simplify the below term as follows:

$$\frac{5 k^2 x^6 y^1 z^{-3} k (x^{-2})^4}{z^0 (5^2)^{-3} y^{-8} y^4 x^3 z^7} = \frac{5^7 k^3 y^5}{x^5 z^{10}}$$

2.2.2 Roots

If n is a (non-zero) natural number and a a real number, then we can define the real number y as the **n^{th} root of a** as long as it satisfies the following equation:

$$y^n = a$$

Every real number a has none, one, or two n^{th} roots. If n is odd, then there is just one solution, i.e., $\sqrt[n]{a}$, which is either positive (if $a > 0$) or negative (if $a < 0$). In the case that n is even, there are two possibilities: if $a \geq 0$, there are two solutions, i.e., $\pm\sqrt[n]{a}$, or, if $a < 0$, there is no solution. If $n = 2$, the n^{th} root is called the square root, and if $n = 3$, it is called the cube root.

For example, the 6th root of -334 ($\sqrt[6]{-334}$) has no real solutions, the cube root of $-39,304$ ($\sqrt[3]{-39,304}$) has just one solution, i.e., -34 , and the 10th root of 32 has two solutions, i.e., $\sqrt{2}$ and $-\sqrt{2}$.

Note furthermore that for every real number a , and with n a (non-zero) natural number, the following identity applies:

$$\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

For instance, $\sqrt{(-6.4)^2} = |-6.4| = 6.4$ and $\sqrt[5]{(8.2)^5} = 8.2$.

For a non-zero real number a and a (non-zero) natural number n , we can write the n^{th} root of a as follows:

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

We can now expand the Laws of Indices with one more rule, whereby m is a (non-zero) natural number:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}$$

For example, $3^{\frac{162}{27}} = \left(3^{\frac{1}{27}}\right)^{162} = (3^{162})^{\frac{1}{27}} = \sqrt[27]{3^{162}} = 729$.

For any real numbers a and b and any (non-zero) natural numbers q , r , and s , Fig. 2.2 lists a number of general properties of roots, which are valid on the condition that for every property the corresponding root exists:

$\triangleright \sqrt[r]{ab} = \sqrt[r]{a}\sqrt[r]{b} = (ab)^{\frac{1}{r}}$	$\triangleright \sqrt[r]{\sqrt[s]{a}} = \sqrt[rs]{a} = (a)^{\frac{1}{rs}}$
$\triangleright \sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \left(\frac{a}{b}\right)^{\frac{1}{r}}$	$\triangleright \sqrt[qr]{a^{sq}} = \sqrt[rs]{a^s} = a^{\frac{s}{r}}$
$\triangleright \sqrt[r]{a^s} = (\sqrt[r]{a})^s = a^{\frac{s}{r}}$	

Figure 2.2: General properties of roots

Let us consider two examples:

$$\left\{ \begin{array}{l} \sqrt[7]{22^9} = \sqrt[7]{22^2 \cdot 22^7} = \sqrt[7]{22^2} \sqrt[7]{22^7} = 22 \sqrt[7]{22^2} \\ \sqrt[9]{6^3 \sqrt[3]{9^2}} = \sqrt[9]{\sqrt[3]{6^9 \cdot 9^2}} = \sqrt[27]{6^9 \cdot 9^2} = \sqrt[27]{2^9 \cdot 3^{13}} = \sqrt[3]{2} \sqrt[27]{3^{13}} \end{array} \right.$$

2.2.3 Logarithms

The **logarithm** is used to answer the following question: how many times has the value of a certain quantity been increased or decreased by a pre-determined factor or step size? Put differently, the logarithm measures the number of times a factor appears in a series of multiplications that consists only of that factor.

The step size is referred to as the *base* of the logarithm. In mathematical language, the base- a logarithm of y is defined as follows, whereby a and y are non-zero real numbers with $a > 0$, $a \neq 1$, and $y > 0$:

$$\log_a y = x \Leftrightarrow a^x = y$$

For example, the base-5 logarithm of 15,625 is equal to $\log_5(15,625) = 6$, because $5^6 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$. In other words, the value of a certain quantity has been increased 6 times by a factor of 5. Another example is the base-7.5 logarithm of 2.37×10^{-3} , which is calculated as $\log_{7.5}(2.37 \times 10^{-3}) = -3.0$, since $7.5^{-3} = \frac{1}{7.5} \cdot \frac{1}{7.5} \cdot \frac{1}{7.5} = 2.37 \times 10^{-3}$, and indicates that the value of a certain quantity has been reduced 3 times with respect to a step size of 7.5.

Fig. 2.3 provides a list of properties of logarithms, whereby $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, and r any real number. Assume furthermore that x and y are well-defined for every property.

➤ $\log_a 1 = 0$	➤ $\log_a(xy) = \log_a x + \log_a y$
➤ $\log_a a = 1$	➤ $\log_a \frac{x}{y} = \log_a x - \log_a y$
➤ $\log_a a^r = r$	➤ $\log_a \frac{1}{y} = -\log_a y$
➤ $\log_a y^r = r \log_a y$	➤ $\log_a y = \frac{\log_b y}{\log_b a}$
➤ $a^{\log_a y} = y$	➤ $\log_a y = \frac{1}{\log_y a}$

Figure 2.3: General properties of logarithms

If the base of a logarithm is equal to the number 10, the logarithm is called the **common**

logarithm, whereas a logarithm with a base equal to Euler's number $e = 2.71828\dots$ is referred to as the **natural logarithm**. The base 10 of the common logarithm is usually not written and the natural logarithm has its own unique notation:

$$\begin{cases} \log_{10} y = \log y \\ \log_e y = \ln y \end{cases}$$

Let us look at three examples:

$$\begin{cases} \log_5 625 = \frac{\log 625}{\log 5} = \frac{\log 5^4}{\log 5} = \frac{4 \cdot \log 5}{\log 5} = 4 \\ \ln \left(\frac{e^x}{e^y} \right) = \ln(e^x) - \ln(e^y) = x - y \\ \log \left(\sqrt[4]{5.23 \times 10^{-12}} \right) = \frac{1}{4} [\log(5.23) + \log(10^{-12})] = \frac{1}{4} [\log(5.23) - 12] = -2.82 \end{cases}$$

2.3 Polynomials

2.3.1 Definitions

Algebraic expressions with a finite number of terms and whereby any exponents of the variables are non-negative integers are called **polynomials**. Polynomials with one, two, or three terms are referred to as monomials, binomials, and trinomials, respectively.

Generally, a polynomial of the variable y takes the following form, with the coefficients b_0, b_1, \dots, b_n real numbers and n a natural number:

$$b_0 + b_1 y + b_2 y^2 + \dots + b_n y^n = \sum_{i=0}^n b_i y^i$$

The **degree** of a polynomial is equal to the highest of the degrees of all the terms, whereby the degree of a term is calculated as the sum of the exponents of all the variables included within the term. In the **standard form** of a polynomial the terms are positioned from left to right in decreasing order of degree.

For instance, the degree of the polynomial $3.4x^3y - 5y^4z^4 + x^3yz^3 + 5.5$ (not in standard form) is equal to 8 since the sum of the exponents of the variables of the second term is the highest of all the terms.

A polynomial is called **homogeneous of degree m** if the degree of every term of the polynomial is equal to m .

2.3.2 Long Division

Apart from multiplying polynomials with each other, polynomials can also be divided by one another through a method called **polynomial long division**. If a polynomial $p(x) = \sum_{i=0}^m b_i x^i$ of degree m is divided by the polynomial $s(x) = \sum_{i=0}^n c_i x^i$ of degree n , whereby $n \leq m$, then we can write:

$$p(x) = q(x) \cdot s(x) + r(x)$$

whereby $q(x)$ represents the **quotient** with a degree equal to $m - n$ and $r(x)$ the **remainder** with a degree smaller than or equal to $n - 1$. The polynomial $p(x)$ and $s(x)$ are called the **dividend** and the **divisor**, respectively.

There are five steps to follow when implementing the method of polynomial long division:

1. Write the dividend $p(x)$ and divisor $s(x)$ in the standard form
2. Divide the first term of the dividend $p(x)$ by the first term of the divisor $s(x)$ and add the result to the final answer $q(x)$
3. Multiply the result of step 2 by the divisor $s(x)$ and write the answer below the dividend $p(x)$
4. Subtract the answer of step 3 from the dividend $p(x)$, resulting in a new polynomial
5. Repeat the steps 2 to 4, replacing thereby at every cycle the dividend $p(x)$ in the description by the newly created polynomial of step 4 until the division has ended

In Fig. 2.4, two examples of polynomial long division are provided whereby the dividend $p(x)$ is displayed in blue, the divisor $s(x)$ in green, the quotient $q(x)$ in orange, and the remainder $r(x)$ in red.

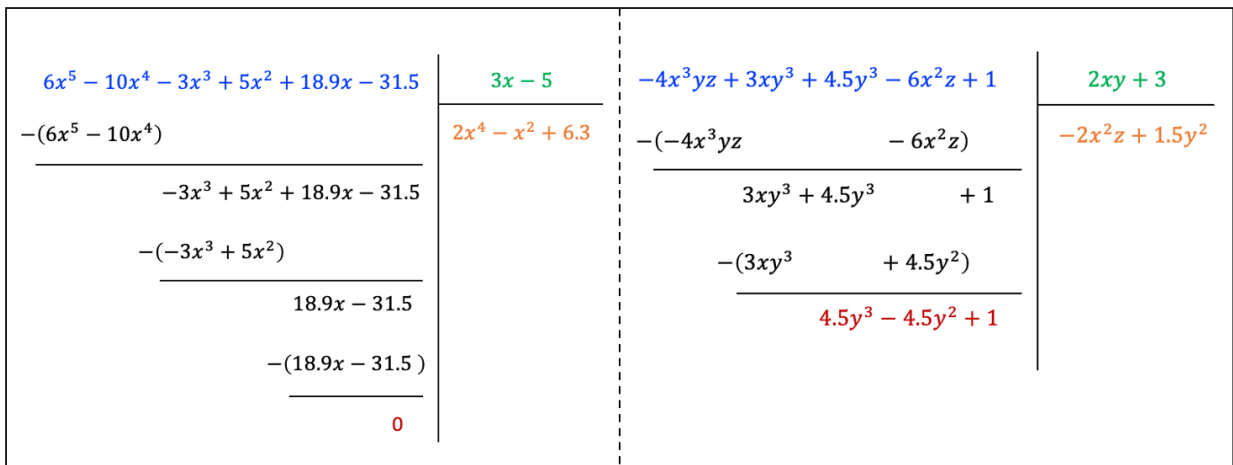


Figure 2.4: Two examples of polynomial long division

The dividend $p(x)$ in each of the above two examples can thus be reformulated as follows:

$$\left\{ \begin{array}{l} 6x^5 - 10x^4 - 3x^3 + 5x^2 + 18.9x - 31.5 = (3x - 5)(2x^4 - x^2 + 6.3) \\ -4x^3yz + 3xy^3 + 4.5y^3 - 6x^2z + 1 = (2xy + 3)(-2x^2z + 1.5y^2) + 4.5y^3 - 4.5y^2 + 1 \end{array} \right.$$

2.3.3 Factoring Polynomials

The process of writing a polynomial as a product of two or more polynomials is called **the factorization of polynomials**. A polynomial is *completely* factored when all the factors cannot be further reduced. Note also that the sum of the degrees of the various factors must be equal to the degree of the initial polynomial. In the first of the above two examples, the degree of the two factors $(3x - 5)$ and $(2x^4 - x^2 + 6.3)$ are 1 and 4, respectively, which gives a total degree of 5, i.e., the degree of the original polynomial.

Factoring polynomials is convenient for a number of reasons: it allows the polynomial to be written in a more simple form; it greatly facilitates the process of solving equations; and it helps to better understand the behaviour of polynomials graphically.

Let us look at six methods to factor polynomials: the largest common factor, grouping, splitting terms, quadratic trinomials, special products, and roots of polynomials. With respect to the first method, the **largest common factor** of a polynomial is isolated from every term. For instance:

$$92x^6y^2z^3 + 16x^5y^2z^2 - 48x^3yz^2 = 4x^3yz^2(23x^3yz + 4x^2y - 12)$$

The second method of **grouping** can be useful when there are at least four terms in the polynomial. In a first step, two or more groups of terms are formed. The groups are then factored by isolating a common factor from each group. If a common factor is present among all the factored groups of the previous step, then, in a final step, the expression is again factored. For example:

$$\begin{aligned} 8x^3 - 18x^2 + 20x - 45 &= [8x^3 + 20x] - [18x^2 + 45] \\ &= [4x(2x^2 + 5)] - [9(2x^2 + 5)] \\ &= (2x^2 + 5)(4x - 9) \end{aligned}$$

The third method of **splitting terms** is typically used for second-degree polynomials of one variable called quadratic polynomials. The most general form of such a quadratic expression is equal to:

$$acx^2 + (ad + bc)x + bd$$

The method of splitting terms consists of splitting the second term (of the first degree) in such a way so that by resorting to the method of grouping the polynomial can be factored.

The above general form can be factored as follows:

$$\begin{aligned}
 acx^2 + (ad + bc)x + bd &= acx^2 + adx + bcx + bd = [acx^2 + adx] + [bcx + bd] \\
 &= [ax(cx + d)] + [b(cx + d)] \\
 &= (ax + b)(cx + d)
 \end{aligned}$$

Consider the following numerical example:

$$\begin{aligned}
 3x^2 - 7x + 2 &= 3x^2 + (-6x - x) + 2 = [3x^2 - 6x] + [-x + 2] \\
 &= [3x(x - 2)] + [-(x - 2)] \\
 &= (3x - 1)(x - 2)
 \end{aligned}$$

The fourth method of **quadratic trinomials** is similar to the previous method of splitting terms but does not involve splitting the (first-degree) second term. That is, although it also takes the general form of a quadratic polynomial $acx^2 + (ad + bc)x + bd$ as a starting point, the method basically consists of applying a trial and error strategy to find the two factors of the (zero-degree) third term, i.e., bd .

Let us look for example at the quadratic trinomial $12x^2 + 17x - 5$. In a first step, when considering the coefficient of the (second-degree) first term $ac = 12$, we identify three possible solutions, which have the form $(1x + b)(12x + d)$, $(3x + b)(4x + d)$, and $(2x + b)(6x + d)$. In the next step, we try different combinations of the (zero-degree) third term $bd = -5$ and insert them in these three possible solutions. Eventually, we find that the only solution to give $17x$ as the (first-degree) second term is equal to $(3x + 5)(4x - 1)$.

The fifth method that we will discuss to factor polynomials relies upon **algebraic identities** also known as special products. Fig. 2.5 lists some of the most used identities, whereby x , y , and z represent real numbers.

➤ $x^2 \pm 2xy + y^2$	$= (x \pm y)^2$
➤ $x^2 - y^2$	$= (x + y)(x - y)$
➤ $x^3 \pm 3x^2y + 3xy^2 \pm y^3$	$= (x \pm y)^3$
➤ $x^3 \pm y^3$	$= (x \pm y)(x^2 \mp xy + y^2)$
➤ $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$	$= (x + y + z)^2$
➤ $x^3 + y^3 + z^3 - 3xyz$	$= \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

Figure 2.5: Algebraic identities

Let us consider the following example:

$$\begin{aligned}
 54x^4 + 27x^3 - 16x - 8 &= [54x^4 + 27x^3] + [-16x - 8] = [27x^3(2x + 1)] + [-8(2x + 1)] \\
 &= (27x^3 - 8)(2x + 1) \\
 &= [(3x)^3 - 2^3](2x + 1) \\
 &= (3x - 2)(9x^2 + 6x + 4)(2x + 1)
 \end{aligned}$$

The sixth and final method entails finding a **root of the polynomial**. The root of a polynomial $p(x)$ is defined as the real number c for which the equation $p(c) = 0$ holds. This furthermore means that the term $x - c$ is a divisor of the polynomial $p(x)$, so that $p(x)$ can be factored by applying the method of polynomial long division.

For instance, with a bit of educated guess work we can see that $c = -2$ is a root of the polynomial $p(x) = 2x^4 + x^3 + 2x^2 + 4x - 24$. If we divide $p(x)$ by $x + 2$, we find the quotient $q(x) = 2x^3 - 3x^2 + 8x - 12$, which can be further factored as follows:

$$\begin{aligned}
 2x^4 + x^3 + 2x^2 + 4x - 24 &= (x + 2)(2x^3 - 3x^2 + 8x - 12) \\
 &= (x + 2)[x^2(2x - 3) + 4(2x - 3)] \\
 &= (x + 2)(2x - 3)(x^2 + 4)
 \end{aligned}$$

With respect to quadratic trinomials, the roots can be calculated by resorting to the method of the **discriminant**. Given the general form of a quadratic trinomial $p(x) = ax^2 + bx + c$, the discriminant D is calculated as $D = b^2 - 4ac$, whereby the roots of $p(x)$ are equal to:

$$x = \begin{cases} \frac{-b \pm \sqrt{D}}{2a} & \text{if } D > 0 \text{ (2 roots)} \\ \frac{-b}{2a} & \text{if } D = 0 \text{ (1 root with multiplicity 2)} \\ \emptyset & \text{if } D < 0 \text{ (0 roots)} \end{cases}$$

For example, the discriminant of the quadratic trinomial $p(x) = 133x^2 - 82.5x + 11$ is equal to $D = b^2 - 4ac = (-82.5)^2 - 4 \cdot 133 \cdot 11 = 954.25 > 0$. This means that the polynomial has two roots, which are calculated as follows:

$$\begin{cases} x_1 = \frac{-b + \sqrt{D}}{2a} = \frac{-(-82.5) + \sqrt{954.25}}{2 \cdot 133} = 0.426 \\ x_2 = \frac{-b - \sqrt{D}}{2a} = \frac{-(-82.5) - \sqrt{954.25}}{2 \cdot 133} = 0.194 \end{cases}$$

The quadratic trinomial $p(x) = 133x^2 - 82.5x + 11$ is then factored as $p(x) = 133 \cdot [(x - x_1)(x - x_2)] = 133 \cdot [(x - 0.426)(x - 0.194)]$. Note that we have to multiply the factored polynomial with the coefficient $a = 133$, because the above formulas of the roots implicitly work with the following adjusted form of the general quadratic trinomial: $a \left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

3 Linear Algebra

Within the field of mathematics, the branch of **linear algebra** studies linear equations, linear transformations, and vectors as well as their representations in the vector space with the assistance of matrices.

3.1 Definitions

A **linear equation** is an equation that has the general form $\sum_{i=1}^n a_i x_i = b$ or $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$, whereby a_i refers to the coefficients, x_i to the unknown variables of the linear equation, and b to a constant. It is called linear because the exponent of each variable is equal to one.

A **vector** is a geometric object that exhibits both a magnitude (or length) and a direction. It is noted with an arrow or a dash above a certain letter, such as \vec{a} or \vec{w} . A **vector space** is the mathematical space in which vectors can be added and multiplied by **scalars**, which are quantities that only have a magnitude but not a direction (such as the real numbers).

A **linear transformation or linear map** is a relationship between two vector spaces whereby the structure of the vector space is preserved, i.e., the operations of vector addition and scalar multiplication. Every linear transformation is always associated with a matrix, so that a linear transformation acting on a vector is equivalent to multiplying its matrix with that vector.

A **matrix** is a rectangular array of elements a_{ij} (with $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$) that are organized in n rows and m columns and represent a mathematical object. A matrix with n rows and m columns is noted as a $n \times m$ -matrix and can be written as (a_{ij}) . We say that the **order** of a matrix is equal to $n \times m$. In linear algebra, the majority of operations or specific properties of mathematical objects are expressed by way of matrices.

3.2 Types of Matrices

The **null matrix** is a $n \times m$ -matrix whereby the elements a_{ij} are all equal to zero.

A **square matrix** is a $n \times m$ -matrix whereby $n = m$. The order of a square matrix is equal to its number of rows (or columns). In a square matrix, the **main diagonal** is composed of the elements a_{ii} . If all the elements of a square matrix are equal to zero except the ones on the main diagonal, the matrix is called a **diagonal matrix**. If only the elements below (above) the main diagonal are all zero, the matrix is known as an **upper (lower) triangular matrix**. The **identity matrix** I_n is the diagonal matrix whereby $a_{ii} = 1$ ($i \in \{1, \dots, n\}$).

A **row matrix** is a matrix with just one row, whereas a **column matrix** has multiple rows and just one column. A row (column) matrix is also called a **row (column) vector**. A **scalar** is a 1×1 -matrix.

The **transpose** of the $n \times m$ -matrix $A = (a_{ij})$ is equal to the $m \times n$ -matrix $A^T = (a_{ji})$.

In other words, the rows (columns) of A become columns (rows) in A^T and vice versa. A square matrix A is called **symmetric** if $A^T = A$ and **antisymmetric** if $A^T = -A$.

$A = \begin{bmatrix} 3 & -1 & 7 \\ 11 & 5 & -4 \end{bmatrix}$	$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
2×3 matrix	Null matrix	Diagonal matrix	Identity matrix
$A = [5 \quad -2 \quad 0 \quad 8]$	$A = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$	$A = \begin{bmatrix} 0 & -2 & 3 \\ -2 & 1 & -5 \\ 3 & -5 & 2 \end{bmatrix}$	$A = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$
Row matrix	Column matrix	Symmetric matrix	Antisymmetric matrix

Figure 3.1: Different types of matrices

3.3 Matrix Properties and Operations

3.3.1 Addition and Multiplication

The **scalar multiplication** is the multiplication of a matrix A by a scalar s . This means that every element a_{ij} of the matrix A is multiplied with the scalar s .

The **addition** of matrices is possible only if the matrices are all of the same order, i.e., they must have the same number of rows and columns. The addition of the $p \times q$ -matrix A and the $p \times q$ -matrix B then results in the $p \times q$ -matrix C , with the elements $c_{ij} = a_{ij} + b_{ij}$ whereby a_{ij} and b_{ij} represent the elements of the matrix A and B , respectively.

The operation of addition allows us to both switch the order of positions of matrices in a sum—this property is called **commutativity**—and randomly group matrices without affecting the result—this second property is referred to as **associativity**.

The addition of matrices has furthermore the null matrix as the **neutral element**, i.e., the addition of matrix A and the null matrix gives back the original matrix A . For the scalar multiplication, the scalar 1 is the neutral element. Multiplying a matrix A with the scalar 0 always results in the null matrix.

The **multiplication of matrices** is possible only if the amount of columns of the matrix on the left-hand side of the product is equal to the amount of rows of the matrix at the right-hand side. The multiplication of the $p \times q$ -matrix A by the $q \times r$ -matrix B then results in the $p \times r$ -matrix C , whereby the elements c_{ik} are defined as $c_{ik} = \sum_{j=1}^q a_{ij}b_{jk}$, with a_{ij} and b_{jk} the elements of the matrix A and B , respectively, and $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$, and $k \in \{1, \dots, r\}$.

For example, the product of the 3×2 -matrix A by the 2×2 -matrix B , which are defined below, results in the 3×2 -matrix C . The element c_{22} , for instance, is then calculated as

$$c_{22} = \sum_{j=1}^2 a_{2j}b_{j2} = a_{21}b_{12} + a_{22}b_{22} = 0 \cdot 0 + 1 \cdot (-1) = -1.$$

$$A \cdot B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & -1 \\ -2 & -6 \end{bmatrix}$$

The operation of multiplication of matrices is associative but *not* commutative. What is more, the identity matrix is the neutral element and multiplication with a null matrix always results in a null matrix.

Fig. 3.2 summarizes the different properties of (scalar) multiplication and addition of matrices, whereby the letters r and s represent scalars, the matrices A , B , and C represent a $m \times n$ -matrix, the matrices $O_{m,n}$ and I_m are defined as the $m \times n$ null matrix and the $m \times m$ identity matrix, respectively, and the matrices D and E have the form of a $n \times p$ -matrix and a $p \times q$ -matrix, respectively. Note that the product $D \cdot A$ is only defined if $p = m$.

<u>Properties of scalar multiplication</u>	
➤ $s \cdot (r \cdot A) = (s \cdot r) \cdot A$	➤ $1 \cdot A = A$
➤ $s \cdot (A + B) = s \cdot A + s \cdot B$	➤ $0 \cdot A = O_{m,n}$
➤ $(s + r) \cdot A = s \cdot A + r \cdot A$	
<u>Properties of addition</u>	<u>Properties of matrix multiplication</u>
➤ $A + B = B + A$	➤ $A \cdot D \neq D \cdot A$
➤ $A + (B + C) = (A + B) + C$	➤ $A \cdot (D \cdot E) = (A \cdot D) \cdot E$
➤ $A + O_{m,n} = A = O_{m,n} + A$	➤ $A \cdot O_{n,p} = O_{m,p} = O_{m,n} \cdot D$
	➤ $I_m \cdot A = A = A \cdot I_n$

Figure 3.2: Properties of (scalar) multiplication and addition of matrices

3.3.2 Transposed, Orthogonal, and Inverse Matrices

Now that we have defined the (scalar) multiplication and addition of matrices, we can look at the properties of **transposed matrices**, which are presented in Fig. 3.3. Note that the letter r represents a scalar, the matrices A and B are $m \times n$ -matrices, and the matrix D represents a $n \times p$ -matrix.

Properties of transposed matrices

➤ $(A^T)^T = A$	➤ $(r \cdot A)^T = r \cdot A^T$
➤ $(A + B)^T = A^T + B^T$	➤ $(A \cdot D)^T = D^T \cdot A^T$

Figure 3.3: Properties of transposed matrices

A square $p \times p$ -matrix L is called an **orthogonal matrix** if $L \cdot L^T = I_p = L^T \cdot L$. For example, the 2×2 matrix L as defined below is an orthogonal matrix.

$$L \cdot L^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = L^T \cdot L$$

A square $n \times n$ -matrix M is referred to as the **inverse matrix** of a given $n \times n$ -matrix P if $P \cdot M = I_n = M \cdot P$. The matrix M is then formulated as $M = P^{-1}$. Note that only square matrices can be inverted and that not all square matrices are invertible.

Square matrices for which the inverse is defined are called **non-singular matrices**, whereas square matrices that are not invertible are called **singular matrices**.

In the case that P is an orthogonal matrix, it follows from the definition that $P^T = P^{-1}$. That is, the inverse of an orthogonal matrix is equal to its transposed matrix. Reversely, if we find for a given matrix S that $S^T = S^{-1}$, we can conclude that S is an orthogonal matrix.

3.3.3 The Trace and Rank of a Matrix

The **trace** of a $m \times m$ square matrix A is equal to the sum of the elements on its main diagonal. In other words, $Tr(A) = \sum_{i=1}^m a_{ii}$. Fig. 3.4 lists the main properties of the trace operator, whereby A and B are $m \times m$ matrices, s and t are scalars, and the matrices C and D represent a matrix of the form $p \times q$ and $q \times p$, respectively.

Properties of the trace operator

➤ $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$	➤ $\text{Tr}(C \cdot D) = \text{Tr}(D \cdot C)$
➤ $\text{Tr}(s \cdot A) = s \cdot \text{Tr}(A)$	➤ $\text{Tr}(A^T) = \text{Tr}(A)$
➤ $\text{Tr}(s \cdot A + t \cdot B) = s \cdot \text{Tr}(A) + t \cdot \text{Tr}(B)$	➤ $\text{Tr}[(A + B)^T] = \text{Tr}(A) + \text{Tr}(B)$

Figure 3.4: Properties of the trace operator

The **rank** of a matrix is equal to the maximum amount of linearly independent rows (columns) of that matrix. Linearly independent rows (columns) means that a row (column) cannot be written as a linear combination of other rows (columns). The rank of a matrix A is denoted by $\text{rank}(A)$.

For example, consider the following three matrices L , M , and N :

$$L = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 2 & 5 \\ -1 & 0 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$$

They all have a rank equal to 2. Given that the rows in L and N are linearly independent, their rank equals 2. Regarding M , the second row is a linear combination of the first and third row: 3 times the third row plus the first row gives the second row. Therefore, matrix M only has two linearly independent rows, which gives a rank equal to 2.

(Non-)Singular matrices can also be defined in terms of their rank. If the rank of a square matrix C is equal to its order ($\text{rank}(C) = \text{order}(C)$), C is a non-singular matrix. Put another way, all the rows of C are linearly independent. Looking back at the above matrices L , M , and N , the only non-singular matrix is N .

Similarly, for a singular matrix B we have that $\text{rank}(B) < \text{order}(B)$, which means that some rows are linearly dependent. From the above example, we know that $\text{rank}(M) = 2 < \text{order}(M) = 3$, so that M is classified as a singular matrix.

3.3.4 Determinants

The **determinant** of a $n \times n$ -matrix is a characteristic scalar value of square matrices. To calculate the determinant, we will rely on the concepts of minor and cofactor. The **minor** of an element a_{ij} is equal to the determinant of the matrix that emerges when the i^{th} row and the j^{th} column are eliminated from the original $n \times n$ -matrix. The **cofactor** of the element a_{ij} is defined as $A_{ij} = (-1)^{i+j} \times (\text{minor of } a_{ij})$.

The determinant of a $n \times n$ -matrix A can then be calculated by applying the technique of **cofactor expansion** along the row i or column j in the following way:

$$\left\{ \begin{array}{l} \text{Along row } i: \quad \det(A) = \sum_{j=1}^n a_{ij} A_{ij} = \sum_{j=1}^n a_{ij} [(-1)^{i+j} \times (\text{minor of } a_{ij})] \\ \\ \text{Along column } j: \quad \det(A) = \sum_{i=1}^n a_{ij} A_{ij} = \sum_{i=1}^n a_{ij} [(-1)^{i+j} \times (\text{minor of } a_{ij})] \end{array} \right.$$

When applying cofactor expansion along a certain row or column of a 2×2 -matrix A , we find that the determinant is calculated as $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. In the above example of section 3.3.3, the determinant of N is equal to $\det(N) = 6 \cdot 3 - 1 \cdot (-2) = 20$.

For a 3×3 -matrix A , we obtain the general formula $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$. For instance, the determinant of the matrix M from section 3.3.3 is equal to $\det(M) = 1 \cdot 2 \cdot 2 + 2 \cdot 5 \cdot (-1) + (-1) \cdot (-2) \cdot 0 - 1 \cdot 5 \cdot 0 - 2 \cdot (-2) \cdot 2 - (-1) \cdot 2 \cdot (-1) = 0$.

If $\det(A) = 0$ ($\det(A) \neq 0$), the matrix is **(non-)singular**. As we have seen in section 3.3.3, the matrix M is singular, which means that $\det(M) = 0$ as calculated in the previous paragraph. In contrast, $\det(N) = 20 \neq 0$, which implies that N is non-singular as already established in section 3.3.3.

Fig. 3.5 provides the properties of determinants, whereby A , B , and C are $p \times p$ matrices, C is non-singular, and r is a scalar.

Properties of determinants

➤ $\det(A \cdot B) = \det(A) \cdot \det(B)$	➤ $\det(r \cdot A) = r^p \cdot \det(A)$
➤ $\det(C^{-1}) = \frac{1}{\det(C)}$	➤ $\det(A^T) = \det(A)$

Figure 3.5: Properties of determinants

3.4 Systems of Linear Equations

3.4.1 Definitions

A **system of linear equations** is a set of n equations with p unknown variables which has the following general form:

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1p}x_p = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2p}x_p = b_2 \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{np}x_p = b_n \end{array} \right.$$

A system of linear equations can be solved for the unknown variables through, for instance, the substitution method or the combination method, but we will turn to matrices to discuss the solutions of such systems.

The above set of linear equations can be rewritten as a product of matrices. If A represents the $n \times p$ **coefficient matrix**, which consists of the coefficients a_{ij} (with $i \in \{1 \dots n\}$ and $j \in \{1 \dots p\}$), X the $p \times 1$ **variable matrix** containing the variables x_j , and B the $n \times 1$ **matrix of the constants** whose elements are the constants b_i , the set of equations takes on the form $A \cdot X = B$ whereby the respective matrices are the following:

$$A \cdot X = B \iff \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1p} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{np} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We also define the $n \times (p + 1)$ **augmented matrix** $A|B$ which adds the column matrix B to the coefficient matrix A :

$$A = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1p} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2p} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{np} & b_n \end{array} \right]$$

3.4.2 The Rank Method

The **rank method** allows us to analyze the solutions of our system of linear equations $A \cdot X = B$ and Fig. 3.6 summarizes the method. Note that p refers to the number of unknown variables.

If the matrix of the constants B is equal to the null matrix ($B = O_{n,1}$), the system of linear equations $A \cdot X = O_{n,1}$ is called **homogeneous**. If $\text{rank}(A) = \text{rank}(A|O_{n,1}) = p$, the only solution to the homogeneous system of linear equations is the null matrix $O_{p,1}$. If $\text{rank}(A) = \text{rank}(A|O_{n,1}) < p$, the set of equations includes solutions different from the null matrix.

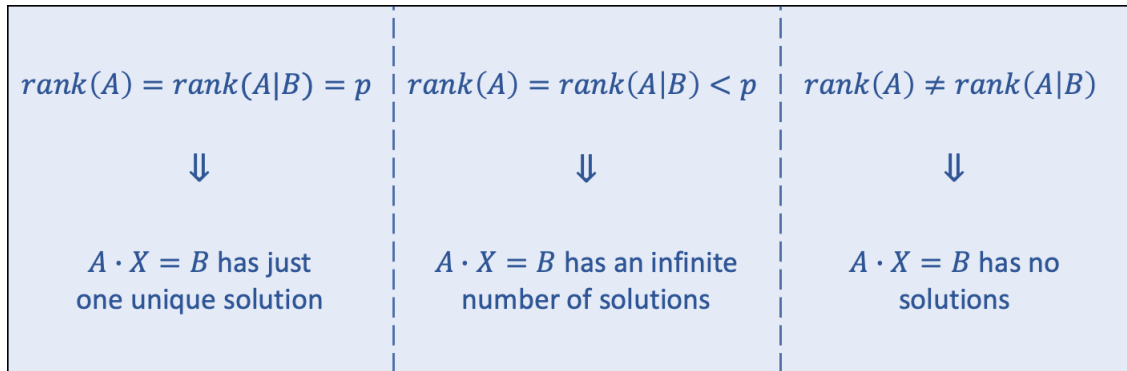


Figure 3.6: The rank method

Let us now suppose for a moment that the coefficient matrix A of the system of linear equations $A \cdot X = B$ is a square $p \times p$ -matrix. If $\det(A) \neq 0$, the matrix A is **non-singular**, which means that $rank(A) = order(A) = p$, so that $A \cdot X = B$ has one unique solution. Given that the inverse of non-singular matrix is defined, we can write the solution as $X = A^{-1} \cdot B$. However, if $\det(A) = 0$, we know that A is **singular**, which implies that $rank(A) < order(A) = p$, so that the system of linear equations $A \cdot X = B$ has either zero or an infinite number of solutions.

3.4.3 Gauss-Jordan Elimination

Consider again the general case whereby A is a $n \times p$ -matrix. To actually solve a system of linear equations $A \cdot X = B$, we will discuss the method of **Gauss-Jordan elimination**, which relies on **elementary row operations**. These operations refer to swapping two rows, multiply one row with a non-zero real number, or adding to a row a multiple of another row.

The starting point of the Gauss-Jordan elimination method is the augmented matrix $A|B$. In a first step, the method entails performing elementary row operations until $A|B$ is written in **row echelon form**. This means that, starting from the first row, every next non-zero row starts with strictly more zeros than the previous row. All the zero rows are thus at the bottom of the matrix. At this point, the rank method will indicate the type of solutions we are dealing with. Note that, when a matrix is in row echelon form, the rank of a matrix is equal to the number of non-zero rows.

In a second step, and if $rank(A) = rank(A|B)$, we calculate the solutions to $A \cdot X = B$ by transforming the row echelon form into a **reduced row echelon form**. This implies that all the leftmost non-zero entries of non-zero rows—these entries are called **leading coefficients**—must be equal to 1 and that all the values above and below the leading coefficients are zero. If A is a non-singular matrix, this means that its reduced row echelon form is equal to the identity matrix.

Let us look at the following example.

$$\begin{cases} 2x + 5y - z + 3u = -1 \\ y + z - u = 2 \\ 2x - y + 6z - 2u = 0 \end{cases} \Leftrightarrow A \cdot X = B \Leftrightarrow \begin{bmatrix} 2 & 5 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 2 & -1 & 6 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

In a first step, we transform the augmented matrix $A|B$ into its row echelon form.

$$A|B = \left[\begin{array}{cccc|c} 2 & 5 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 2 & -1 & 6 & -2 & 0 \end{array} \right] \xrightarrow{(R_3 \rightarrow -R_1 + R_3)} \sim \left[\begin{array}{cccc|c} 2 & 5 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & -6 & 7 & -5 & 1 \end{array} \right]$$

$$\xrightarrow{(R_3 \rightarrow 6R_2 + R_3)} \sim \left[\begin{array}{cccc|c} 2 & 5 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 13 & -11 & 13 \end{array} \right]$$

Since there are no zero rows in the last matrix, the rank method tells us that $\text{rank}(A) = \text{rank}(A|B) = 3 < 4$, whereby 4 is the number of unknown variables. This system of linear equations has thus an infinite number of solutions. To write down the solutions, we first transform the row echelon form into the reduced echelon form.

$$\left[\begin{array}{cccc|c} 2 & 5 & -1 & 3 & -1 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 13 & -11 & 13 \end{array} \right] \xrightarrow{(R_1 \rightarrow \frac{R_1}{2} \ \& \ R_3 \rightarrow \frac{R_3}{13})} \sim \left[\begin{array}{cccc|c} 1 & \frac{5}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -\frac{11}{13} & 1 \end{array} \right]$$

$$\xrightarrow{(R_1 \rightarrow -\frac{5}{2}R_2 + R_1)} \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 4 & -\frac{11}{2} \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & -\frac{11}{13} & 1 \end{array} \right]$$

$$\xrightarrow{\substack{(R_2 \rightarrow -R_3 + R_2) \\ (R_1 \rightarrow 3R_3 + R_1)}} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{19}{13} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{2}{13} & 1 \\ 0 & 0 & 1 & -\frac{11}{13} & 1 \end{array} \right]$$

$$\Leftrightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & \frac{19}{13} \\ 0 & 1 & 0 & -\frac{2}{13} \\ 0 & 0 & 1 & -\frac{11}{13} \end{array} \right] \cdot \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{cases} x + \frac{19}{13}u = -\frac{5}{2} \\ y - \frac{2}{13}u = 1 \\ z - \frac{11}{13}u = 1 \end{cases}$$

If the variable u is treated as a parameter, the last set of equations provides the solution to the system of linear equations $A \cdot X = B$.

3.4.4 Cramer's Rule

In case that the coefficient matrix A of our system of linear equations $A \cdot X = B$ is a $p \times p$ non-singular matrix, we also have the option, besides the method of Gauss-Jordan elimination, to fall back on **Cramer's rule** to solve this set of linear equations.

If the matrix A_i (with $i \in \{1 \dots p\}$) is defined as the matrix whereby the i^{th} column of A is

replaced by the matrix of constants B , the solution to $A \cdot X = B$ according to Cramer's rule is equal to:

$$\left(\frac{\det(A_1)}{\det(A)}, \frac{\det(A_2)}{\det(A)}, \dots, \frac{\det(A_p)}{\det(A)} \right)$$

Remember that $\det(A) \neq 0$ given that A is non-singular. Let us consider the following example:

$$\begin{cases} -x + 3y = 2 \\ 2x + 4y = -3 \end{cases} \Leftrightarrow A \cdot X = B \Leftrightarrow \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Since $\det(A) = -1 \cdot 4 - 2 \cdot 3 = -10 \neq 0$, the coefficient matrix A is non-singular and we can apply Cramer's rule. The matrices A_1 and A_2 are defined as follows:

$$A_1 = \begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

Their determinant is equal to $\det(A_1) = 17$ and $\det(A_2) = -1$, respectively, so that the solution of our system of linear equations is equal to:

$$\begin{cases} x = \frac{\det(A_1)}{\det(A)} = -\frac{17}{10} \\ y = \frac{\det(A_2)}{\det(A)} = \frac{1}{10} \end{cases}$$

3.5 Inverse Matrices

We have seen in section 3.3.2 that an inverse matrix is defined for non-singular matrices. In this section we discuss how to actually calculate the inverse matrix. We will explore two different methods. Suppose hereby that A is a non-singular $p \times p$ -matrix.

3.5.1 Two Methods

With respect to the **first method**, we start by transforming A into an augmented matrix $A|I_p$ whereby the part added to the right of A consists of the identity matrix I_p . In a next step, we apply elementary row operations to the matrix $A|I_p$ so that the matrix A is converted into its reduced row echelon form—which in the case of a non-singular matrix is equivalent to the identity matrix. The matrix that appears to the right of the reduced row echelon form is then equal to the inverse matrix A^{-1} .

Let us look at the following example.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \Rightarrow A|I_3 = \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ -1 & 2 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{(R_2 \leftrightarrow R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ -1 & 2 & 1 & 0 & 1 & 0 \end{array} \right] \xrightarrow{(R_3 \rightarrow R_1 + R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{(R_3 \rightarrow \frac{1}{2}R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right] \xrightarrow{(R_3 \rightarrow -R_2 + R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & -1 \end{array} \right]$$

$$\xrightarrow{(R_3 \rightarrow -\frac{1}{2}R_3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array} \right] \xrightarrow{(R_2 \rightarrow -3R_3 + R_2)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{array} \right] \xrightarrow{(R_1 \rightarrow -R_3 + R_1)}$$

The inverse matrix A^{-1} is then equal to:

$$A^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{3}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

For the **second method**, we have to introduce a new type of matrix called the **adjacent matrix**. This matrix is found by first replacing every element of A by its corresponding cofactor (see section 3.3.4) and subsequently taking the transpose. The adjacent matrix of A is denoted by $\text{adj}(A)$.

The inverse matrix A^{-1} is then calculated as $A^{-1} = \frac{\text{adj}(A)}{\det(A)}$. Consider the following example:

$$A = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} \Rightarrow \text{adj}(A) = \begin{bmatrix} -1 & -2 & 7 \\ 0 & 18 & 0 \\ -3 & 12 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \begin{bmatrix} -\frac{1}{18} & -\frac{1}{9} & \frac{7}{18} \\ 0 & 1 & 0 \\ -\frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix}$$

Let us consider, for instance, the element $a_{32} = 12$ of $\text{adj}(A)$. This value is calculated as $a_{32} = [(-1)^{2+3} \times (\text{minor of } a_{23} \text{ of matrix } A)]$, whereby the minor of a_{23} of matrix A is equal to $1 \cdot 3 - 3 \cdot 5 = -12$. Given that $\det(A) = 18$, the element a_{32} of A^{-1} is then calculated as $\frac{12}{18} = \frac{2}{3}$.

3.5.2 Solving Systems of Linear Equations

The inverse matrix can also be used to solve systems of linear equations. If the $p \times p$ coefficient matrix A of the system $A \cdot X = B$ is non-singular, the solution can be found as follows:

$$A \cdot X = B \Leftrightarrow A^{-1} \cdot A \cdot X = A^{-1} \cdot B \Leftrightarrow I_p \cdot X = A^{-1} \cdot B \Leftrightarrow X = A^{-1} \cdot B$$

Let us look at the following example:

$$\begin{cases} y - 6z = -2 \\ x + y = 3 \\ -5y + 2z = 7 \end{cases} \Leftrightarrow A \cdot X = B \Leftrightarrow \begin{bmatrix} 0 & 1 & -6 \\ 1 & 1 & 0 \\ 0 & -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}$$

Given that $\det(A) = 28 \neq 0$, the matrix A is non-singular, so that an inverse is defined. The inverse A^{-1} can be calculated through any of the two methods discussed in the previous section and is equal to:

$$A^{-1} = \begin{bmatrix} \frac{1}{14} & 1 & \frac{3}{14} \\ -\frac{1}{14} & 0 & -\frac{3}{14} \\ -\frac{5}{28} & 0 & -\frac{1}{28} \end{bmatrix}$$

The solution to our system of linear equations is then found when A^{-1} is multiplied by the matrix of the constants B :

$$X = A^{-1} \cdot B = \begin{bmatrix} \frac{1}{14} & 1 & \frac{3}{14} \\ -\frac{1}{14} & 0 & -\frac{3}{14} \\ -\frac{5}{28} & 0 & -\frac{1}{28} \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{61}{14} \\ -\frac{19}{14} \\ \frac{3}{28} \end{bmatrix}$$

3.6 Eigenvectors and Eigenvalues

If T represents a linear transformation from a vector space to that same vector space, we can define an **eigenvector** as a non-null vector that does not change its direction when T is acting on it. The length of the eigenvector can however change and the amount by which it does is called the **eigenvalue**.

If \vec{x} is an eigenvector of T , we can write $T(\vec{x}) = \lambda\vec{x}$, whereby the scalar λ is called the eigenvalue. Given that every linear transformation is associated with a matrix, the above equation can be written as $A \cdot X = \lambda \cdot X$, with A a square $n \times n$ -matrix and X the column matrix containing the coordinates of \vec{x} .

The equation $A \cdot X = \lambda \cdot X$ is equivalent to the equation $(A - \lambda \cdot I_n) \cdot X = O_{n,1}$. As discussed in section 3.4.2, in order to obtain a solution other than the null matrix to a homogeneous system of equations ($X \neq O_{n,1}$), we know that the condition $\text{rank}(A - \lambda \cdot I_n) = \text{rank}[(A - \lambda \cdot I_n) | O_{n,1}] < n$ must be fulfilled. In other words, the matrix $(A - \lambda \cdot I_n)$ must be singular and thus $\det(A - \lambda \cdot I_n) = 0$.

The equation $\det(A - \lambda \cdot I_n) = 0$ is called the **characteristic equation** of the matrix A .

For the cases $n = 2$ and $n = 3$, the characteristic equation can be found through the following formulas:

$$\begin{cases} n = 2: & \lambda^2 - \text{Tr}(A) + \det(A) = 0 \\ n = 3: & -\lambda^3 + \text{Tr}(A) \cdot \lambda^2 - \text{Tr}(\text{adj}(A)) \cdot \lambda + \det(A) = 0 \end{cases}$$

The eigenvalues are found by solving the characteristic equation. Each eigenvalue corresponds with an eigenvector whose coordinates are calculated by inserting the respective eigenvalue into the equation $(A - \lambda \cdot I_n) \cdot X = O_{n,1}$.

Let us look at the below example. The trace and the determinant of the matrix A is equal to $\text{Tr}(A) = 3$ and $\det(A) = -8$, respectively. The trace of the adjacent matrix of A is equal to $\text{Tr}(\text{adj}(A)) = -2$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \text{adj}(A) = \begin{bmatrix} -3 & -4 & 5 \\ -2 & 0 & -2 \\ 1 & -4 & 1 \end{bmatrix}$$

The characteristic equation of A obtains the following form:

$$-\lambda^3 + 3 \cdot \lambda^2 - (-2) \cdot \lambda + (-8) = 0 \Leftrightarrow (\lambda - 2) \cdot \left(\lambda - \frac{1 + \sqrt{17}}{2} \right) \cdot \left(\lambda - \frac{1 - \sqrt{17}}{2} \right) = 0$$

In other words, the eigenvalues of A are equal to $\lambda_1 = 2$, $\lambda_2 = \frac{1 + \sqrt{17}}{2}$, and $\lambda_3 = \frac{1 - \sqrt{17}}{2}$. Let us now find the eigenvector \vec{x}_1 that corresponds with the eigenvalue $\lambda_1 = 2$.

$$\begin{aligned} (A - \lambda_1 \cdot I_3) \cdot X = O_{3,1} &\Leftrightarrow \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Leftrightarrow \begin{cases} -x + 2y - z = 0 \\ -y + 2z = 0 \\ -x + 2y - z = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} x = 3z \\ y = 2z \end{cases} \end{aligned}$$

This means that we can arbitrarily choose a value for z to determine the eigenvector \vec{x}_1 . If we set $z = 1$, then the eigenvector is equal to $\vec{x}_1 = (3, 2, 1)$.

If we go through the same process for the eigenvalues λ_2 and λ_3 , we find the eigenvectors $\vec{x}_2 = \left(1, \frac{1 + \sqrt{17}}{4}, 1 \right)$ and $\vec{x}_3 = \left(1, \frac{1 - \sqrt{17}}{4}, 1 \right)$.

The eigenvectors that correspond to different eigenvalues are linearly independent. If the

$n \times n$ -matrix A of the linear transformation T has n distinct eigenvalues, we can say that A is **diagonalizable**. This means that A can be written as $A = M \cdot D \cdot M^{-1}$, with M a $n \times n$ -matrix whose columns contain the eigenvectors of A and with D a $n \times n$ diagonal matrix whose main diagonal consists of the eigenvalues of A .

In our above example we have 3 distinct eigenvalues, so A is diagonalizable and can be written as:

$$A = M \cdot D \cdot M^{-1} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & \frac{1+\sqrt{17}}{4} & \frac{1-\sqrt{17}}{4} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1+\sqrt{17}}{2} & 0 \\ 0 & 0 & \frac{1-\sqrt{17}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{7+\sqrt{17}}{4\sqrt{17}} & \frac{2}{\sqrt{17}} & \frac{5+3\sqrt{17}}{4\sqrt{17}} \\ \frac{7-\sqrt{17}}{4\sqrt{17}} & -\frac{2}{\sqrt{17}} & \frac{-5+3\sqrt{17}}{4\sqrt{17}} \end{bmatrix}$$

A square $n \times n$ -matrix can still be diagonalizable if the number of distinct eigenvalues is lower than n , but only if the algebraic multiplicity is equal to the geometric multiplicity for each eigenvalue. The **algebraic multiplicity** of an eigenvalue is equal to the multiplicity of the corresponding root of the characteristic equation and the **geometric multiplicity** is the number of eigenvectors that are associated with an eigenvalue.

For example, consider the below matrix A , for which $\det(A) = -2$, $Tr(A) = 0$, and $Tr(\text{adj}(A)) = -3$.

$$A = \begin{bmatrix} \frac{-1+\sqrt{13}}{2} & -1 & 0 \\ 1 & -\frac{1+\sqrt{13}}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \text{adj}(A) = \begin{bmatrix} -\frac{1+\sqrt{13}}{2} & 1 & 0 \\ -1 & \frac{-1+\sqrt{13}}{2} & 0 \\ -1 & \frac{-1+\sqrt{13}}{2} & -2 \end{bmatrix}$$

The characteristic equation is equal to $-\lambda^3 + 0 \cdot \lambda^2 - (-3) \cdot \lambda + (-2) = 0 \Leftrightarrow (\lambda - 1)^2 \cdot (\lambda + 2) = 0$. The eigenvalue $\lambda_1 = 1$ has thus an algebraic multiplicity equal to 2 and that of the eigenvalue $\lambda_2 = -2$ is equal to 1.

The eigenvalue λ_1 produces just one eigenvector, i.e., $\vec{x}_1 = \left(\frac{3+\sqrt{13}}{2}, 1, 1\right)$, so that its geometric multiplicity equals 1. Since the geometric and algebraic multiplicity are not equal ($1 \neq 2$), the matrix A is not diagonalizable.

4 Plane Geometry

Geometry is the study of the properties of space and **Euclidean geometry** is geometry that is based on the five axioms of Euclid. Euclidean geometry encompasses plane geometry and solid geometry. **Plane geometry** is geometry of two-dimensional Euclidean spaces, whereas **solid geometry** (see section 6) discusses three-dimensional Euclidean spaces. Euclidean spaces are mathematical spaces that aim to represent physical space and consist of an infinite number of points.

4.1 Vectors

4.1.1 Coordinates and Coordinate Spaces

A **Euclidean plane** is the two-dimensional Euclidean space whereby the position of a point is determined by a pair of two values. If the values are real numbers, the set of all pairs is denoted by \mathbb{R}^2 and is called a **real coordinate space**. The elements of this real coordinate space are referred to as **coordinate vectors**, which have been introduced in section 3.1, and the values of these pairs are called **coordinates**. The real coordinate space is thus a vector space and we will refer to that space from now on as *the* Euclidean plane.

A **basis** of a vector space V is defined as a finite set of vectors within V so that every vector of V can be uniquely constructed as a finite linear combination of these basis vectors. The coordinates of a vector are then equal to the coefficients of this linear combination. The number of basis vectors pertaining to a base is called the **dimension of the vector space** and does not change. A basis of the Euclidean plane always contains two basis vectors, so that it has a dimension of 2.

For example, a basis E of the Euclidean plane could be $E = \{(3, 2), (-1, 1)\}$. If $\vec{e}_1 = (3, 2)$ and $\vec{e}_2 = (-1, 1)$, then the vector $\vec{a} = (2, 4)$ can be written as $\vec{a} = 2\vec{e}_1 + 4\vec{e}_2$. If we would now choose a second basis $B = \{(1, 0), (0, 1)\}$ with $\vec{b}_1 = (1, 0)$ and $\vec{b}_2 = (0, 1)$, we can express the basis vectors of E in terms of those of B : $\vec{e}_1 = 3\vec{b}_1 + 2\vec{b}_2$ and $\vec{e}_2 = -\vec{b}_1 + \vec{b}_2$. The vector \vec{a} with respect to basis B then becomes $\vec{a} = 2(3\vec{b}_1 + 2\vec{b}_2) + 4(-\vec{b}_1 + \vec{b}_2) = 2\vec{b}_1 + 8\vec{b}_2$ or $\vec{a} = (2, 8)$.

To graphically construct vectors in the Euclidean plane, we need to define the **origin or the null vector** of the plane which is a point that, once chosen, remains fixed and is denoted by \vec{o} with coordinates $\vec{o} = (0, 0)$. The vector $\vec{v} = (a, b)$ is then constructed by drawing an arrow that starts in the origin and ends in the point with coordinates (a, b) .

The coordinate axes of a **coordinate system** consist of the straight lines that connect the origin with the respective points associated with the basis vectors. If the coordinate axes in the Euclidean

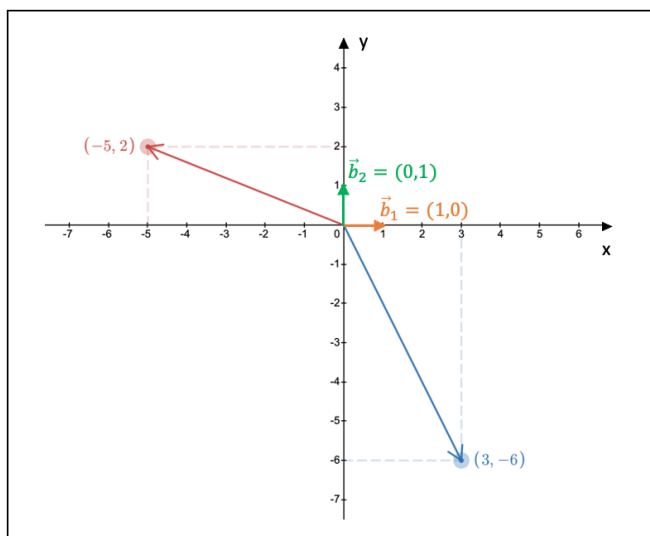


Figure 4.1: The Cartesian coordinate system

plane are oriented perpendicularly and if the length of the basis vectors is equal to 1, we refer to the coordinate system as an **orthonormal** or a **Cartesian coordinate system**.

4.1.2 Vector Operations and Properties

Given that the Euclidean plane is a vector space, we can add vectors together and perform scalar multiplication. The **sum vector** $\vec{c} = \vec{a} + \vec{b}$ of two vectors \vec{a} and \vec{b} that do not lie on the same straight line is drawn as the diagonal, with the starting point in the origin, of the parallelogram spanned by the two vectors \vec{a} and \vec{b} . If \vec{a} and \vec{b} lie on the same line, the sum vector $\vec{c} = \vec{a} + \vec{b}$ also lies on that line.

Multiplying the vector \vec{a} with a **scalar** $\lambda \in \mathbb{R}$ results in the vector $\vec{b} = \lambda\vec{a}$, which lies on the same line as vector \vec{a} .

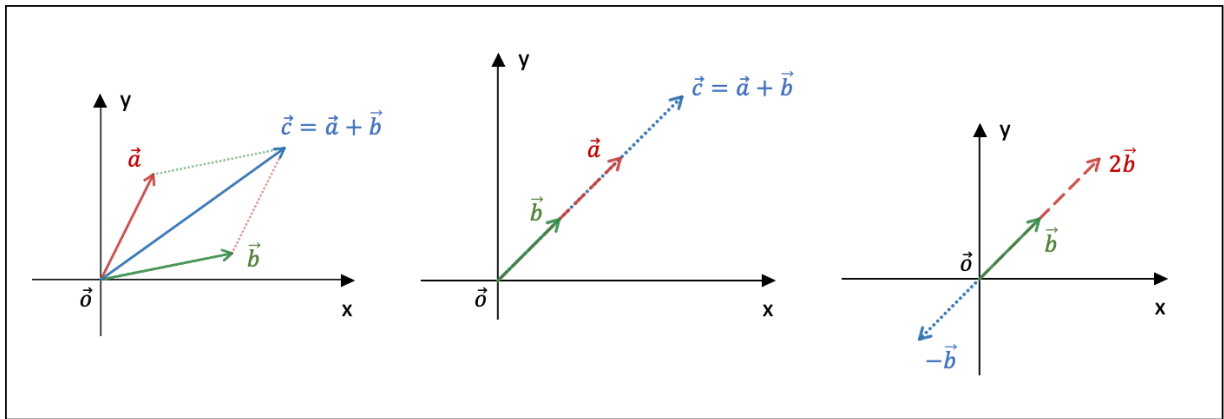


Figure 4.2: Vector addition and scalar multiplication

From the above it becomes clear that two vectors \vec{a} and \vec{b} are **parallel** to each other ($\vec{a} \parallel \vec{b}$) if one vector can be found by applying scalar multiplication to the other vector, i.e., $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \mu\vec{a}$. For instance, the red-coloured vector, let's call it \vec{c} , in the coordinate system at the right-hand side in Fig. 4.2 is parallel to the green-coloured vector \vec{b} because $\vec{c} = 2\vec{b}$ or $\vec{b} = \frac{1}{2}\vec{c}$.

Fig. 4.3 summarizes the properties of vector addition and scalar multiplication whereby λ and μ represent scalars.

In terms of coordinates, the sum of the two vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is equal to $\vec{c} = (a_1 + b_1, a_2 + b_2)$ and the scalar multiplication of $\vec{a} = (a_1, a_2)$ with the real scalar λ is defined as $\lambda \cdot \vec{a} = (\lambda a_1, \lambda a_2)$.

The **norm** of a vector \vec{a} indicates its length and is denoted by $\|\vec{a}\|$. The **distance** between two vectors \vec{a} and \vec{b} is written as $d(\vec{a}, \vec{b})$. The norm of a vector \vec{a} can also be interpreted as the distance between the origin and the vector \vec{a} , i.e., $d(\vec{o}, \vec{a})$.

<u>Properties of vector addition</u>	
➤ $\vec{a} + \vec{b} = \vec{b} + \vec{a}$	➤ $\vec{a} + \vec{o} = \vec{a} = \vec{o} + \vec{a}$
➤ $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$	➤ $\vec{a} + (-\vec{a}) = \vec{o} = (-\vec{a}) + \vec{a}$
<u>Properties of scalar multiplication</u>	<u>Combined properties</u>
➤ $\lambda \cdot (\mu \cdot \vec{a}) = (\lambda \cdot \mu) \cdot \vec{a}$	➤ $\lambda \cdot (\vec{a} + \vec{b}) = \lambda \cdot \vec{a} + \lambda \cdot \vec{b}$
➤ $1 \cdot \vec{a} = \vec{a}$	➤ $(\lambda + \mu) \cdot \vec{a} = \lambda \cdot \vec{a} + \mu \cdot \vec{a}$

Figure 4.3: Properties of vector addition and scalar multiplication

In terms of coordinates, the norm of vector $\vec{a} = (a_1, a_2)$ and the distance between vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ can be calculated as follows:

$$\begin{cases} \|\vec{a}\| = \sqrt{a_1^2 + a_2^2} \\ d(\vec{a}, \vec{b}) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \end{cases}$$

Fig. 4.4 lists the properties of the norm and distance in the Euclidean plane, whereby λ represents a real scalar ($\lambda \in \mathbb{R}$).

<u>Properties of the norm</u>	<u>Properties of the distance</u>
➤ $\ \vec{a}\ \geq 0$	➤ $d(\vec{a}, \vec{b}) \geq 0$
➤ $\ \vec{a}\ = 0 \Leftrightarrow \vec{a} = \vec{o}$	➤ $d(\vec{a}, \vec{b}) = 0 \Leftrightarrow \vec{a} = \vec{b}$
➤ $\ \vec{a} + \vec{b}\ \leq \ \vec{a}\ + \ \vec{b}\ $	➤ $d(\vec{a}, \vec{b}) \leq d(\vec{a}, \vec{c}) + d(\vec{c}, \vec{b})$
➤ $\ \lambda \cdot \vec{a}\ = \lambda \cdot \ \vec{a}\ $	

Figure 4.4: Properties of the norm and distance

The **dot product** between two vectors \vec{a} and \vec{b} represents a real number and provides information about their norm and the angle between the two vectors. The dot product is defined as $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos \theta$. If one of the two vectors equals the null vector, the dot product becomes zero.

The dot product $\vec{a} \cdot \vec{b}$ is greater (less) than zero if and only if the angle θ is acute (obtuse). If $\vec{a} \neq \vec{o}$, $\vec{b} \neq \vec{o}$, and $\vec{a} \cdot \vec{b} = 0$, the two vectors \vec{a} and \vec{b} are oriented perpendicularly, i.e., $\theta = 90^\circ$.

In terms of coordinates, the dot product between the vectors $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is equal to $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$.

4.2 Straight Lines

4.2.1 The Equation of a Straight Line

Consider two points \vec{a} and \vec{b} , as depicted in Fig. 4.5. Between the points \vec{o} and $\vec{b} - \vec{a}$ we can construct the line L_0 , whereby the vector $\vec{b} - \vec{a}$ is called the **directional vector** of the line L_0 .

We can also draw a unique line between the points \vec{a} and \vec{b} , i.e., the line L . A point \vec{x} belongs to L if the vector $\vec{x} - \vec{a}$ belongs to the line L_0 . Since the directional vector $\vec{b} - \vec{a}$ is parallel to the directional vector $\vec{x} - \vec{a}$, we can write that $\vec{x} - \vec{a} = \lambda \cdot (\vec{b} - \vec{a})$, with λ a real scalar. The **vector equation** of the straight line L can then be written generally as $\vec{x} = \vec{a} + \lambda \cdot (\vec{b} - \vec{a})$, with $\vec{b} - \vec{a}$ the directional vector of L (which is also the directional vector of L_0).

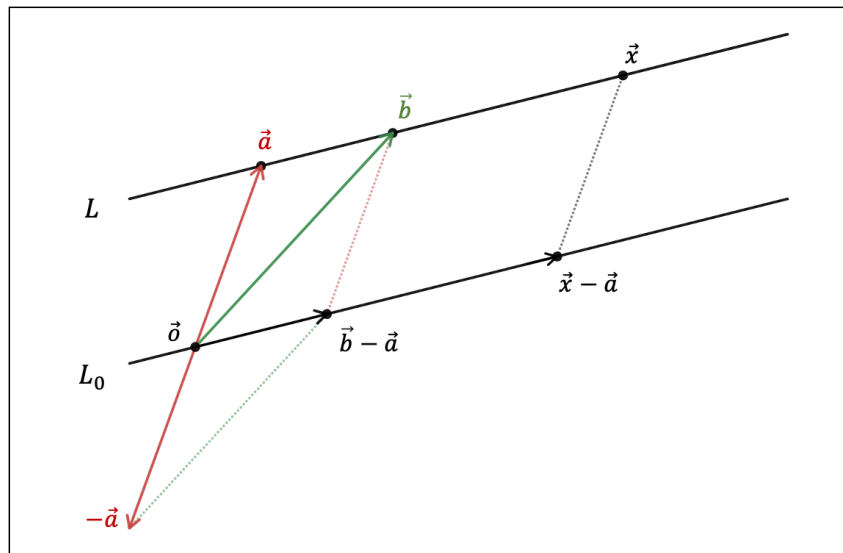


Figure 4.5: Straight line L going through points \vec{a} and \vec{b}

When we write the vector equation of the straight line L in terms of the vector components, we then obtain the **parametric equation** of L (whereby $\vec{x} = (x, y)$, $\vec{a} = (a_1, a_2)$, and $\vec{b} = (b_1, b_2)$):

$$\begin{cases} x = a_1 + \lambda \cdot (b_1 - a_1) \\ y = a_2 + \lambda \cdot (b_2 - a_2) \end{cases}$$

If we eliminate the parameter λ from the parametric equation of L , we obtain the **Cartesian equation** of a straight line:

$$y = a_2 + \left(\frac{x - a_1}{b_1 - a_1} \right) \cdot (b_2 - a_2) \Leftrightarrow y = \left(\frac{b_2 - a_2}{b_1 - a_1} \right) \cdot x + \left[a_2 - \left(\frac{b_2 - a_2}{b_1 - a_1} \right) \cdot a_1 \right]$$

$$\Leftrightarrow y = mx + c$$

whereby $m = \frac{b_2 - a_2}{b_1 - a_1}$ is called the **slope** of L and $c = a_2 - m \cdot a_1$ the **y-intercept** of L .

The Cartesian equation can also be written in a different but equivalent form:

$$y = a_2 + \left(\frac{x - a_1}{b_1 - a_1} \right) \cdot (b_2 - a_2) \Leftrightarrow (b_1 - a_1) \cdot y = (b_1 - a_1) \cdot a_2 + (x - a_1)(b_2 - a_2)$$

$$\Leftrightarrow (b_2 - a_2) \cdot x + (a_1 - b_1) \cdot y + (b_1 a_2 - a_1 b_2) = 0$$

$$\Leftrightarrow ax + by + c = 0$$

whereby $a = b_2 - a_2$, $b = a_1 - b_1$, and c is the same y-intercept as above ($c = a_2 - m \cdot a_1$). Notice that $(-b, a)$ is a directional vector of L and that the slope m is equal to $m = -\frac{a}{b}$. In other words, if $\vec{v} = (v_1, v_2)$ is a directional vector of a line, with $v_1 = (b_1 - a_1)$ and $v_2 = (b_2 - a_2)$, the slope of the line is equal to $m = \frac{v_2}{v_1}$.

For instance, if we are given the Cartesian equation $2x + 6y - 1 = 0$, we know that $(-6, 2)$ is a directional vector of the straight line, that the slope is equal to $m = -\frac{1}{3}$, and that $c = -1$.

In a second example, given the two points $\vec{a} = (-3.4, 7.0)$ and $\vec{b} = (2.9, -0.1)$, we can calculate the slope m of the straight line going through these two points as $m = \frac{b_2 - a_2}{b_1 - a_1} = \frac{-0.1 - 7.0}{2.9 - (-3.4)} = -1.1$. The Cartesian equation of the line is then written as $y = mx + [a_2 - ma_1] = -1.1x + [7.0 - (-1.1) \cdot (-3.4)] = -1.1x + 3.2$. The parametric equation is equal to:

$$\begin{cases} x = -3.4 + 6.3\lambda \\ y = 7.0 - 7.1\lambda \end{cases}$$

The vector equation of this straight line takes the form of $(x, y) = (-3.4, 7.0) + \lambda \cdot (6.3, -7.1)$.

4.2.2 Perpendicular and Intersecting Lines

To find the equation of the straight line $L_\perp : y_\perp = m_\perp x + c$ that is **perpendicular to a given straight line** $L : y = mx + c$, we have to calculate m_\perp .

The directional vector of L_\perp and L is equal to $\vec{v}_\perp = (1, m_\perp)$ and $\vec{v} = (1, m)$, respectively. The

lines L and L_{\perp} are perpendicularly oriented if their directional vectors are perpendicular. This means that their dot product must be zero: $\vec{v}_{\perp} \cdot \vec{v} = 1 \cdot 1 + m_{\perp} \cdot m = 0$.

In other words, the slope of L_{\perp} can be calculated with the assistance of the relationship $m_{\perp} \cdot m = -1 \Leftrightarrow m_{\perp} = -\frac{1}{m}$.

For example, given a straight line with Cartesian equation $L : y = 6x - 3.5$, the line $L_{\perp} : y_{\perp} = m_{\perp}x + c$, which is perpendicular to the given line, must have a slope m_{\perp} equal to $m_{\perp} = -\frac{1}{m} = -\frac{1}{6}$. The choice of intercept is irrelevant since it is the slope that determines the direction of a straight line. Therefore, the line L_{\perp} has the general form of $L_{\perp} : y_{\perp} = -\frac{1}{6}x + c$.

More generally, let us consider the Cartesian equation $ax + by + c = 0$ of line L . As discussed earlier, the directional vector \vec{v} of L is equal to $\vec{v} = (-b, a)$. The coordinates of the directional vector $\vec{v}_{\perp} = (x, y)$ of the line L_{\perp} perpendicular to L can be found through the definition of the dot product: $\vec{v} \cdot \vec{v}_{\perp} = 0 \Leftrightarrow (-b, a) \cdot (x, y) = 0 \Leftrightarrow -bx + ay = 0$. This means that $(x, y) = (a, b)$.

The directional vector \vec{v}_{\perp} that is perpendicular to a given straight line $L : ax + by + c = 0$ has thus coordinates $\vec{v}_{\perp} = (a, b)$ and is called the **normal vector** of L .

If we take for instance the Cartesian equation $L : -3x + 5y - 2 = 0$, the normal vector \vec{n} has coordinates $\vec{n} = (a, b) = (-3, 5)$. Given a directional vector $\vec{v} = (-b, a) = (-5, -3)$ of L , we can indeed see that the dot product between \vec{n} and \vec{v} is zero.

The **point of intersection** between two straight lines can be found via the method of substitution, combination, or via matrices.

Let us consider an example. Given two straight lines with Cartesian equation $L_1 : 3x - 2y + 8 = 0$ and $L_2 : -x + y - 7 = 0$ (see Fig. 4.6), we will determine their point of intersection via the three methods mentioned above.

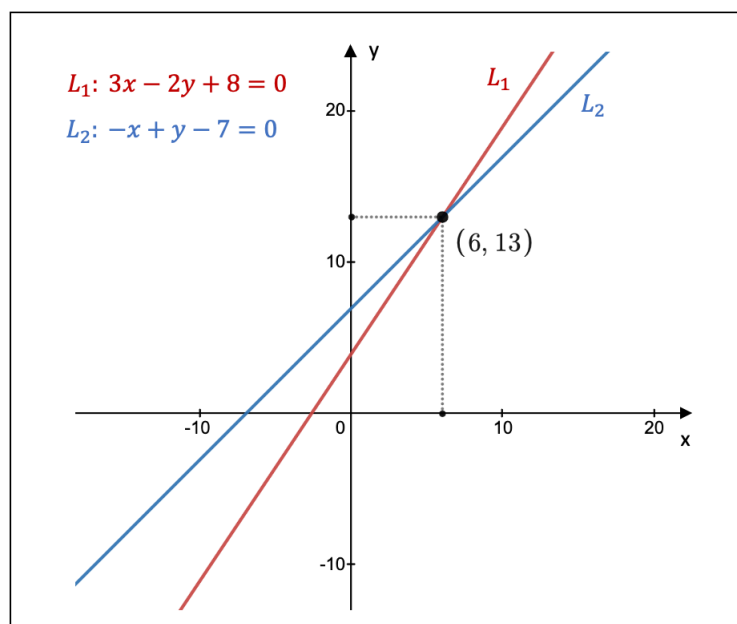


Figure 4.6: Two intersecting straight lines

With respect to the substitution method, we can for instance write y in terms of x in the equation of L_1 and substitute it into the equation of L_2 :

$$3x - 2y + 8 = 0 \Leftrightarrow y = \frac{3}{2}x + 4 \xrightarrow{\text{in } L_2} -x + \left[\frac{3}{2}x + 4 \right] - 7 = 0$$

$$\Leftrightarrow x = 6$$

Inserting $x = 6$ into either L_1 or L_2 gives us the coordinates of the intersection point: $(6,13)$. Using the combination method, we obtain the solution as follows:

$$\begin{array}{r} 1 \times (L_1 : 3x - 2y + 8 = 0) \\ +2 \times (L_2 : -x + y - 7 = 0) \\ \hline x - 6 = 0 \Leftrightarrow x = 6 \end{array}$$

Finally, we can also find the solution with the assistance of matrices. If we construct the augmented matrix $A|B$, we see that $\text{rank}(A) = \text{rank}(A|B) = 2$. Given that 2 is equal to the number of unknown variables, we know that there is just one unique solution. Since A is non-singular, it follows that A is invertible, so that we can calculate the solution X via $A \cdot X = B \Leftrightarrow X = A^{-1} \cdot B$:

$$A|B = \left[\begin{array}{cc|c} 3 & -2 & -8 \\ -1 & 1 & 7 \end{array} \right] \Rightarrow X = A^{-1} \cdot B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \end{bmatrix}$$

4.2.3 Distances between Points and Lines

Suppose there is a point $\vec{q} = (q_1, q_2)$ and a straight line $L : ax + by + c = 0$, so that \vec{q} does not belong to L . The distance $d(\vec{q}, L)$ is equal to the shortest distance between \vec{q} and L and is calculated via the following formula:

$$d(\vec{q}, L) = \frac{|aq_1 + bq_2 + c|}{\sqrt{a^2 + b^2}}$$

For example, suppose we wish to calculate the distance $d(\vec{q}, L)$ between the point $\vec{q} = (5, 4)$ and the straight line $L : 6x + y - 9 = 0$, as shown in Fig. 4.7.

Using the above formula, we obtain the following value for the distance $d(\vec{q}, L)$:

$$d(\vec{q}, L) = \frac{|aq_1 + bq_2 + c|}{\sqrt{a^2 + b^2}} = \frac{|6 \cdot 5 + 1 \cdot 4 - 9|}{\sqrt{6^2 + 1^2}} = \frac{25}{\sqrt{37}}$$

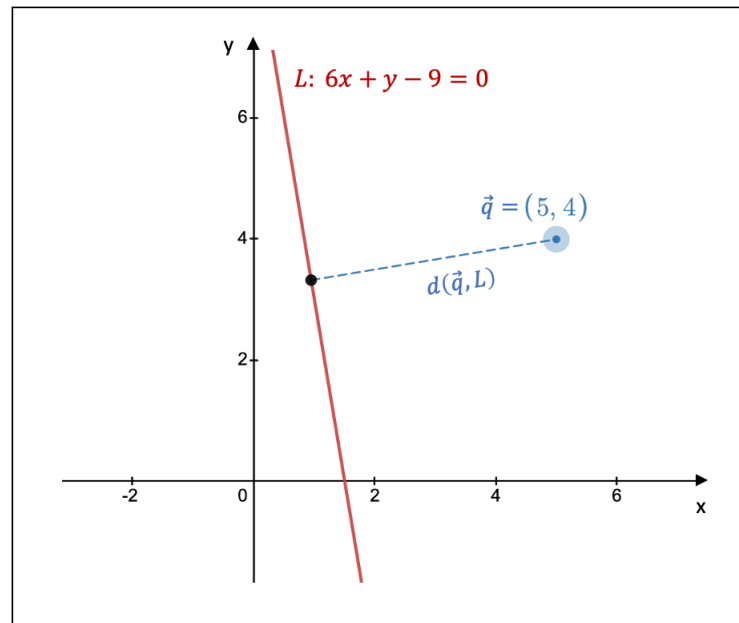


Figure 4.7: Distance between a point and a straight line

In case we wish to calculate the distance between two parallel lines, we can first choose one point that belongs to one of the two lines and then use the above formula to calculate the distance from that point to the other straight line.

4.3 Conic Sections

4.3.1 General Properties

Conic sections are curves that are formed when a plane intersects with a three-dimensional shape that consists of two right circular cones placed in such a way that the two vertices of the cones touch each other (called a double-napped right circular cone). The possible curves that emerge from this intersection are a circle, an ellipse, a parabola, and a hyperbola.

The (simplified) general equation for a conic section can be written as follows:

$$ax^2 + by^2 + cx + dy + e = 0$$

If a conic section exists, we can distinguish three cases (note that in the case of a parabola either a or b is zero, but not both):

$$\begin{cases} ab > 0 : & \text{The conic section is a circle or an ellipse} \\ ab = 0 : & \text{The conic section is a parabola} \\ ab < 0 : & \text{The conic section is a hyperbola} \end{cases}$$

For instance, the equation $-2x^2 - 3y^2 - x + 5y + 2 = 0$ represents a circle or an ellipse, given that $ab = (-2) \cdot (-3) = 6 > 0$.

4.3.2 The Circle

If $ab > 0$ and $a = b$, the conic section is defined as a **circle**, but only if the right-hand side of the equation of the circle in the **standard form** is greater than zero. The standard form is the following:

$$(x - m_1)^2 + (y - m_2)^2 = r^2$$

with $\vec{m} = (m_1, m_2)$ the position vector of the centre of the circle and r the radius of the circle.

For example, consider the conic equation $4x^2 + 4y^2 + 2x - y + \frac{1}{16} = 0$. Since $ab = 16 > 0$ and $a = b = 4$, we are dealing with a circle but only if the right-hand side of the standard form is greater than zero. The standard form is obtained as follows:

$$\begin{aligned} 4x^2 + 4y^2 + 2x - y + \frac{1}{16} = 0 &\Leftrightarrow (4x^2 + 2x) + (4y^2 - y) = -\frac{1}{16} \\ &\Leftrightarrow \left[4\left(x + \frac{1}{4}\right)^2 - \frac{1}{4} \right] + \left[4\left(y - \frac{1}{8}\right)^2 - \frac{1}{16} \right] = -\frac{1}{16} \\ &\Leftrightarrow \left(x + \frac{1}{4}\right)^2 + \left(y - \frac{1}{8}\right)^2 = \frac{1}{16} \end{aligned}$$

In other words, given that the right-hand side of the standard form is greater than zero ($\frac{1}{16} > 0$), the conic equation indeed represents a circle, whose centre is located at $\vec{m} = \left(-\frac{1}{4}, \frac{1}{8}\right)$ and whose radius measures $r = \frac{1}{4}$.

As a counterexample, even though $ab > 0$ and $a = b$ for the conic equation $-6x^2 - 6y^2 - 5x + 7y - 4 = 0$, its standard form is equal to $\left(x + \frac{5}{12}\right)^2 + \left(y - \frac{7}{12}\right)^2 = -\frac{11}{72}$, which does not correspond to the equation of a circle.

As a different exercise, let us now find the equation of the tangent line T in point $\vec{p} = (1, -1)$ at the circle with equation $2x^2 + 2y^2 - x + 5y + 2 = 0$ (see Fig. 4.8). In a first step, we write the equation of the circle in its standard form: $\left(x - \frac{1}{4}\right)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{5}{8}$.

A directional vector of the line R that connects the point \vec{p} with the centre of the circle $\vec{m} = \left(\frac{1}{4}, -\frac{5}{4}\right)$ is $\vec{v} = \vec{p} - \vec{m} = \left(\frac{3}{4}, \frac{1}{4}\right)$. Given that the tangent line T is perpendicular to the line R , a directional vector of T is equal to $\vec{v}_\perp = \left(\frac{1}{4}, -\frac{3}{4}\right)$, which gives a slope equal to $m_\perp = -3$.

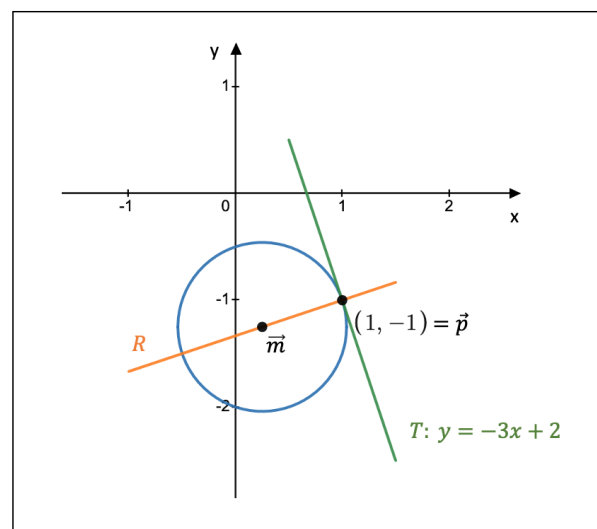


Figure 4.8: The tangent line at a circle

The tangent line T has thus the form $T : y = -3x + b_i$. Given that $\vec{p} = (1, -1)$ belongs to the circle, we find that $b_i = 2$. The final equation of the tangent line T is therefore $T : y = -3x + 2$.

4.3.3 The Ellipse

If $ab > 0$ and $a \neq b$, the conic section is defined as an **ellipse**, but only if the right-hand side of the equation of the ellipse in the standard form is greater than zero. The standard form is equal to:

$$\left\{ \begin{array}{l} \text{Horizontal ellipse: } \frac{(x - m_1)^2}{r_1^2} + \frac{(y - m_2)^2}{r_2^2} = 1 \\ \text{Vertical ellipse: } \frac{(x - m_1)^2}{r_2^2} + \frac{(y - m_2)^2}{r_1^2} = 1 \end{array} \right.$$

with $\vec{m} = (m_1, m_2)$ the position vector of the centre of the ellipse, r_1 the length of the semi-major axis, and r_2 the length of the semi-minor axis. Since the major axis is longer than the minor axis, we have that $r_1 > r_2$.

Note also that a horizontal (vertical) ellipse implies that the major axis is horizontally (vertically) oriented. We will work in this section with a horizontal ellipse, unless stated otherwise.

The **foci** of an ellipse are two fixed points \vec{f}_1 and \vec{f}_2 located on the major axis at equal distance f_c from the ellipse's centre with the following property: the sum of the distances s_1 and s_2 from any point on the ellipse to the two foci remains constant and is equal to the length of the major axis. In other words: $s_1 + s_2 = 2r_1$.

The above property together with the Pythagorean theorem allows us to find the **focal length** f_c between the centre point \vec{m} and the foci. If we consider the point $(m_1, r_2 + m_2)$ on the ellipse, we can determine the distances s_1 and s_2 as $s_1 = s_2 = \sqrt{r_2^2 + f_c^2}$. The distance f_c is then derived as follows:

$$\begin{aligned} s_1 + s_2 = 2r_1 &\Leftrightarrow \sqrt{r_2^2 + f_c^2} + \sqrt{r_2^2 + f_c^2} = 2r_1 \Leftrightarrow r_2^2 + f_c^2 = r_1^2 \\ &\Leftrightarrow f_c = \sqrt{r_1^2 - r_2^2} \end{aligned}$$

Another property of the ellipse is its **eccentricity** e_c , which is a value between 0 and 1 and describes how circular an ellipse is. For $e_c = 0$, the ellipse becomes a circle. We can calculate the value through the formula $e_c = \frac{f_c}{r_1} = \sqrt{1 - (r_2^2/r_1^2)}$.

Let us look at the ellipse with the equation $E : 3x^2 + 9y^2 + 12x - 26 = 0$, as depicted in Fig. 4.9.

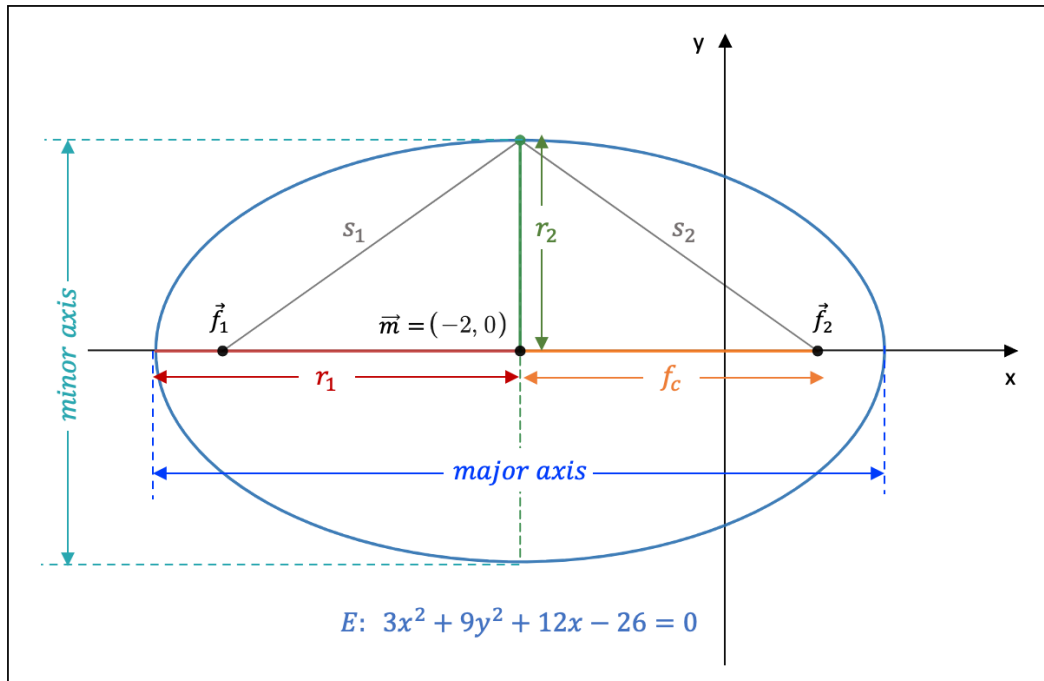


Figure 4.9: The ellipse with equation $E : 3x^2 + 9y^2 + 12x - 26 = 0$

The standard form of ellipse E is equal to $\frac{(x+2)^2}{38/3} + \frac{y^2}{38/9} = 1$. The centre of the ellipse has coordinates $\vec{m} = (-2, 0)$ and the length of the semi-major axis r_1 and semi-minor axis r_2 is equal to $r_1 = \sqrt{\frac{38}{3}}$ and $r_2 = \frac{\sqrt{38}}{3}$, respectively. The focal length f_c is calculated as $f_c = \sqrt{r_1^2 - r_2^2} = \sqrt{\left(\frac{38}{3}\right) - \left(\frac{38}{9}\right)} = \frac{2}{3}\sqrt{19}$ and the ellipse has an eccentricity of $e_c = \frac{f_c}{r_1} = \left(\frac{2}{3}\sqrt{19}\right) / \left(\sqrt{\frac{38}{3}}\right) = \sqrt{\frac{2}{3}}$.

Let us check whether the condition of the foci $s_1 + s_2 = 2r_1$ indeed holds. At the point $(-2, \frac{\sqrt{38}}{3})$, we know that $s_1 = s_2$. The distance s_1 is calculated as $s_1 = \sqrt{r_2^2 + f_c^2}$, and given that $f_c^2 = r_1^2 - r_2^2$ we find that $s_1 = \sqrt{r_2^2 + r_1^2 - r_2^2} = r_1$. We see that the condition of the foci indeed holds, because $s_1 + s_2 = s_1 + s_1 = 2s_1 = 2r_1$.

4.3.4 The Parabola

If only the x or y variable is squared in the general conic equation, i.e., either $a = 0$ or $b = 0$ so that $ab = 0$, we are dealing with a **parabola**. The standard form is equal to:

$$\begin{cases} \text{Horizontal parabola: } x = s \cdot (y - m_2)^2 + m_1 \\ \text{Vertical parabola: } y = s \cdot (x - m_1)^2 + m_2 \end{cases}$$

whereby $\vec{m} = (m_1, m_2)$ indicates the position of the parabola's **vertex**. With respect to a horizontal parabola, if $s > 0$ ($s < 0$) the vertex will be the leftmost (rightmost) point of the parabola, i.e., it opens up to the right (left). In case of a vertical parabola, if $s > 0$ ($s < 0$) the vertex will be the lowest (highest) point of the parabola, i.e., it opens up upwards (downwards).

The **axis of symmetry** of a horizontal (vertical) parabola is equal to $y = m_2$ ($x = m_1$). Every parabola also has a **focus point** \vec{f}_p , which lies on the axis of symmetry at a distance p from its vertex, inside the parabola.

The line segment that goes through the focus point and is perpendicular to the axis of symmetry and whose endpoints belong to the parabola is called the **focal chord**. The length of the focal chord is equal to $4p$, whereby $p = \frac{1}{4|s|}$ (we take the absolute value of s , since we treat p as a distance).

A parabola can be described as the set of points that are at equal distances from the focus point and the **directrix**, which is the line perpendicular to the axis of symmetry located at a distance p from the parabola's vertex at the opposite side of the focus point.

To find out more about how to manipulate a parabola's width and position, please refer to section 7.6.3.

Consider the following example of a parabola with the conic equation $P : y^2 + 16x - 32y + 320 = 0$, which is shown in Fig. 4.10. The standard form is equal to $P : x = -\frac{1}{16}(y - 16)^2 - 4$.

The standard form tells us that we are dealing with a horizontal parabola, whose vertex is located at the point $\vec{m} = (-4, 16)$. Since $s = -\frac{1}{16} < 0$ the parabola opens up to the left. Its axis of symmetry is a horizontal line with equation $y = 16$.

The distance p is found to be $p = \frac{1}{4|s|} = \frac{16}{4} = 4$. This means that the focus point is positioned at $\vec{f}_p = (-4 - p, 16) = (-8, 16)$ and that the focal chord has a length equal to $4p = 16$. The directrix is a vertical line with equation $x = -4 + p = 0$, i.e., the y-axis.

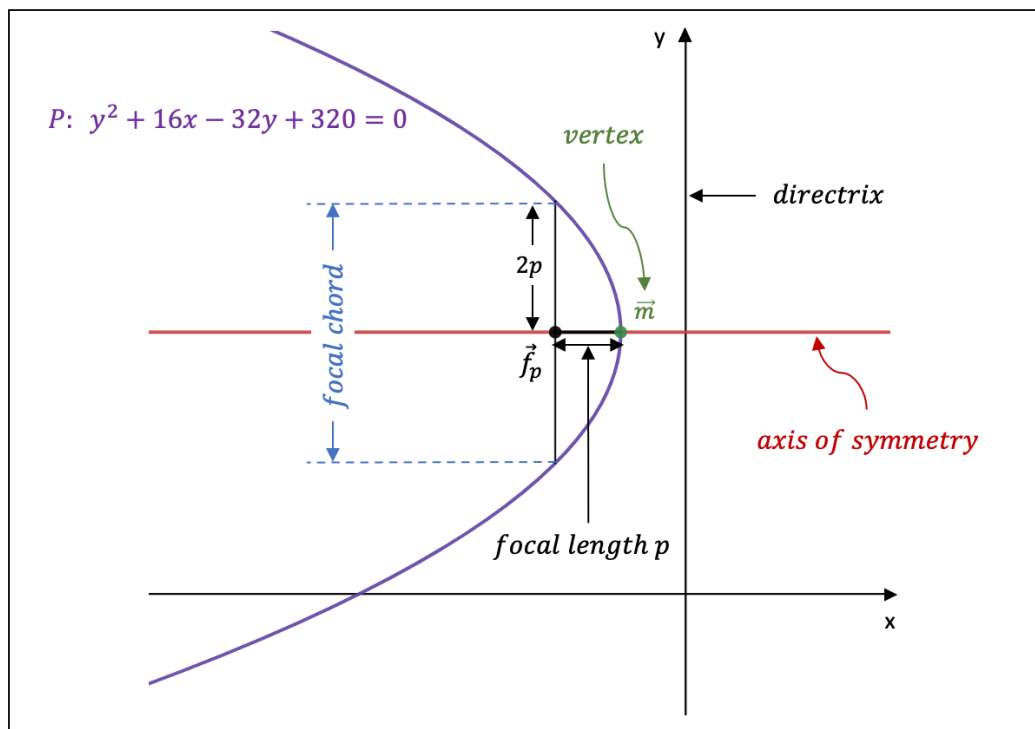


Figure 4.10: The parabola with equation $P : y^2 + 16x - 32y + 320 = 0$

4.3.5 The Hyperbola

If $ab < 0$, the conic section is a **hyperbola**. The standard form is equal to:

$$\left\{ \begin{array}{l} \text{Horizontal hyperbola:} \\ \text{Vertical hyperbola:} \end{array} \right. \quad \begin{array}{l} \frac{(x - m_1)^2}{d_1^2} - \frac{(y - m_2)^2}{d_2^2} = 1 \\ \frac{(y - m_2)^2}{d_1^2} - \frac{(x - m_1)^2}{d_2^2} = 1 \end{array}$$

whereby the centre of the hyperbola is located at $\vec{m} = (m_1, m_2)$.

With respect to the horizontal (vertical) hyperbola, the **transverse axis** is a symmetry axis that is oriented horizontally (vertically) and intersects the two **branches** of the hyperbola in two points called the **vertices**.

The distance d_1 is equal to the distance from the hyperbola's centre \vec{m} to a vertex \vec{v}_t . The coordinates of the vertices of a horizontal hyperbola are equal to $\vec{v}_t = (m_1 \pm d_1, m_2)$, whereas those of a vertical hyperbola are $\vec{v}_t = (m_1, m_2 \pm d_1)$.

The **conjugate axis** goes through the hyperbola's center and stands perpendicular to the transverse axis. The **co-vertices** \vec{v}_c lie on the conjugate axis at a distance d_2 from the centre \vec{m} . Their coordinates with respect to a horizontal and vertical hyperbola are equal to $\vec{v}_c = (m_1, m_2 \pm d_2)$ and $\vec{v}_c = (m_1 \pm d_2, m_2)$, respectively.

Every hyperbola has two **foci**, which sit on the transverse axis at a distance $f_c = \sqrt{d_1^2 + d_2^2}$ from the centre \vec{m} . The coordinates of the foci for a horizontal hyperbola are $\vec{f}_{1,2} = (m_1 \pm f_c, m_2)$, whereas the coordinates for a vertical hyperbola are equal to $\vec{f}_{1,2} = (m_1, m_2 \pm f_c)$.

A hyperbola has furthermore two asymptotes (see section 7.3). For a horizontal hyperbola, the equations of the asymptotes are $A_{1,2} : y - m_2 = \pm \frac{d_2}{d_1}(x - m_1)$, while for a vertical hyperbola the asymptotes have the equations $A_{1,2} : y - m_2 = \pm \frac{d_1}{d_2}(x - m_1)$.

In general, a hyperbola can be described as the set of points for which the absolute value of the difference of the distances s_1 and s_2 between a point on the hyperbola and the foci is equal to the distance between the two vertices. In other words: $|s_1 - s_2| = 2d_1$.

Consider as an example the conic equation $H : -x^2 + 2y^2 - 28x + 48y - 64 = 0$, which is depicted in Fig. 4.11. Since $ab = (-1) \cdot 2 = -2 < 0$, we are indeed dealing with a hyperbola.

The standard form is equal to $H : \frac{(y+12)^2}{78} - \frac{(x+14)^2}{156} = 1$, from which we can deduce that the hyperbola H is a vertical hyperbola with its centre positioned at $\vec{m} = (-14, -12)$.

The transverse axis $x = -14$ is thus vertically oriented, whereas the conjugate axis $y = -12$ is a horizontal line.

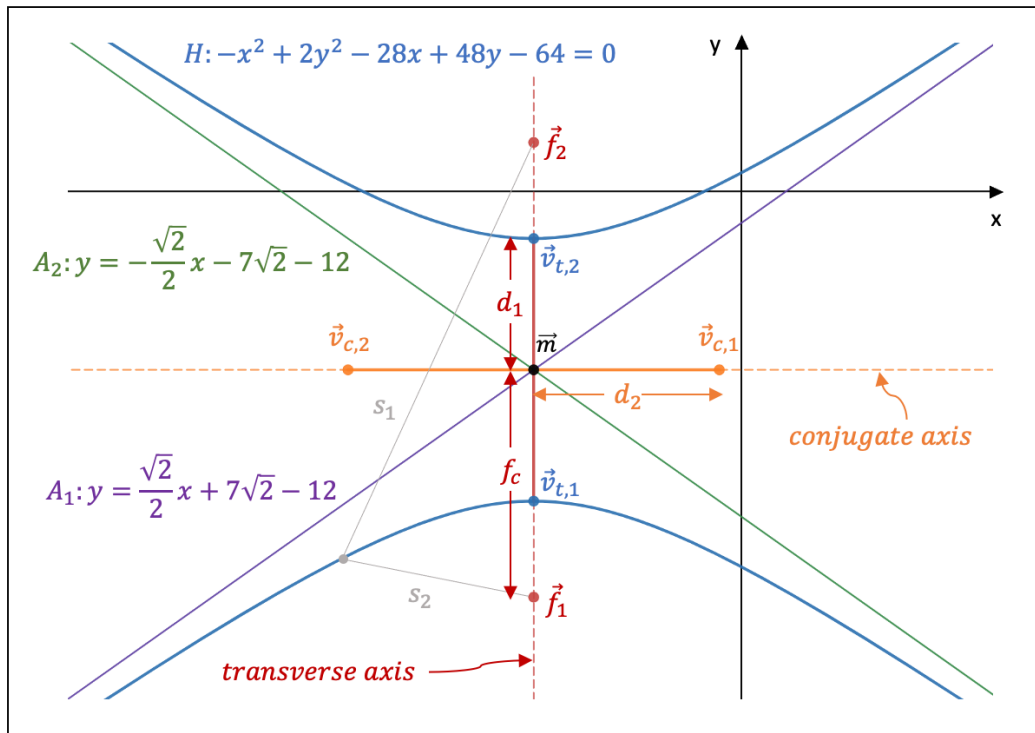


Figure 4.11: The hyperbola with equation $H : -x^2 + 2y^2 - 28x + 48y - 64 = 0$

The distances d_1 and d_2 are equal to $d_1 = \sqrt{78}$ and $d_2 = 2\sqrt{39}$, respectively. The focal length f_c is calculated as $f_c = \sqrt{d_1^2 + d_2^2} = \sqrt{78 + 156} = \sqrt{234}$, so that the foci are positioned at $\vec{f}_1 = (m_1, m_2 - f_c) = (-14, -12 - \sqrt{234})$ and $\vec{f}_2 = (m_1, m_2 + f_c) = (-14, -12 + \sqrt{234})$.

The vertices have the coordinates $\vec{v}_{t,1} = (m_1, m_2 - d_1) = (-14, -12 - \sqrt{78})$ and $\vec{v}_{t,2} = (m_1, m_2 + d_1) = (-14, -12 + \sqrt{78})$, whereas the co-vertices are located at $\vec{v}_{c,1} = (m_1 + d_2, m_2) = (-14 + 2\sqrt{39}, -12)$ and $\vec{v}_{c,2} = (m_1 - d_2, m_2) = (-14 - 2\sqrt{39}, -12)$.

Finally, the asymptotes for this vertical hyperbola are $A_1 : y - m_2 = \frac{d_1}{d_2}(x - m_1) \Leftrightarrow y + 12 = \frac{\sqrt{78}}{2\sqrt{39}}(x + 14) \Leftrightarrow y = \frac{\sqrt{2}}{2}x + 7\sqrt{2} - 12$ and $A_2 : y - m_2 = -\frac{d_1}{d_2}(x - m_1) \Leftrightarrow y + 12 = -\frac{\sqrt{78}}{2\sqrt{39}}(x + 14) \Leftrightarrow y = -\frac{\sqrt{2}}{2}x - 7\sqrt{2} - 12$.

5 Trigonometry

Trigonometry is the branch within the field of mathematics that studies the sides and angles in triangles as well as their relationships.

5.1 Definitions

An **angle** is the figure that is created when two lines come together in a point called the **vertex**. If we wish to identify the angle between two points A and C at point B , which is the vertex, we denote the angle by $\angle ABC$, $\sphericalangle ABC$, \widehat{ABC} , or a letter, preferably from the Greek alphabet.

The angle can be measured according to three different systems: the decimal system, the sexagesimal system, or the circular system.

In the **decimal system**, the angle is expressed in degrees ($^\circ$) whereby the value of the angle follows the decimal system. For instance, we can write $\alpha = 54.3^\circ$ or $\sphericalangle NMP = 22.0^\circ$.

In the **sexagesimal system**, every degree ($^\circ$) is divided into 60 equal elements called minutes ($'$) and every minute is further divided into 60 equal parts called seconds ($''$). For example, consider the angle $\widehat{HTA} = 45.36^\circ$, which is equal to $45^\circ + 0.36^\circ$. The part 0.36° corresponds with $0.36 \cdot 60 = 21.6$ minutes and, similarly, the part 0.6 minutes corresponds to $0.6 \cdot 60 = 36$ seconds. Therefore, the angle $\widehat{HTA} = 45.36^\circ$ is written in the sexagesimal system as $\widehat{HTA} = 45^\circ 21' 36''$.

In the **circular system**, an angle is expressed in radians (rad), whereby $360^\circ = 2\pi$ rad. For instance, an angle β equal to $\beta = 65^\circ$ is written in the circular system as $\beta = 65 \cdot \frac{\pi}{180} = \frac{13\pi}{36}$ rad. Conversely, an angle γ equal to $\gamma = \frac{3\pi}{10}$ rad is equivalent to $\gamma = \frac{3\pi}{10} \cdot \frac{180}{\pi} = 54^\circ$ in the decimal system.

Fig. 5.1 provides an example of how to define angles between points. Given that the lines KL and NM are parallel as well as KN and LM , we know that $\sphericalangle KNM = \sphericalangle KLM$ and that $\sphericalangle NKL = \sphericalangle NML$. So, $\widehat{KLM} = 125^\circ$ or $\widehat{KLM} = 125^\circ 0' 0''$ or $\widehat{KLM} = \frac{25\pi}{36}$ rad. The angle $\sphericalangle NML$ is equal to $\sphericalangle NML = 55^\circ$ or $\sphericalangle NML = 55^\circ 0' 0''$ or $\sphericalangle NML = \frac{11\pi}{36}$ rad.

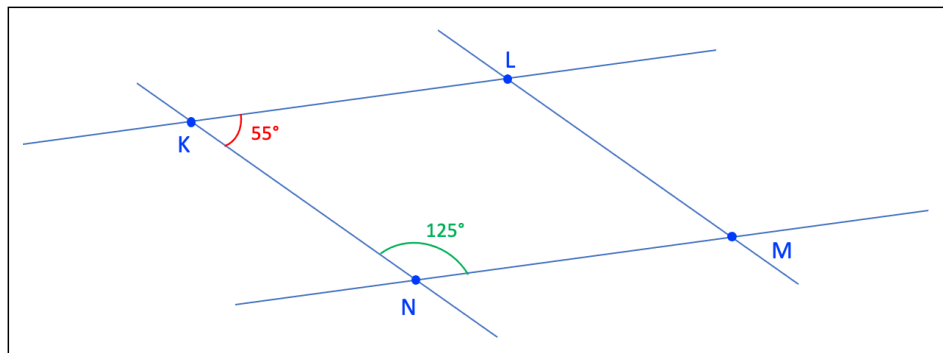


Figure 5.1: Angles between parallel lines

The angles in Fig. 5.1 are not oriented. The convention that is usually upheld is: the **counterclockwise** direction corresponds with **positive** angles whereas **clockwise** implies **negative** angles. Fig. 5.2 provides an overview of the different types of angles.

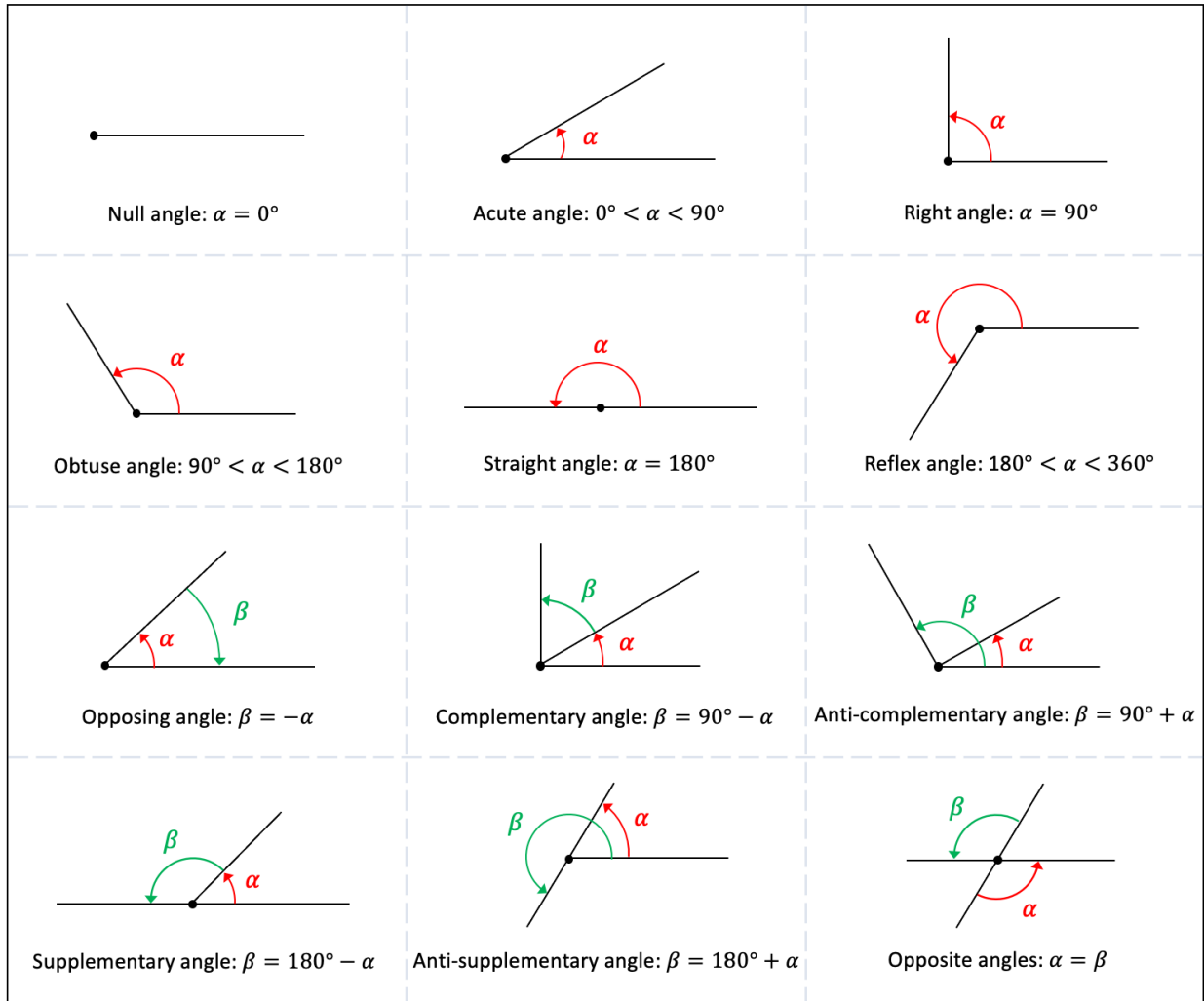


Figure 5.2: Overview of different types of angles

Consider an orthonormal coordinate system in the Euclidean plane. The circle with radius equal to 1 and with its centre located in the origin is called the **trigonometric or unit circle**.

The coordinate system divides the trigonometric circle up into 4 equal parts called **quadrants**, which are denoted by Roman numbers and arranged in ascending order in the counterclockwise direction.

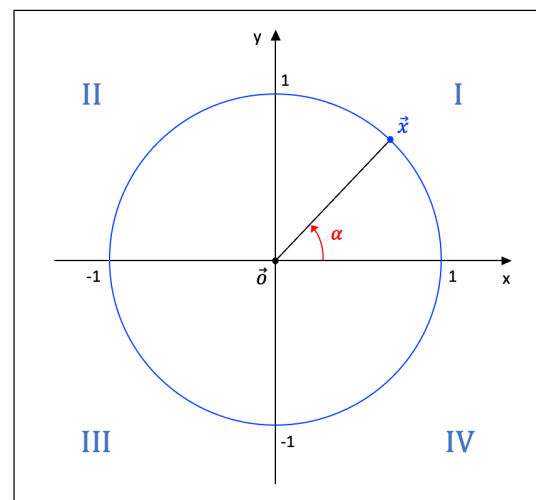


Figure 5.3: The trigonometric circle

5.2 Trigonometric Numbers

5.2.1 Angles

Consider a point \vec{x} on the unit circle that makes an angle θ with the horizontal. The **trigonometric numbers** $\cos \theta$ and $\sin \theta$ are defined as the x -coordinate and y -coordinate of the point \vec{x} . Applying the Pythagorean theorem, we find the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$.

Based on the above two trigonometric numbers, we can derive four more trigonometric numbers, i.e., the tangent (\tan), the cotangent (\cot), the cosecant (\csc), and the secant (\sec). The definitions are provided below, whereby the number in the respective denominator is well-defined, in that it does not take on a zero value:

$$\begin{aligned} \bullet \tan \theta &= \frac{\sin \theta}{\cos \theta} & \bullet \csc \theta &= \frac{1}{\sin \theta} \\ \bullet \cot \theta &= \frac{\cos \theta}{\sin \theta} & \bullet \sec \theta &= \frac{1}{\cos \theta} \end{aligned}$$

Fig. 5.4 shows how to retrieve the six trigonometric numbers from a graph of the unit circle. The coordinates of the point \vec{x} gives us $\cos \theta$ and $\sin \theta$. The y -coordinate (x -coordinate) of the point of intersection between the vertical line $x = 1$ (horizontal line $y = 1$) and the line going through \vec{x} and the origin is equal to $\tan \theta$ ($\cot \theta$). The x -intercept (y -intercept) of the line tangent to the circle at the point \vec{x} is equal to $\sec \theta$ ($\csc \theta$).

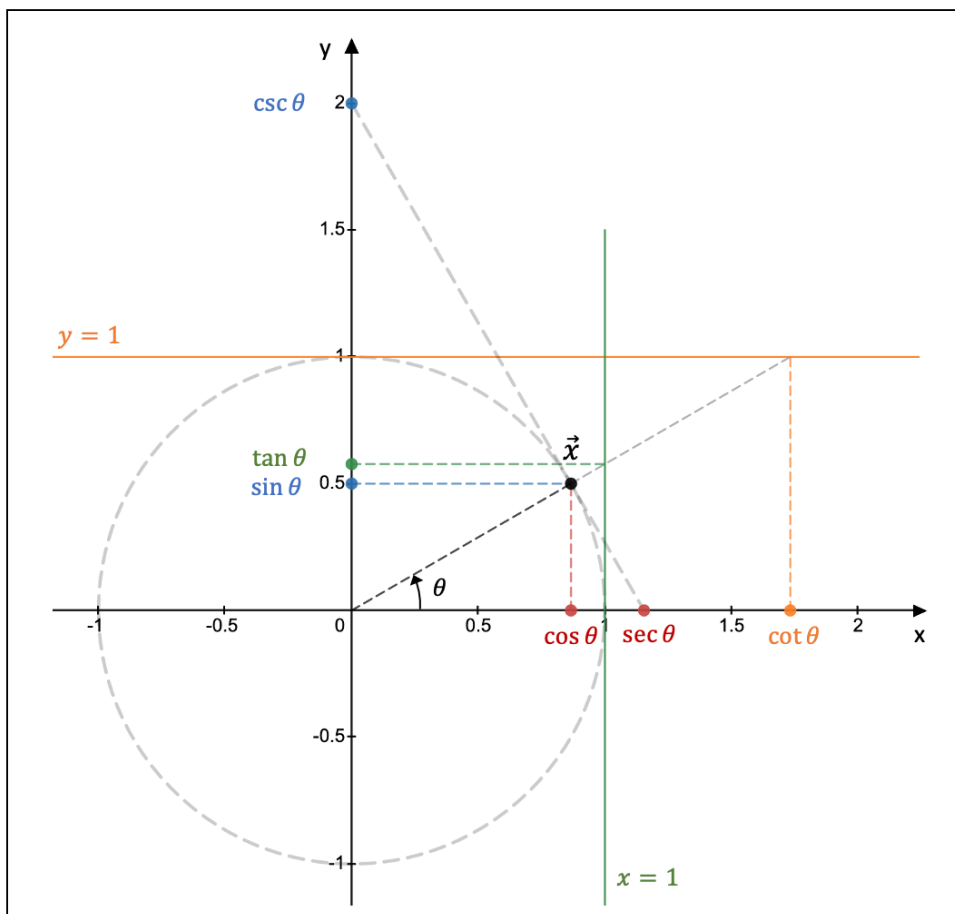


Figure 5.4: Trigonometric numbers and the unit circle

The angle in Fig. 5.4 is equal to $\theta = \frac{\pi}{6}$ rad. Under this angle, we find that the point \vec{x} has the coordinates $\vec{x} = (\cos \theta, \sin \theta) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using the above definitions for the remaining four trigonometric numbers, we calculate that $\tan \theta = \frac{\sqrt{3}}{3}$, $\cot \theta = \sqrt{3}$, $\sec \theta = \frac{2\sqrt{3}}{3}$, and $\csc \theta = 2$.

There is another but equivalent way to read the trigonometric numbers off a unit circle and it is based on right triangles, as shown in Fig. 5.5. Note that here the different trigonometric numbers represent distances.

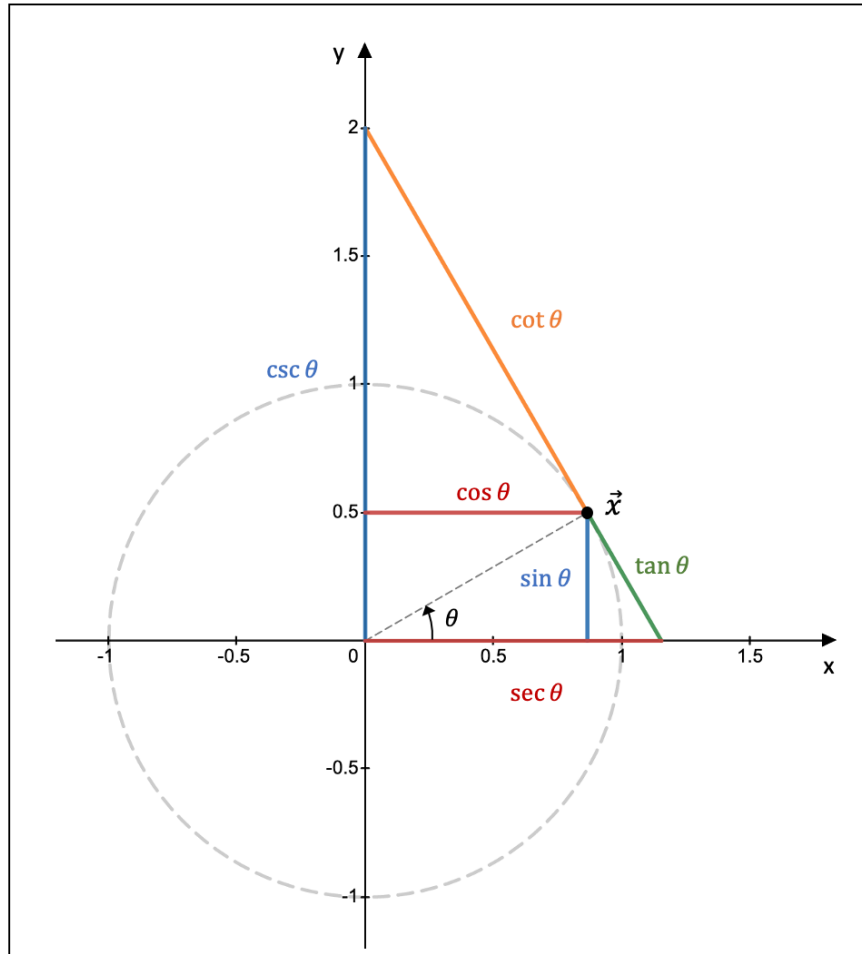


Figure 5.5: An alternative way to read trigonometric numbers

From Fig. 5.4 and Fig. 5.5 we see that the trigonometric numbers remain the same with every full rotation of 2π rad. This means that we can replace the angle θ with the expression $\theta + 2k\pi$ (with k a whole number; $k \in \mathbb{Z}$) and still obtain the same values.

What is more, for any given angle θ the trigonometric numbers $\cos \theta$ and $\sin \theta$ are restricted to the interval $[-1, 1]$. The trigonometric numbers $\sec \theta$ and $\tan \theta$ ($\csc \theta$ and $\cot \theta$) can take on any value, as long as the angle θ is not equal to $\theta = \frac{\pi}{2} + k\pi$ ($\theta = k\pi$), with $k \in \mathbb{Z}$.

Fig. 5.6 lists the trigonometric numbers of some commonly used angles, whereas Fig. 5.7 depicts the trigonometric numbers of five special angles: opposing angles, complementary angles, anti-complementary angles, supplementary angles, and anti-supplementary angles.

θ	$\sin \theta$	$\cos \theta$	$\csc \theta$	$\sec \theta$	$\tan \theta$	$\cot \theta$
0	0	1	/	1	0	/
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\sqrt{2}$	1	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{2\sqrt{3}}{3}$	2	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	1	/	/	0

Figure 5.6: The trigonometric numbers of some common angles

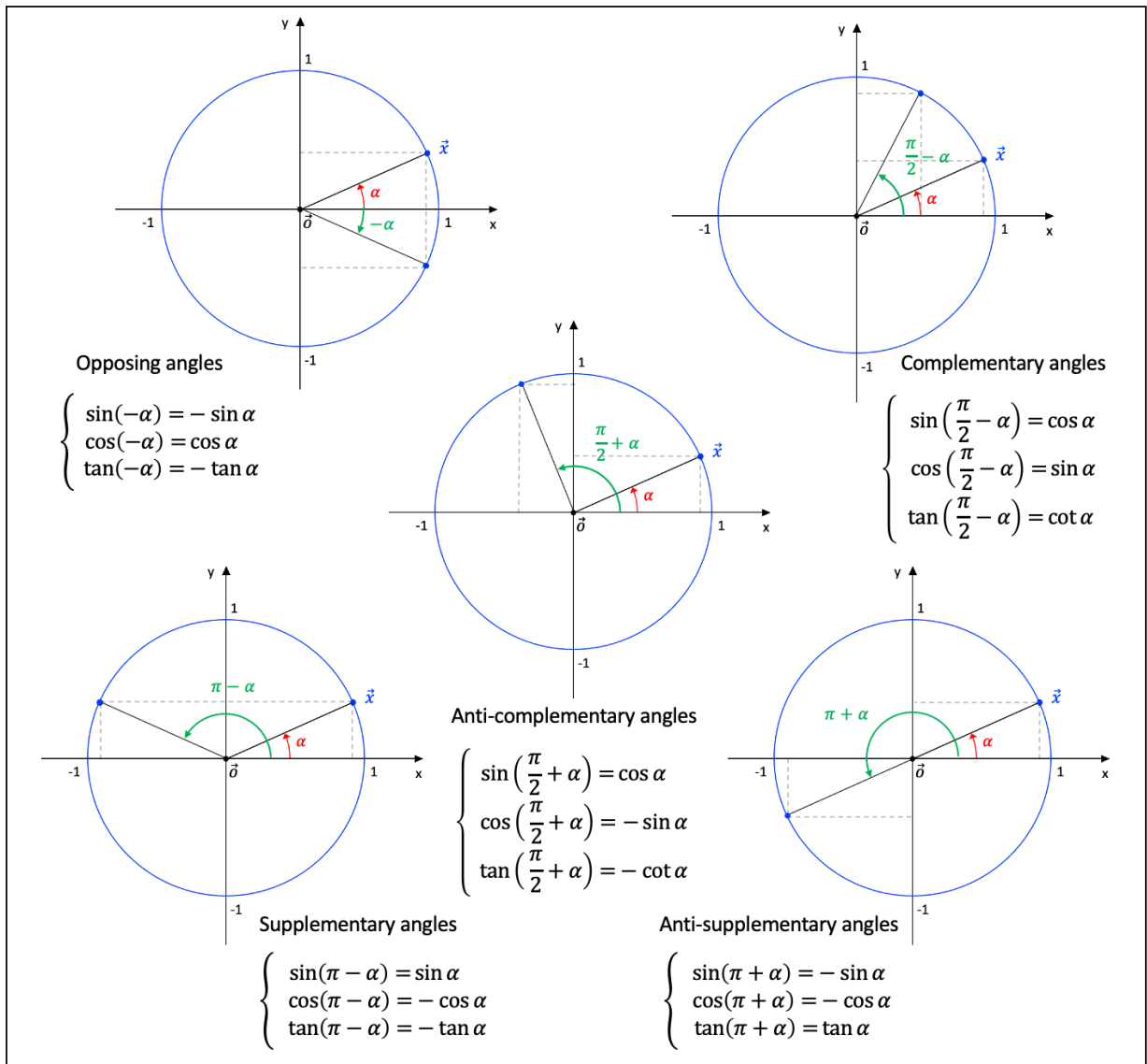


Figure 5.7: Trigonometric numbers of special angles

5.2.2 Triangles

In Euclidean geometry, the sum of the angles inside a triangle is always equal to 180° .

In a **right triangle**, one of the angles of the triangle is equal to 90° , which is appropriately called the right angle. The sum of the two remaining angles is then equal to 90° , i.e., they are complementary angles.

If we consider one of the angles in a right triangle other than the right angle, we call the **opposite side** the side that does not form part of the chosen angle, the **adjacent side** the shortest side of the chosen angle, and the **hypotenuse** the longest side of the chosen angle. The hypotenuse is also the longest side in a right triangle.

With these definitions we can construct the so-called **trigonometric ratios** in terms of trigonometric numbers (see left-hand side of Fig. 5.8).

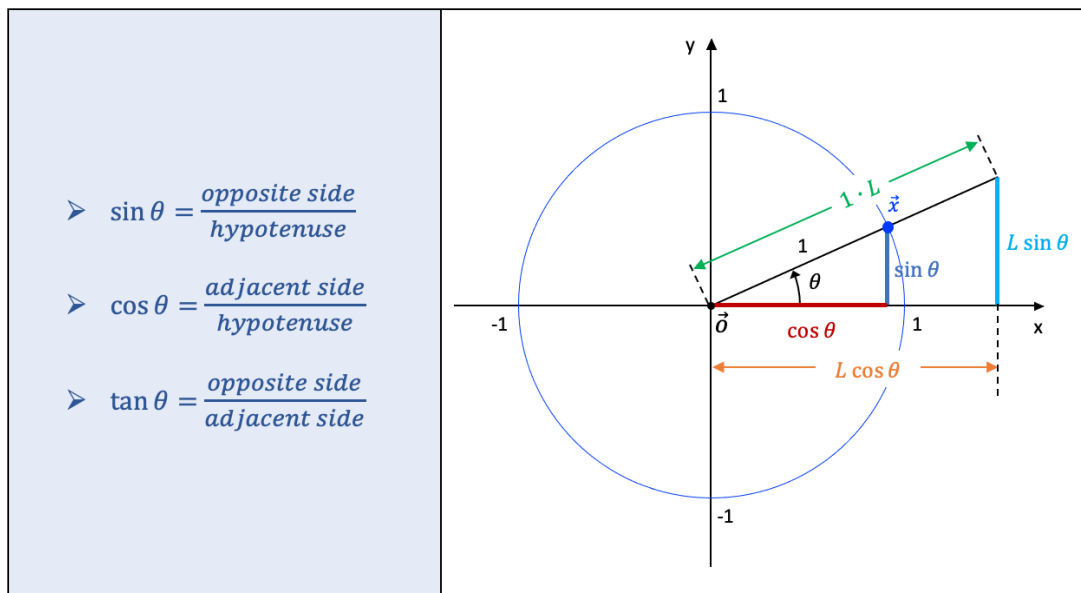


Figure 5.8: Trigonometric ratios

We can interpret these ratios more intuitively by looking at the right-hand side of Fig. 5.8. Since we are dealing with the unit circle, the hypotenuse of the right triangle within the circle is equal to the radius of the circle, which has a length of 1. As we have seen, the y -coordinate of the point \vec{x} is equal to the trigonometric number $\sin \theta$, which is also equal to the length of the opposite side—with respect to the angle θ —in the right triangle under consideration. Similarly, the trigonometric number $\cos \theta$ is equal to the adjacent side.

Let us now multiply the hypotenuse with length 1 by a factor L , so that its length becomes $1 \cdot L = L$. If we consider the right triangle with hypotenuse L , we see that the opposite side also has grown by a factor L and is equal to $L \cdot \sin \theta$. This is precisely what the trigonometric ratio for the sine of θ is telling us, i.e., the length of the opposite side ($L \cdot \sin \theta$) is equal to the length of the hypotenuse (L) multiplied by the trigonometric number related to the sine of the angle ($\sin \theta$). A similar reasoning can be applied to the adjacent side of the angle θ .

In case we are dealing with a **random triangle**, there are two sets of trigonometric relationships on which we can rely to calculate the sides and angles of a triangle: the **law of cosines** and the **law of sines**.

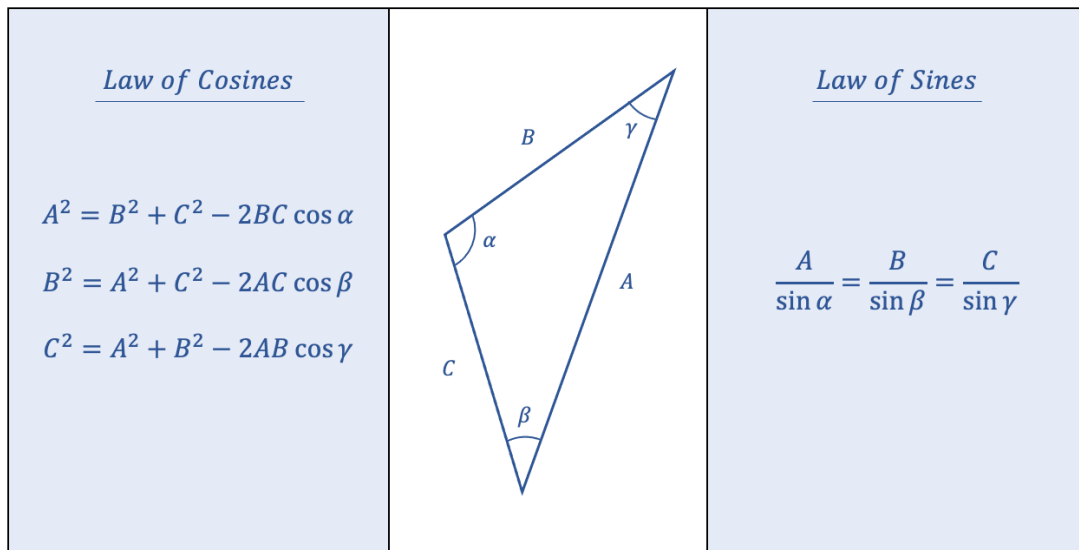


Figure 5.9: Law of cosines and law of sines

Bear in mind that in certain circumstances the law of sines may produce more than one solution. This occurs when two sides and an angle are given whereby the angle is not the angle formed by the two sides.

For example, consider Fig. 5.10 whereby we know that $A = 5$, $B = 9$, and $\alpha = 32^\circ$. We are asked to find the third side and the two other angles.

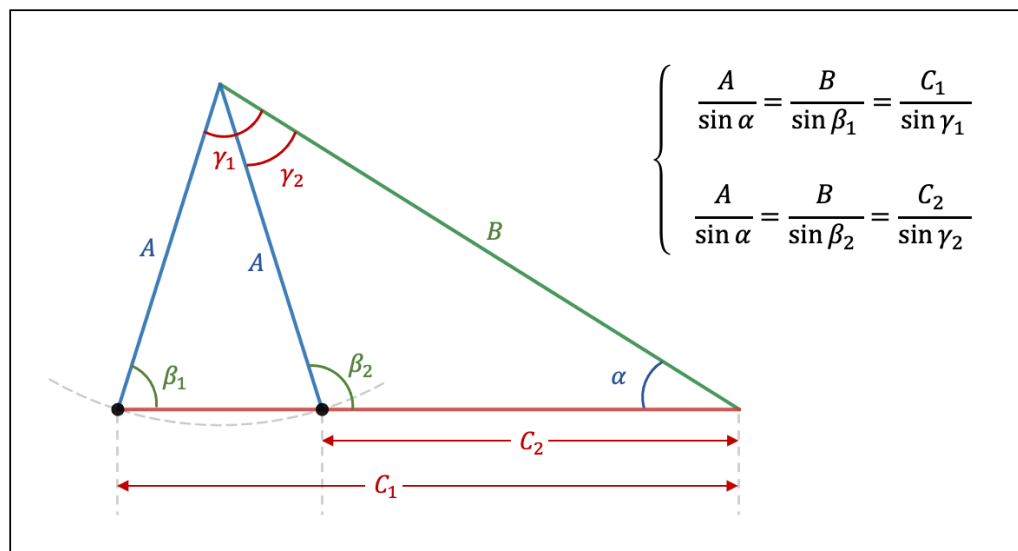


Figure 5.10: Example of the law of sines

In a first step, we write the law of sines for the triangle ABC_1 :

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta_1} = \frac{C_1}{\sin \gamma_1} \Leftrightarrow \frac{5}{\sin 32^\circ} = \frac{9}{\sin \beta_1} = \frac{C_1}{\sin \gamma_1}$$

We then find that $\sin \beta_1 = \frac{9}{5} \sin(32^\circ) \Leftrightarrow \beta_1 = \sin^{-1}\left(\frac{9}{5} \sin(32^\circ)\right) = 72.5^\circ$. Since the sum of the angles in a triangle equals 180° , the angle γ_1 is calculated as follows:

$$180^\circ = \alpha + \beta_1 + \gamma_1 \Leftrightarrow \gamma_1 = 180^\circ - \alpha - \beta_1 = 180^\circ - 32^\circ - 72.5^\circ = 75.5^\circ$$

However, given that supplementary angles produce the same trigonometric number *sin*, i.e., $\sin \beta_1 = \sin(180^\circ - \beta_1)$ (see Fig. 5.7), we find a second angle $\beta_2 = 180^\circ - \beta_1 = 180^\circ - 72.5^\circ = 107.5^\circ$ that satisfies the equation $\sin \beta_2 = \frac{9}{5} \sin(32^\circ)$. The third angle γ_2 is now equal to:

$$180^\circ = \alpha + \beta_2 + \gamma_2 \Leftrightarrow \gamma_2 = 180^\circ - \alpha - \beta_2 = 180^\circ - 32^\circ - 107.5^\circ = 40.5^\circ$$

The length of the third side in both cases is found as follows:

$$\begin{cases} C_1 = \frac{5}{\sin(32^\circ)} \sin \gamma_1 = \frac{5}{\sin(32^\circ)} \sin(75.5^\circ) = 9.13 \\ C_2 = \frac{5}{\sin(32^\circ)} \sin \gamma_2 = \frac{5}{\sin(32^\circ)} \sin(40.5^\circ) = 6.13 \end{cases}$$

5.3 Trigonometric Formulas

If we divide the **Pythagorean trigonometric identity** $\cos^2 \theta + \sin^2 \theta = 1$ by $\cos^2 \theta$ and $\sin^2 \theta$, respectively, we find the following equivalent identities (note that $\cos \theta \neq 0$ and $\sin \theta \neq 0$ for the respective case):

$$\begin{cases} 1 + \tan^2 \theta = \sec^2 \theta \\ \cot^2 \theta + 1 = \csc^2 \theta \end{cases}$$

Another set of trigonometric formulas are the **double-angle formulas**, which facilitate the transition between trigonometric numbers of single and double angles (see Fig. 5.11, whereby $\cos 2\theta \neq 0$ in the expression of the tangent).

$\sin 2\theta = 2 \sin \theta \cos \theta$	$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ $= 2 \cos^2 \theta - 1$ $= 1 - 2 \sin^2 \theta$	$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
--	--	--

Figure 5.11: Double-angle formulas

Based on the double-angle formula for the cosine, we can derive the **Carnot's formulas** as well as the **half-angle formulas** (whereby $\cos \frac{\theta}{2} \neq 0$ in the expression of the tangent):

<u>Carnot's Formulas</u>	<u>Half-angle Formulas</u>
$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$	$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$
$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$	$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$
	$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$

Figure 5.12: Half-angle formulas

Another useful set of trigonometric formulas are the **angle addition and subtraction formulas** (whereby $\cos(\theta - \phi) \neq 0$ and $\cos(\theta + \phi) \neq 0$ for the respective expression of the tangent) and the **Simpson's formulas**:

<u>Angle addition formulas</u>	<u>Angle subtraction formulas</u>
$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$	$\sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi$
$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$	$\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$
$\tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi}$	$\tan(\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$
<u>Simpson's formulas</u>	
$\sin \theta + \sin \phi = 2 \sin \left(\frac{\theta + \phi}{2} \right) \cos \left(\frac{\theta - \phi}{2} \right)$	$2 \sin \theta \cos \phi = \sin(\theta - \phi) + \sin(\theta + \phi)$
$\sin \theta - \sin \phi = 2 \sin \left(\frac{\theta - \phi}{2} \right) \cos \left(\frac{\theta + \phi}{2} \right)$	$2 \sin \phi \cos \theta = \sin(\phi - \theta) + \sin(\theta + \phi)$
$\cos \theta + \cos \phi = 2 \cos \left(\frac{\theta + \phi}{2} \right) \cos \left(\frac{\theta - \phi}{2} \right)$	$2 \sin \theta \sin \phi = \cos(\theta - \phi) - \cos(\theta + \phi)$
$\cos \theta - \cos \phi = 2 \sin \left(\frac{\phi - \theta}{2} \right) \sin \left(\frac{\theta + \phi}{2} \right)$	$2 \cos \theta \cos \phi = \cos(\theta - \phi) + \cos(\theta + \phi)$

Figure 5.13: Angle addition, angle subtraction and Simpson's formulas

Based on the double-angle formula for the cosine and the half-angle formulas, we find a final set of trigonometric formulas called the **tangent half-angle substitution formulas**:

$$t = \tan \frac{\theta}{2} \quad \Rightarrow \quad \begin{cases} \cos \theta = \frac{1-t^2}{1+t^2} \\ \sin \theta = \frac{2t}{1+t^2} \\ \tan \theta = \frac{2t}{1-t^2} \end{cases}$$

Figure 5.14: The tangent half-angle substitution formulas

5.4 Cyclometric Numbers

As discussed in section 5.2, given a certain angle θ we can find a unique trigonometric number. However, the reverse operation, i.e., finding an angle given a trigonometric number, does not produce a unique solution. If we restrict the interval of the angle, we can define the **cyclometric numbers**: the arc sine, the arc cosine, the arc secant, the arc cosecant, the arc tangent, and the arc cotangent:

$$\begin{cases} x \in [-1, 1] \\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases} \quad \& \quad \sin y = x \quad \Leftrightarrow \quad y = \arcsin x$$

$$\begin{cases} x \in [-1, 1] \\ y \in [0, \pi] \end{cases} \quad \& \quad \cos y = x \quad \Leftrightarrow \quad y = \arccos x$$

$$\begin{cases} x \in]-\infty, -1] \cup [1, +\infty[\\ y \in \left[0, \frac{\pi}{2}\right[\cup \left[\frac{\pi}{2}, \pi\right] \end{cases} \quad \& \quad \sec y = x \quad \Leftrightarrow \quad y = \operatorname{arcsec} x$$

$$\begin{cases} x \in]-\infty, -1] \cup [1, +\infty[\\ y \in \left[-\frac{\pi}{2}, 0\right[\cup \left[0, \frac{\pi}{2}\right] \end{cases} \quad \& \quad \csc y = x \quad \Leftrightarrow \quad y = \operatorname{arccsc} x$$

$$\begin{cases} x \in \mathbb{R} \\ y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\end{cases} \quad \& \quad \tan y = x \quad \Leftrightarrow \quad y = \arctan x$$

$$\begin{cases} x \in \mathbb{R} \\ y \in]0, \pi[\end{cases} \quad \& \quad \cot y = x \quad \Leftrightarrow \quad y = \operatorname{arccot} x$$

Figure 5.15: The cyclometric numbers

For example, $y = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ because $x = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $-1 \leq -\frac{\sqrt{2}}{2} \leq 1$, and $-\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}$.

If we do not restrict the interval of the angle, we find multiple solutions for the cyclometric numbers. Let us consider, for instance, the equation $\cos y = \frac{\sqrt{3}}{2}$, for which we find that $y = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$. Given that adding any multiple of 2π to the angle does not change the trigonometric number, we can write $y = \frac{\pi}{6} + 2k\pi$ (with $k \in \mathbb{Z}$).

However, Fig. 5.7 tells us that $\cos y = \cos(-y)$, which means that we have another solution equal to $-y = \frac{\pi}{6} + 2k\pi \Leftrightarrow y = -\frac{\pi}{6} + 2k\pi$ (with $k \in \mathbb{Z}$).

Similarly, Fig. 5.7 also tells us that $\sec(-y) = \sec y$, $\sin(\pi - y) = \sin y$, $\csc(\pi - y) = \csc y$, $\tan(\pi + y) = \tan y$, and $\cot(\pi + y) = \cot y$. We can therefore formulate the following general solutions (with $a \in [-1, 1]$, $b \in]-\infty, -1] \cup [1, \infty[$, $c \in \mathbb{R}$, and $k \in \mathbb{Z}$):

$\sin y = a$	\Leftrightarrow	$\begin{cases} y = \arcsin a + 2k\pi \\ y = \pi - \arcsin a + 2k\pi \end{cases}$
$\cos y = a$	\Leftrightarrow	$\begin{cases} y = \arccos a + 2k\pi \\ y = -\arcsin a + 2k\pi \end{cases}$
$\csc y = b$	\Leftrightarrow	$\begin{cases} y = \operatorname{arccsc} b + 2k\pi \\ y = \pi - \operatorname{arccsc} b + 2k\pi \end{cases}$
$\sec y = b$	\Leftrightarrow	$\begin{cases} y = \operatorname{arcsec} b + 2k\pi \\ y = -\operatorname{arcsec} b + 2k\pi \end{cases}$
$\tan y = c$	\Leftrightarrow	$y = \arctan c + k\pi$
$\cot y = c$	\Leftrightarrow	$y = \operatorname{arccot} c + k\pi$

Figure 5.16: Multiple solutions for the cyclometric numbers

Another kind of example is the equation $\sec(4x + 2) - \sec(x - 2) = 0$, which we would like to solve for x . We find one solution by setting the arguments equal to each other, keeping in mind that any multiple of 2π is also a solution: $4x + 2 = x - 2 + 2k\pi \Leftrightarrow x = \frac{1}{3}(2k\pi - 4)$ (with $k \in \mathbb{Z}$).

However, given that $\sec(-y) = \sec y$, we also have a second solution, which is equal to $4x + 2 = -(x - 2) + 2k\pi \Leftrightarrow x = \frac{2}{5}k\pi$ (with $k \in \mathbb{Z}$).

Since cyclometric numbers are angles, we can calculate **the trigonometric numbers of cyclometric numbers**. The trivial cases are, for instance, $\sin(\arcsin x) = x$ or $\csc(\operatorname{arccsc} x) = x$.

Let us consider a non-trivial example, such as $\sin(\operatorname{arcsec} x)$. Given that $\sec(\operatorname{arcsec} x) = x$, we need to find a way to express the trigonometric number *sin* in terms of the trigonometric number *sec*:

$$\sin^2 \theta + \cos^2 \theta = 1 \Leftrightarrow \sin^2 \theta + \frac{1}{\sec^2 \theta} = 1 \Leftrightarrow \sin \theta = \pm \sqrt{1 - \frac{1}{\sec^2 \theta}}$$

In our example $\theta = \operatorname{arcsec} x$ and from the definitions (see Fig. 5.15) we know that $\theta = \operatorname{arcsec} x \in [0, \pi] \setminus \{\frac{\pi}{2}\}$. This means that the sine of θ will always be positive, because the angle θ is located in the first or second quadrant. We can therefore write (with $x \in]-\infty, -1] \cup [1, \infty[$):

$$\sin(\operatorname{arcsec} x) = \sqrt{1 - \frac{1}{\sec^2(\operatorname{arcsec} x)}} = \sqrt{1 - \frac{1}{x^2}}$$

For instance, if we are asked to calculate $\sin(\operatorname{arcsec}(-6.5))$, we simply use the above expression with $x = -6.5$, so that $\sin(\operatorname{arcsec}(-6.5)) = \sqrt{1 - \frac{1}{(-6.5)^2}} = 0.988$.

Applying the same method to other cyclometric numbers, we find for instance the following expressions (whereby x is considered well-defined for the respective trigonometric number):

- $\sin(\operatorname{arccsc} x) = \frac{1}{x}$
- $\cos(\arcsin x) = \sqrt{1 - x^2}$
- $\tan(\arcsin x) = \frac{x}{\sqrt{1 - x^2}}$
- $\sin(\arctan x) = \frac{x}{\sqrt{1 + x^2}}$
- $\cos(\arctan x) = \frac{1}{\sqrt{1 + x^2}}$
- $\tan(\operatorname{arccos} x) = \frac{\sqrt{1 - x^2}}{x}$
- $\sin(\operatorname{arccos} x) = \sqrt{1 - x^2}$
- $\cos(\operatorname{arcsec} x) = \frac{1}{x}$
- $\tan(\operatorname{arccsc} x) = \pm \frac{1}{\sqrt{x^2 - 1}}$
- $\sin(\operatorname{arccot} x) = \frac{1}{\sqrt{1 + x^2}}$
- $\cos(\operatorname{arccot} x) = \frac{x}{\sqrt{1 + x^2}}$
- $\tan(\operatorname{arcsec} x) = \pm \sqrt{x^2 - 1}$

Another useful application is the case where one finds a **linear combination of a cosine and a sine of the same angle**. The linear combination can then be rewritten in terms of just a cosine: $a \cos \theta + b \sin \theta = c \cos(\theta + \phi)$, with $a \neq 0$.

The coefficient c is equal to $c = \operatorname{sgn}(a)\sqrt{a^2 + b^2}$, whereby $\operatorname{sgn}(a)$ indicates the sign of the coefficient a , and the angle ϕ , called the phase shift, is equal to $\phi = \arctan\left(-\frac{b}{a}\right)$.

For example, the linear combination $-3 \cos \theta + 9 \sin \theta$ can be written as $-3\sqrt{10} \cos(\theta + 1.25)$, whereby $c = -\sqrt{(-3)^2 + 9^2} = -3\sqrt{10}$ and $\phi = \arctan\left(-\frac{9}{-3}\right) = 1.25$ rad.

5.5 Polar Coordinates

Based on the trigonometric ratios, as discussed in section 5.2.2, we can define the **polar coordinates**, which are characterized by the distance r and the angle θ .

If a Cartesian coordinate system is chosen, the distance r represents the distance between the origin and a point \vec{p} (with $\vec{p} \neq \vec{o}$), while the angle θ is measured from the positive side of the x -axis in counterclockwise direction.

If a point \vec{p} has Cartesian coordinates $\vec{p} = (x, y)$, the polar coordinates are equal to $\vec{p} = (r, \theta)$ and the relationship between the Cartesian and polar coordinates are provided by the trigonometric ratios: $x = r \cos \theta$ and $y = r \sin \theta$.

The norm of the vector \vec{p} , which represents the distance between the origin and the point \vec{p} , is equal to $\|\vec{p}\| = \sqrt{x^2 + y^2} = \sqrt{(r \cos \theta)^2 + (r \sin \theta)^2} = \sqrt{r^2(\cos^2 \theta + \sin^2 \theta)} = \sqrt{r^2 \cdot 1} = r$. In other words, the distance r , which is the first coordinate of the polar coordinates (r, θ) , is equal to $r = \sqrt{x^2 + y^2}$.

The angle θ can then be found through one of the following expressions (with $\theta \in [0, 2\pi[$):

$$\begin{cases} \theta = \arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right) \\ \theta = \arcsin\left(\frac{y}{r}\right) = \arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

Note that the angle θ can only be found through the cyclometric number \arctan if $x \neq 0$, i.e., the point cannot be located on the y -axis.

Let us, for instance, find the polar coordinates $\vec{p} = (r, \theta)$ of the point \vec{p} with Cartesian coordinates $\vec{p} = (-2, -4)$.

The distance r is equal to $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-4)^2} = 2\sqrt{5}$.

The angle θ is calculated as $\theta = \arccos\left(\frac{-2}{2\sqrt{5}}\right) = \arccos\left(-\frac{1}{\sqrt{5}}\right) = 2.03$ rad. Since $\cos(-\theta) = \cos \theta$, the other solution is equal to $\theta = -2.03$ rad, but given that $\theta \in [0, 2\pi[$, the second solution becomes $\theta = -2.03 + 2\pi = 4.25$ rad. As we know that point \vec{p} lies in the third quadrant, the angle θ must be equal to $\theta = 4.25$ rad.

The polar coordinates of \vec{p} are thus $\vec{p} = (r, \theta) = (2\sqrt{5}, 4.25)$.

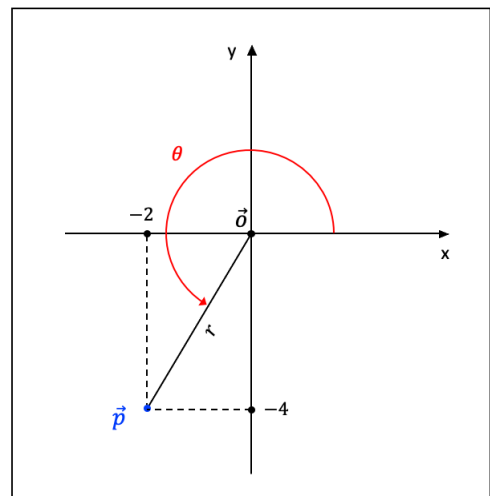


Figure 5.17: Polar coordinates

We can describe curves in the Euclidean plane not only with Cartesian coordinates, but also with polar coordinates. The equation of a curve expressed in polar coordinates is called a **polar equation**.

For instance, let us look at the polar equation $r = \sqrt{\frac{36}{1+5\cos(2\theta)}}$. To understand with which type of curve we are dealing with, we transition from polar coordinates to Cartesian coordinates via the trigonometric ratios:

$$\begin{aligned} r = \sqrt{\frac{36}{1+5\cos(2\theta)}} &\Leftrightarrow r^2 = \frac{36}{1+5\cos(2\theta)} &\Leftrightarrow r^2 = \frac{36}{1+5(\cos^2\theta - \sin^2\theta)} \\ & &\Leftrightarrow r^2 = \frac{36}{1+5\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)} \\ & &\Leftrightarrow 1 = \frac{36}{(x^2+y^2)+5(x^2-y^2)} \\ & &\Leftrightarrow \frac{x^2}{6} - \frac{y^2}{9} = 1 \end{aligned}$$

This is the equation of a horizontal hyperbola, whereby its center is located at the origin. The denominator of the polar equation becomes zero if $\theta = \frac{1}{2} \arccos\left(-\frac{1}{5}\right) + k\pi = 0.886 + k\pi$ rad or if $\theta = -\frac{1}{2} \arccos\left(-\frac{1}{5}\right) + k\pi = -0.886 + k\pi$ rad. This means that if we let the angle θ take on all the values in the interval $]0.886, 7.17[\setminus \{2.26, 4.03, 5.40\}$, the polar equation will display the entire curve.

6 Solid Geometry

As already alluded to in the introduction of section 4, **solid geometry** is Euclidean geometry in three spatial dimensions. The basic concepts of plane geometry are still applicable in solid geometry, including the Pythagorean theorem, the definition of the distance between two points and the norm of a vector, the dot product between two vectors, the perpendicular position between vectors, and the concept of parallel vectors.

6.1 The Equation of Straight Lines and Euclidean Planes

Three **non-collinear points**, i.e., points that do not lie on the same straight line, define a unique Euclidean plane. Consider, for instance, the points \vec{v}_1 and \vec{v}_2 , which are neither the null vector nor a multiple of each other. Together with the origin \vec{o} of our orthonormal coordinate system, these three points define a plane α . The vectors \vec{v}_1 and \vec{v}_2 are referred to as the **directional vectors** of α .

We can express a point $\vec{q} \in \alpha$ as $\vec{q} = \lambda\vec{v}_1 + \mu\vec{v}_2$ with λ and μ real scalars. Take now another point $\vec{p} \notin \alpha$. There is a unique plane β with the same directional vectors \vec{v}_1 and \vec{v}_2 as α and whereby $\vec{p} \in \beta$. We can now find the point $\vec{x} \in \beta$ so that $\vec{x} - \vec{p} = \vec{q} = \lambda\vec{v}_1 + \mu\vec{v}_2 \Leftrightarrow \vec{x} = \vec{p} + \lambda\vec{v}_1 + \mu\vec{v}_2$. This last equation is called the **vector equation** of plane β .

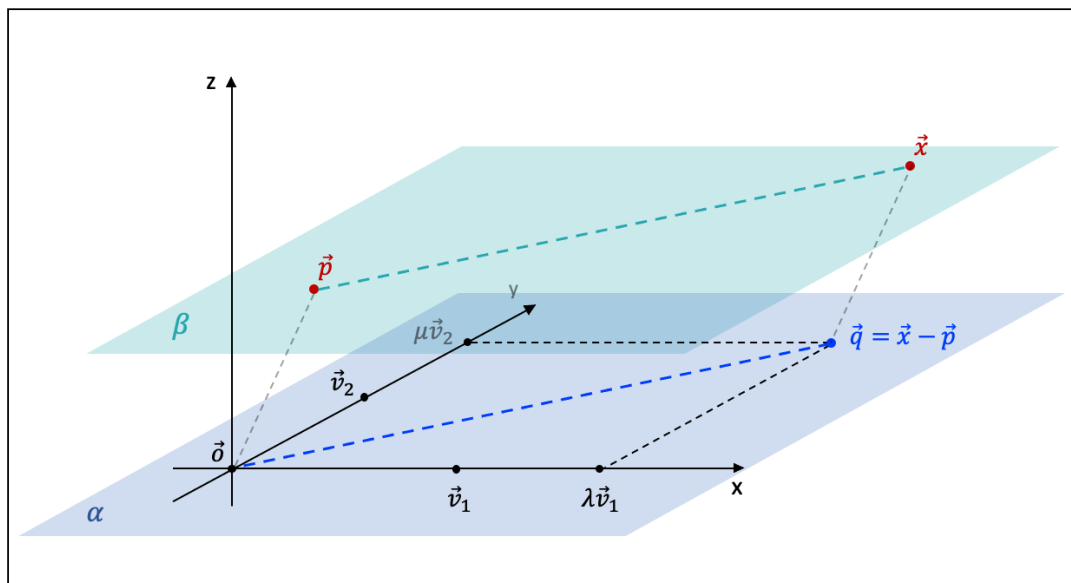


Figure 6.1: Two parallel Euclidean planes α and β

If instead of two directional vectors and the origin, we are given three non-collinear points \vec{a} , \vec{b} , and \vec{c} , we can construct the vector equation of the plane γ defined by these three points as follows:

$$\vec{x} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

whereby $(\vec{b} - \vec{a})$ and $(\vec{c} - \vec{a})$ are two directional vectors of the plane γ .

The vector equation of a straight line between two points \vec{a} and \vec{b} in three-dimensional space remains the same as discussed in section 4.2.1, i.e., $\vec{x} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

The **parametric equation** of the plane γ is constructed by writing out the individual components of the vector equation, whereby $\vec{x} = (x, y, z)$, $\vec{a} = (a_x, a_y, a_z)$, $\vec{b} = (b_x, b_y, b_z)$, and $\vec{c} = (c_x, c_y, c_z)$:

$$\begin{cases} x = a_x + \lambda(b_x - a_x) + \mu(c_x - a_x) \\ y = a_y + \lambda(b_y - a_y) + \mu(c_y - a_y) \\ z = a_z + \lambda(b_z - a_z) + \mu(c_z - a_z) \end{cases}$$

The parametric equation of a straight line in three-dimensional space also remains the same as in section 4.2.1, but with an additional dimension:

$$\begin{cases} x = a_x + \lambda(b_x - a_x) \\ y = a_y + \lambda(b_y - a_y) \\ z = a_z + \lambda(b_z - a_z) \end{cases}$$

The **Cartesian equation** of plane γ is found by eliminating the parameters λ and μ from the parametric equation:

$$\begin{aligned} (x - a_x) \cdot [(b_y - a_y)(c_z - a_z) - (b_z - a_z)(c_y - a_y)] + (y - a_y) \cdot [(b_z - a_z)(c_x - a_x) - (b_x - a_x)(c_z - a_z)] \\ + (z - a_z) \cdot [(b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)] = 0 \end{aligned}$$

If we look closely, we see that the above equation can be written with the assistance of the determinant (see section 3.3.4), whereby we use the notation $\det(A) = |A|$ and whereby we apply the technique of cofactor expansion along the first row:

$$\begin{vmatrix} x - a_x & y - a_y & z - a_z \\ b_x - a_x & b_y - a_y & b_z - a_z \\ c_x - a_x & c_y - a_y & c_z - a_z \end{vmatrix} = 0$$

If we would make the substitutions $a = (b_y - a_y)(c_z - a_z) - (b_z - a_z)(c_y - a_y)$, $b = (b_z - a_z)(c_x - a_x) - (b_x - a_x)(c_z - a_z)$, $c = (b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)$, and $d = -a_x a - a_y b - a_z c$, we find the general Cartesian equation of a plane:

$$ax + by + cz + d = 0$$

The above general equation can also be found if we consider a normal vector $\vec{n} = (a, b, c)$ of the plane γ . Any directional vector of the plane must be perpendicular to \vec{n} , so that their dot product equals zero (see section 4.1.2). If we choose the vector $\vec{x} - \vec{a}$ as a directional vector, we obtain:

$$\begin{aligned}
\vec{n} \cdot (\vec{x} - \vec{a}) = 0 &\Leftrightarrow (a, b, c) \cdot (x - a_x, y - a_y, z - a_z) = 0 \\
&\Leftrightarrow a(x - a_x) + b(y - a_y) + c(z - a_z) = 0 \\
&\Leftrightarrow ax + by + cz + (-a_x a - a_y b - a_z c) = 0 \\
&\Leftrightarrow ax + by + cz + d = 0
\end{aligned}$$

Similarly, the set of Cartesian equations of a straight line in three-dimensional space is found by eliminating the parameter λ from its parametric equation:

$$\frac{x - a_x}{b_x - a_x} = \frac{y - a_y}{b_y - a_y} = \frac{z - a_z}{b_z - a_z}$$

This system of Cartesian equations can also be written as:

$$\begin{cases} (b_y - a_y)x + (a_x - b_x)y = a_x(b_y - a_y) + a_y(a_x - b_x) \\ (b_z - a_z)y + (a_y - b_y)z = a_y(b_z - a_z) + a_z(a_y - b_y) \end{cases}$$

In other words, the above set of two Cartesian equations, i.e., the equation of a straight line, can be interpreted as the intersection of two planes.

Let us, for example, find the system of Cartesian equations that describes the straight line L formed by the intersection of plane α and β , given that $\vec{a}_1 = (-1, -2, 0)$, $\vec{a}_2 = (6, -4, -1)$, and $\vec{a}_3 = (1, 0, 3)$ belong to α and that $\vec{b}_1 = (0, -2, 8)$, $\vec{b}_2 = (2, 0, 1)$, and $\vec{b}_3 = (-3, 3, -7)$ belong to β .

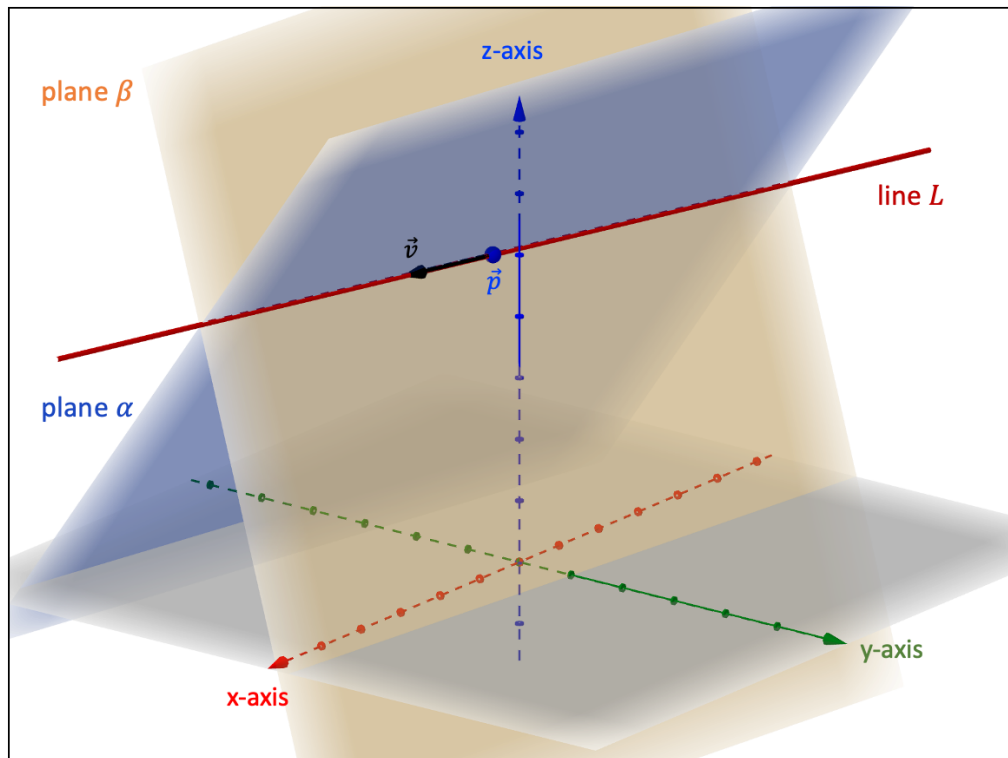


Figure 6.2: Two intersecting planes α and β

In a first step, we determine some directional vectors of plane α and β :

$$\alpha : \begin{cases} \vec{v}_{\alpha 1} = \vec{a}_2 - \vec{a}_1 = (6, -4, -1) - (-1, -2, 0) = (7, -2, -1) \\ \vec{v}_{\alpha 2} = \vec{a}_3 - \vec{a}_1 = (1, 0, 3) - (-1, -2, 0) = (2, 2, 3) \end{cases}$$

$$\beta : \begin{cases} \vec{v}_{\beta 1} = \vec{b}_2 - \vec{b}_1 = (2, 0, 1) - (0, -2, 8) = (2, 2, -7) \\ \vec{v}_{\beta 2} = \vec{b}_3 - \vec{b}_1 = (-3, 3, -7) - (0, -2, 8) = (-3, 5, -15) \end{cases}$$

The Cartesian equations of plane α and β can be found via the method of the discriminant:

$$\alpha : \begin{vmatrix} x+1 & y+2 & z \\ 7 & -2 & -1 \\ 2 & 2 & 3 \end{vmatrix} = 0 \Leftrightarrow 4x + 23y - 18z + 50 = 0$$

$$\beta : \begin{vmatrix} x & y+2 & z-8 \\ 2 & 2 & -7 \\ -3 & 5 & -15 \end{vmatrix} = 0 \Leftrightarrow 5x + 51y + 16z - 26 = 0$$

The system of Cartesian equations that represents the line L is then equal to:

$$\begin{cases} 4x + 23y - 18z + 50 = 0 \\ 5x + 51y + 16z - 26 = 0 \end{cases} \Leftrightarrow x = \begin{pmatrix} y + \frac{166}{643} \\ -\frac{77}{643} \end{pmatrix} = \begin{pmatrix} z - \frac{1574}{643} \\ \frac{89}{1286} \end{pmatrix}$$

In other words, the point $\vec{p} = \left(0, -\frac{166}{643}, \frac{1574}{643}\right)$ is a point that belongs to L and the vector $\vec{v} = \left(1, -\frac{77}{643}, \frac{89}{1286}\right)$ is a directional vector of L . Note that the starting point of vector \vec{v} in Fig. 6.2 is the point \vec{p} , not the origin \vec{o} .

6.2 Relative Position of Straight Lines and Planes

6.2.1 Straight Lines

Two straight lines in three-dimensional space are **parallel** if their directional vector are a multiple of each other.

Consider, for example, the lines L_1 and L_2 with the following parametric equation:

$$L_1 : \begin{cases} x = 1 - \frac{5}{3}t \\ 2y = 3 + 5t \\ z = -1 - 2t \end{cases} \quad L_2 : \begin{cases} 3x = 1 + 4r \\ y = -2r \\ 5z = -3 + 8r \end{cases}$$

The directional vector of L_1 and L_2 is equal to $\vec{v}_1 = \left(-\frac{5}{3}, \frac{5}{2}, -2\right)$ and $\vec{v}_2 = \left(\frac{4}{3}, -2, \frac{8}{5}\right)$, respectively. The lines L_1 and L_2 are parallel given that $\vec{v}_1 = -\frac{5}{4} \cdot \vec{v}_2$.

To determine whether two lines **intersect**, there are a couple of methods we can apply. Let us examine the various methods with an example. Consider the parametric equations of the straight lines R_1 and R_2 :

$$R_1 : \begin{cases} x = 2 + 3t \\ y = 3 - t \\ z = -1 + 2t \end{cases} \quad R_2 : \begin{cases} x = 5 - r \\ y = -3 + 2r \\ z = -19 + 6r \end{cases}$$

One method is to set the respective equations of both lines equal to each other and if we find a solution for both parameters t and r , the lines intersect. We first set the x - and y -equations equal to each other:

$$\begin{cases} x: & 2 + 3t = 5 - r \Leftrightarrow t = 1 - \frac{1}{3}r \\ y: & 3 - t = -3 + 2r \Leftrightarrow t = 6 - 2r \end{cases} \Rightarrow 1 - \frac{1}{3}r = 6 - 2r \Leftrightarrow r = 3$$

This gives us a value of the parameter t equal to $t = 0$. The lines intersect if this set of values for r and t does not produce contradictions with respect to the z -equations:

$$-1 + 2t = -19 + 6r \Leftrightarrow -1 + 2 \cdot 0 = -19 + 6 \cdot 3 \Leftrightarrow -1 = -1$$

This result means that the lines intersect. Inserting the values $t = 0$ and $r = 3$ in the equations gives us the point of intersection $\vec{p} = (2, 3, -1)$.

Suppose instead that we are given the system of Cartesian equations for the lines R_1 and R_2 :

$$R_1 : \begin{cases} x + 3y - 11 = 0 \\ 2y + z - 5 = 0 \end{cases} \quad R_2 : \begin{cases} 2x + y - 7 = 0 \\ 3y - z - 10 = 0 \end{cases}$$

The second method is similar to the first one: we find a value for the x - and y -coordinate and see whether it produces any contradictions with respect to the z -coordinate. From the first equation in both sets of equations, we obtain, for instance, a y -coordinate:

$$\begin{cases} x = -3y + 11 \\ x = -\frac{1}{2}y + \frac{7}{2} \end{cases} \Rightarrow -3y + 11 = -\frac{1}{2}y + \frac{7}{2} \Leftrightarrow y = 3$$

This gives us an x -coordinate equal to $x = 2$. Given these two values, both sets of equations produce a z -coordinate equal to $z = -1$, so that we find no contradictions and the lines R_1 and R_2 indeed intersect.

A final method is a criteria to determine whether two lines intersect and relies on the determinant. If two lines intersect, there is always a plane we can construct to which both lines belong. In a first step, we then identify two points on each line and create three directional vectors. We know that these vectors have to be linearly dependent, because only two directional vectors span a Euclidean plane. In other words, if two lines intersect, the determinant of the matrix with the three directional vectors introduced as rows (or columns) must equal zero.

Regarding line R_1 , we choose the points $\vec{p}_1 = (2, 3, -1)$ and $\vec{p}_2 = (5, 2, 1)$, while for line R_2 we

pick the points $\vec{p}_3 = (1, 5, 5)$ and $\vec{p}_4 = (0, 7, 11)$. Next, we create, for instance, the directional vectors $\vec{v}_1 = \vec{p}_2 - \vec{p}_1 = (5, 2, 1) - (2, 3, -1) = (3, -1, 2)$, $\vec{v}_2 = \vec{p}_3 - \vec{p}_1 = (1, 5, 5) - (2, 3, -1) = (-1, 2, 6)$, and $\vec{v}_3 = \vec{p}_4 - \vec{p}_1 = (0, 7, 11) - (2, 3, -1) = (-2, 4, 12)$. We now calculate the determinant of the matrix constructed with these three directional vectors:

$$\begin{vmatrix} 3 & -1 & -2 \\ -1 & 2 & 4 \\ 2 & 6 & 12 \end{vmatrix} = 0$$

This result means that the four points \vec{p}_1 , \vec{p}_2 , \vec{p}_3 , and \vec{p}_4 are co-planar, so the lines R_1 and R_2 must intersect. We can then use one of the first two methods to calculate the point of intersection.

Two lines in three-dimensional space are **perpendicular** to each other if they intersect and if the dot product of their directional vectors produces a value equal to zero.

Let us for example find the straight line S_2 that is perpendicular to S_1 and intersects with S_1 at the point $\vec{p} = (-3, 0, 5)$. The line S_1 is described by the following set of equations:

$$S_1 : \begin{cases} x + 3y + 3 = 0 \\ 7y + z - 5 = 0 \end{cases}$$

We rewrite this as:

$$y = \frac{-x - 3}{3} = \frac{-z + 5}{7}$$

A directional vector of S_1 is equal to $\vec{v}_1 = (-3, 1, -7)$. Line S_2 is perpendicular to S_1 if the dot product of their directional vectors is zero, i.e., $\vec{v}_1 \cdot \vec{v}_2 = (-3, 1, -7) \cdot (v_{2x}, v_{2y}, v_{2z}) = 0$. We can, for instance, set \vec{v}_2 equal to $\vec{v}_2 = (-1, 4, 1)$. The parametric equation of S_2 is then equal to:

$$S_2 : \begin{cases} x = -3 - \lambda \\ y = 4\lambda \\ z = 5 + \lambda \end{cases}$$

Finally, lines that are not parallel and do not intersect are called **skew lines**.

Let us consider the straight lines E_1 and E_2 :

$$E_1 : \begin{cases} x - 6y + 1 = 0 \\ -y + 8z - 15 = 0 \end{cases} \quad E_2 : \begin{cases} -3x + 13y - 2 = 0 \\ 6y - 18z + 5 = 0 \end{cases}$$

We will apply the criteria based on the determinant to determine whether these lines intersect or not. With respect to line E_1 , we choose the points $\vec{p}_1 = (5, 1, 2)$ and $\vec{p}_2 = (-1, 0, \frac{15}{8})$, and regarding line E_2 , we identify the points $\vec{p}_3 = (2, \frac{8}{13}, \frac{113}{234})$ and $\vec{p}_4 = (0, \frac{2}{13}, \frac{77}{234})$.

We then create, for instance, the directional vectors $\vec{v}_1 = \vec{p}_1 - \vec{p}_2 = (5, 1, 2) - (-1, 0, \frac{15}{8}) = (6, 1, \frac{1}{8})$, $\vec{v}_2 = \vec{p}_1 - \vec{p}_3 = (5, 1, 2) - (2, \frac{8}{13}, \frac{113}{234}) = (3, \frac{5}{13}, \frac{355}{234})$, and $\vec{v}_3 = \vec{p}_1 - \vec{p}_4 = (5, 1, 2) - (0, \frac{2}{13}, \frac{77}{234}) = (5, \frac{11}{13}, \frac{391}{234})$.

The determinant of the matrix composed of the directional vectors as row vectors is calculated to be:

$$\begin{vmatrix} 6 & 1 & \frac{1}{8} \\ 3 & \frac{5}{13} & \frac{355}{234} \\ 5 & \frac{11}{13} & \frac{391}{234} \end{vmatrix} = -\frac{140}{117}$$

Given that the determinant is not equal to zero, the four points \vec{p}_1 , \vec{p}_2 , \vec{p}_3 , and \vec{p}_4 are not co-planar so that the lines E_1 and E_2 do not intersect.

However, they can still be parallel. Let us rewrite the equations of E_1 and E_2 as follows:

$$E_1 : y = \frac{x+1}{6} = 8z - 15 \qquad E_2 : y = \frac{3x+2}{13} = \frac{18z-5}{6}$$

A directional vector of E_1 and E_2 is, for instance, $\vec{v}_1 = (6, 1, \frac{1}{8})$ and $\vec{v}_2 = (\frac{13}{3}, 1, \frac{1}{3})$, respectively. Since both vectors are not a multiple of each other, the lines E_1 and E_2 are not parallel. We can thus conclude that the lines are skew.

6.2.2 Planes

Two planes $\alpha : a_1x + a_2y + a_3z + d_1 = 0$ and $\beta : b_1x + b_2y + b_3z + d_2 = 0$ are **parallel** if any two respective normal vectors \vec{n}_α and \vec{n}_β are parallel. In other words, α and β are parallel if \vec{n}_α and \vec{n}_β are a multiple of each other.

As discussed in section 6.1, the coefficients of the variables x , y , and z in the equation of a plane make up the coordinates of a normal vector. As such, we can write that $\vec{n}_\alpha = (a_1, a_2, a_3)$ and $\vec{n}_\beta = (b_1, b_2, b_3)$.

For example, consider the planes $\alpha : -x + 3y - 5z + 3 = 0$ and $\beta : 3x - 9y + 15z - 11 = 0$. We choose two normal vectors $\vec{n}_\alpha = (-1, 3, -5)$ and $\vec{n}_\beta = (3, -9, 15)$. The planes α and β are parallel, given that $(-3) \cdot \vec{n}_\alpha = \vec{n}_\beta$.

Two planes α and β **intersect** if they are not parallel. This means that α and β intersect if any two respective normal vectors \vec{n}_α and \vec{n}_β are not a multiple of each other.

For example, the planes $\alpha : x + 2y - 3z + 29 = 0$ and $\beta : -3x + 2y + z - 29 = 0$ intersect, since $\vec{n}_\alpha = (1, 2, -3)$ and $\vec{n}_\beta = (-3, 2, 1)$ are not a multiple of each other.

Two planes α and β are **perpendicular** if any two respective normal vectors \vec{n}_α and \vec{n}_β are perpendicular, i.e., their dot product must equal zero.

For instance, the planes $\alpha : 3x - y + 5z + 5 = 0$ and $\beta : x - 2y - z - 26 = 0$ are perpendicularly oriented, because $\vec{n}_\alpha \cdot \vec{n}_\beta = (3, -1, 5) \cdot (1, -2, -1) = 3 \cdot 1 + (-1) \cdot (-2) + 5 \cdot (-1) = 0$.

A straight line L is **parallel** to a plane $\alpha : a_1x + a_2y + a_3z + d_1 = 0$ if any directional vector $\vec{v} = (v_x, v_y, v_z)$ of L is perpendicular to any normal vector $\vec{n}_\alpha = (a_1, a_2, a_3)$ of α . In other words, L is parallel to α if the dot product of \vec{v} and \vec{n}_α equals zero, i.e., $\vec{v} \cdot \vec{n}_\alpha = v_x a_1 + v_y a_2 + v_z a_3 = 0$.

For example, consider the line L described by the below set of Cartesian equations and the plane $\alpha : x - 3y - z + 34 = 0$.

$$L : \begin{cases} x + 3y + 1 = 0 \\ 6y + z - 14 = 0 \end{cases}$$

Let us first rewrite the above set of equations as follows:

$$y = \frac{-x - 1}{3} = \frac{-z + 14}{6}$$

A directional vector of L is then equal to $\vec{v} = (-3, 1, -6)$. We find that the dot product between \vec{v} and \vec{n}_α is equal to $\vec{v} \cdot \vec{n}_\alpha = (-3, 1, -6) \cdot (1, -3, -1) = -3 - 3 + 6 = 0$. The line L and the plane α are thus parallel to each other.

A straight line L is **perpendicular** to a plane α if any directional vector \vec{v} of L is parallel to any normal vector \vec{n}_α of α . This means that L and α are perpendicular if \vec{v} and \vec{n}_α are a multiple of each other.

Consider, for instance, the line R with the below system of Cartesian equations and the plane $\alpha : 12x + 14y - 10z + 1 = 0$.

$$R : \begin{cases} 7x - 6y + 1 = 0 \\ 5x + 6z - 1 = 0 \end{cases}$$

We rewrite this set of equations in the following way:

$$x = \frac{6y - 1}{7} = \frac{-6z + 1}{5}$$

A directional vector of R is equal to $\vec{v} = (1, \frac{7}{6}, -\frac{5}{6})$ and a normal vector of α is $\vec{n}_\alpha = (12, 14, -10)$. Given that $\vec{n}_\alpha = 12 \cdot \vec{v}$, we know that the line R and the plane α are oriented perpendicularly.

6.2.3 Angles

The **angle between two non-parallel lines** can be found through the definition of the dot product (see section 4.1.2) of the two respective directional vectors of the straight lines. The angle is always an acute or a right angle.

For example, consider the lines L_1 and L_2 with the following parametric equations:

$$L_1 : \begin{cases} x = 1 + 8t \\ y = 4 - 11t \\ z = -6 + 9t \end{cases} \quad L_2 : \begin{cases} x = -7 - 15r \\ y = 21 + r \\ z = -1 + 16r \end{cases}$$

Given the directional vectors $\vec{v}_1 = (8, -11, 9)$ and $\vec{v}_2 = (-15, 1, 16)$ of L_1 and L_2 , respectively, we find the angle θ between the two lines as follows:

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos \theta \Leftrightarrow \theta = \arccos \left(\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} \right) = \arccos \left(\frac{13}{\sqrt{266} \cdot \sqrt{482}} \right) = 87.9^\circ$$

The **angle between two intersecting planes** is equal to the angle between two respective normal vectors, whereby the angle is either acute or right.

Consider, for instance, the planes $\alpha_1 : 2x - 7y + 8z + 4 = 0$ and $\alpha_2 : x + y - 14z - 4 = 0$. We choose a normal vector $\vec{n}_1 = (2, -7, 8)$ and $\vec{n}_2 = (1, 1, -14)$ for the planes α_1 and α_2 , respectively. The angle θ between the two planes is equal to:

$$\theta = \arccos \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} \right) = \arccos \left(-\frac{117}{\sqrt{117} \cdot \sqrt{198}} \right) = 140.2^\circ$$

However, given that the angle must be acute or right, the correct angle θ is equal to $\theta = 180^\circ - 140^\circ = 39.8^\circ$.

The **angle between an intersecting line and plane** is equal to the complement of the angle between a directional vector of the line and a normal vector of the plane. Bear in mind that we always have to make sure that the angles under consideration are acute or right.

Consider, for example, the line E with the below set of Cartesian equations that intersects the plane $\beta : x - y + z + 5 = 0$.

$$E : \frac{x - 5}{2} = \frac{y + 1}{3} = \frac{z + 6}{-5}$$

A directional vector of E is $\vec{v} = (2, 3, -5)$ and a normal vector of β is $\vec{n} = (1, -1, 1)$. The angle θ_1 between \vec{v} and \vec{n} is calculated to be:

$$\theta_1 = \arccos \left(\frac{\vec{v} \cdot \vec{n}}{\|\vec{v}\| \|\vec{n}\|} \right) = \arccos \left(-\frac{6}{\sqrt{38} \cdot \sqrt{3}} \right) = 124.2^\circ$$

This angle is not acute, so the adjusted angle θ_1 is equal to $\theta_1 = 180^\circ - 124.2^\circ = 55.8^\circ$. The angle θ_2 between the line E and the plane β is then equal to $\theta_2 = 90^\circ - 55.8^\circ = 34.2^\circ$.

6.3 The Cross Product

6.3.1 Definition

An orthonormal (Cartesian) coordinate system is an example of a **right-handed or positively oriented coordinate system**, whereby the three coordinate axes x , y , and z are perpendicular to one another and positioned according to the right-hand rule.

The right-hand rule states that when the thumb, the index finger, and the middle finger of the right hand are oriented perpendicular with respect to each other, the thumb points in the positive x -direction, the index finger in the positive y -direction, and the middle finger in the positive z -direction.

The vector \vec{c}_p is defined as the **cross product** of the vectors \vec{a} and \vec{b} , which is denoted by $\vec{c}_p = \vec{a} \times \vec{b}$, whereby:

1. the vector \vec{c}_p is perpendicular to both \vec{a} and \vec{b} ; put differently, the vector \vec{c}_p is a normal vector of the plane spanned by the vectors \vec{a} and \vec{b}
2. the vector \vec{c}_p is oriented in such a way so that the set $(\vec{c}_p, \vec{a}, \vec{b})$ forms a right-handed system
3. the norm of the vector \vec{c}_p is equal to the area of the parallelogram spanned by the vectors \vec{a} and \vec{b}

Fig. 6.3 tells us that the area A of the parallelogram spanned by the vectors \vec{a} and \vec{b} is equal to $A = h \cdot \|\vec{b}\|$. Based on right-angle trigonometry, we also see that $h = \|\vec{a}\| \sin(\pi - \theta)$. Moreover, as we have seen in section 5.2.1, the trigonometric number $\sin(\pi - \theta)$ is equal to $\sin \theta$. Therefore, we can write the area A of the parallelogram as $A = \|\vec{a}\| \|\vec{b}\| \sin \theta$.

Since the norm of the cross product $\vec{c}_p = \vec{a} \times \vec{b}$ is equal to the area A , we find that $\|\vec{c}_p\| = \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$.

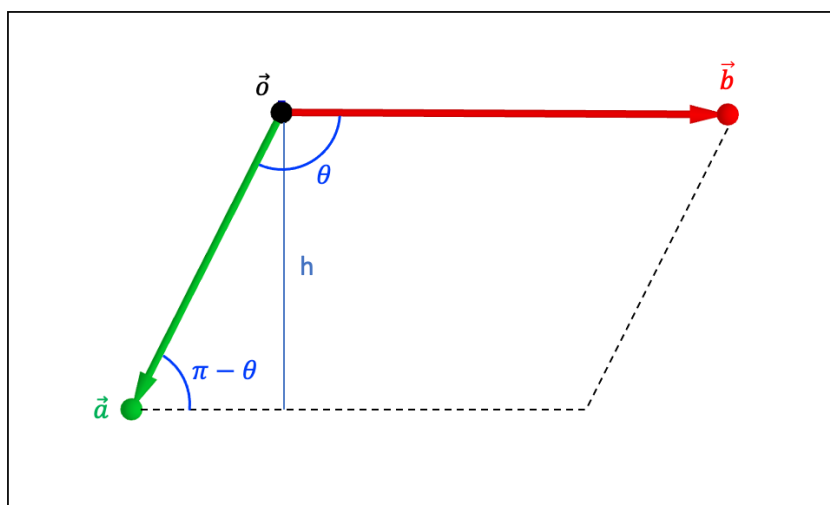


Figure 6.3: The norm of a cross product

6.3.2 Properties and Applications

Fig. 6.4 provides some of the main properties of the cross product. Note that $k \in \mathbb{R}$.

$\vec{a} \times \vec{b} = \vec{o} \Leftrightarrow \vec{a} \parallel \vec{b}$	$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$	$(\vec{b} \times \vec{c}) \times \vec{a} = \vec{c}(\vec{a} \cdot \vec{b}) - \vec{b}(\vec{a} \cdot \vec{c})$
$(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$	$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2(\vec{b} \times \vec{a})$
$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$	$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$
$(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$	$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

Figure 6.4: Properties of the cross product

Consider a Cartesian coordinate system with \vec{i}_x , \vec{i}_y , and \vec{i}_z the unit vector of the respect coordinate axis. We then find the following results for the cross product of the different combinations between the unit vectors:

$$\begin{array}{lll}
 \bullet \vec{i}_x \times \vec{i}_x = \vec{o} & \bullet \vec{i}_x \times \vec{i}_y = \vec{i}_z & \bullet \vec{i}_x \times \vec{i}_z = -\vec{i}_y \\
 \bullet \vec{i}_y \times \vec{i}_y = \vec{o} & \bullet \vec{i}_y \times \vec{i}_z = \vec{i}_x & \bullet \vec{i}_y \times \vec{i}_x = -\vec{i}_z \\
 \bullet \vec{i}_z \times \vec{i}_z = \vec{o} & \bullet \vec{i}_z \times \vec{i}_x = \vec{i}_y & \bullet \vec{i}_z \times \vec{i}_y = -\vec{i}_x
 \end{array}$$

If we write the vectors \vec{a} and \vec{b} in terms of their coordinates, i.e., $\vec{a} = a_1\vec{i}_x + a_2\vec{i}_y + a_3\vec{i}_z$ and $\vec{b} = b_1\vec{i}_x + b_2\vec{i}_y + b_3\vec{i}_z$, the cross product obtains the following form:

$$\begin{aligned}
 \vec{a} \times \vec{b} &= (a_1\vec{i}_x + a_2\vec{i}_y + a_3\vec{i}_z) \times (b_1\vec{i}_x + b_2\vec{i}_y + b_3\vec{i}_z) \\
 &= a_1\vec{i}_x \times b_1\vec{i}_x + a_1\vec{i}_x \times b_2\vec{i}_y + a_1\vec{i}_x \times b_3\vec{i}_z + a_2\vec{i}_y \times b_1\vec{i}_x + a_2\vec{i}_y \times b_2\vec{i}_y + a_2\vec{i}_y \times b_3\vec{i}_z \\
 &\quad + a_3\vec{i}_z \times b_1\vec{i}_x + a_3\vec{i}_z \times b_2\vec{i}_y + a_3\vec{i}_z \times b_3\vec{i}_z \\
 &= (a_2\vec{i}_y \times b_3\vec{i}_z - b_2\vec{i}_y \times a_3\vec{i}_z) + (a_1\vec{i}_x \times b_3\vec{i}_z - b_1\vec{i}_x \times a_3\vec{i}_z) + (a_1\vec{i}_x \times b_2\vec{i}_y - b_1\vec{i}_x \times a_2\vec{i}_y) \\
 &= (a_2b_3 - b_2a_3) (\vec{i}_y \times \vec{i}_z) + (a_1b_3 - b_1a_3) (\vec{i}_x \times \vec{i}_z) + (a_1b_2 - b_1a_2) (\vec{i}_x \times \vec{i}_y) \\
 &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i}_x - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{i}_y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{i}_z
 \end{aligned}$$

$$= \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

In other words, the cross product of \vec{a} and \vec{b} is found by calculating the determinant of the above matrix, so that we can write $\vec{a} \times \vec{b} = (a_2b_3 - b_2a_3, b_1a_3 - a_1b_3, a_1b_2 - b_1a_2)$.

If we want to calculate the dot product $(\vec{a} \times \vec{b}) \cdot \vec{c}$, we find the following result:

$$\begin{aligned} (\vec{a} \times \vec{b}) \cdot \vec{c} &= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (c_1, c_2, c_3) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned}$$

The expression $(\vec{a} \times \vec{b}) \cdot \vec{c}$ can be interpreted geometrically. The volume V of the parallelepiped shown in Fig. 6.5 is equal to $V = \|\vec{a} \times \vec{b}\| \cdot h$. Based on right-triangle trigonometry, we see that the height h is equal to $h = \|\vec{c}\| \cos \theta$. The volume V then becomes: $V = \|\vec{a} \times \vec{b}\| \|\vec{c}\| \cos \theta$, which is the definition of the dot product between the vectors $\vec{a} \times \vec{b}$ and \vec{c} .

In other words, the absolute value of the expression $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is equal to the volume of the parallelepiped spanned by the non co-planar vectors \vec{a} , \vec{b} , and \vec{c} : $V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$.

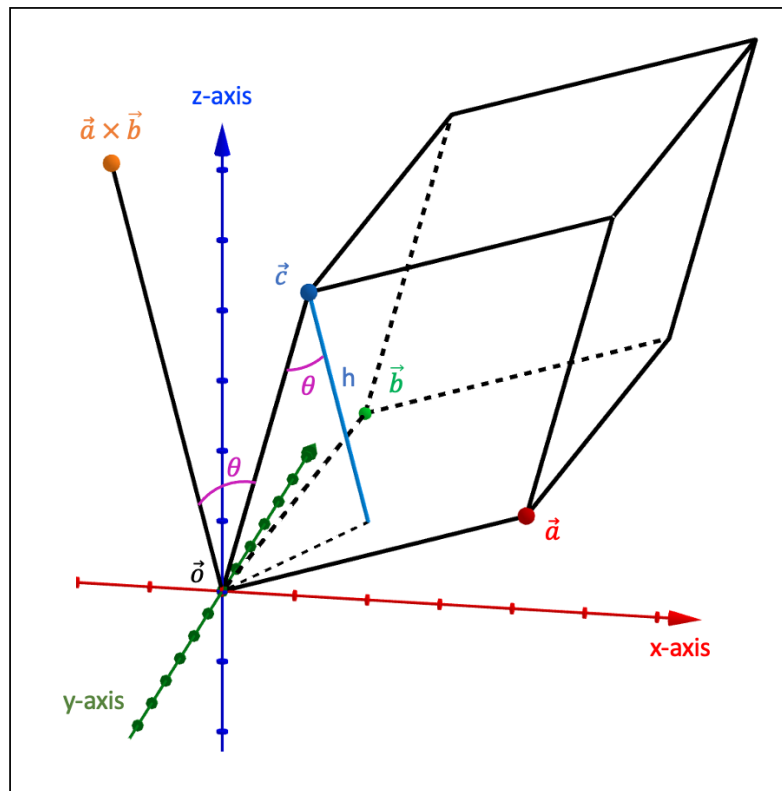


Figure 6.5: The geometric interpretation of $(\vec{a} \times \vec{b}) \cdot \vec{c}$

The dot product and the cross product can also be expressed with the assistance of the **Kronecker delta** and the **Levi-Civita symbol**, respectively. They are defined as follows:

$$\text{Kronecker delta symbol } \delta_{ij} = \begin{cases} 1 & \Leftrightarrow i = j \\ 0 & \Leftrightarrow i \neq j \end{cases}$$

$$\text{Levi-Civita symbol } \epsilon_{ijk} = \begin{cases} 1 & \text{for an even number of permutations of the indices} \\ -1 & \text{for an odd number of permutations of the indices} \\ 0 & \text{if at least two indices are equal} \end{cases}$$

whereby $i, j, k \in \{1, 2, 3\}$ and we define $\epsilon_{123} = 1$. For example, $\delta_{23} = 0$, $\delta_{11} = 1$, $\epsilon_{122} = 0$, $\epsilon_{312} = 1$, and $\epsilon_{213} = -1$.

If \vec{e}_1 , \vec{e}_2 , and \vec{e}_3 represent the three unit vectors of an orthonormal coordinate system, we can write their dot product and cross product as follows:

$$\begin{cases} \vec{e}_i \cdot \vec{e}_j = \delta_{ij} \\ \vec{e}_i \times \vec{e}_j = \sum_{k=1}^3 \epsilon_{ijk} \vec{e}_k \end{cases}$$

For instance, $\vec{e}_2 \cdot \vec{e}_1 = \delta_{21} = 0$ and $\vec{e}_3 \times \vec{e}_1 = \epsilon_{311}\vec{e}_1 + \epsilon_{312}\vec{e}_2 + \epsilon_{313}\vec{e}_3 = 0 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2 + 0 \cdot \vec{e}_3 = \vec{e}_2$.

The dot product and the cross product between two vectors $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$ and $\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3$ can be written in the following way:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \left(\sum_{i=1}^3 a_i \vec{e}_i \right) \cdot \left(\sum_{j=1}^3 b_j \vec{e}_j \right) & \vec{a} \times \vec{b} &= \left(\sum_{i=1}^3 a_i \vec{e}_i \right) \times \left(\sum_{j=1}^3 b_j \vec{e}_j \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 a_i b_j (\vec{e}_i \cdot \vec{e}_j) & &= \sum_{i=1}^3 \sum_{j=1}^3 a_i b_j (\vec{e}_i \times \vec{e}_j) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 a_i b_j \delta_{ij} & &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} a_i b_j \vec{e}_k \end{aligned}$$

6.4 Distances

6.4.1 Distance between a Point and a Straight Line

The **distance between a point \vec{p} and a straight line L** in three-dimensional space can be found as follows. We first construct the plane α to which both the point \vec{p} and the line L belong. Next, we identify the directional vector \vec{v}_\perp that is perpendicular both to the normal \vec{n} of the plane and to a directional vector \vec{v} of the line. We then write down the set of equations that describes the line L_\perp , which has the vector \vec{v}_\perp as directional vector and goes

through point \vec{p} . In a next step, we determine the point of intersection \vec{s} between L and L_{\perp} . Finally, we calculate the distance between \vec{p} and \vec{s} .

Let us consider an example. We wish to calculate the distance between the point $\vec{p} = (-1, 5, -7)$ and the straight line L with the following Cartesian equations:

$$L : \frac{x-2}{-1} = \frac{y+1}{-3} = \frac{z-9}{4}$$

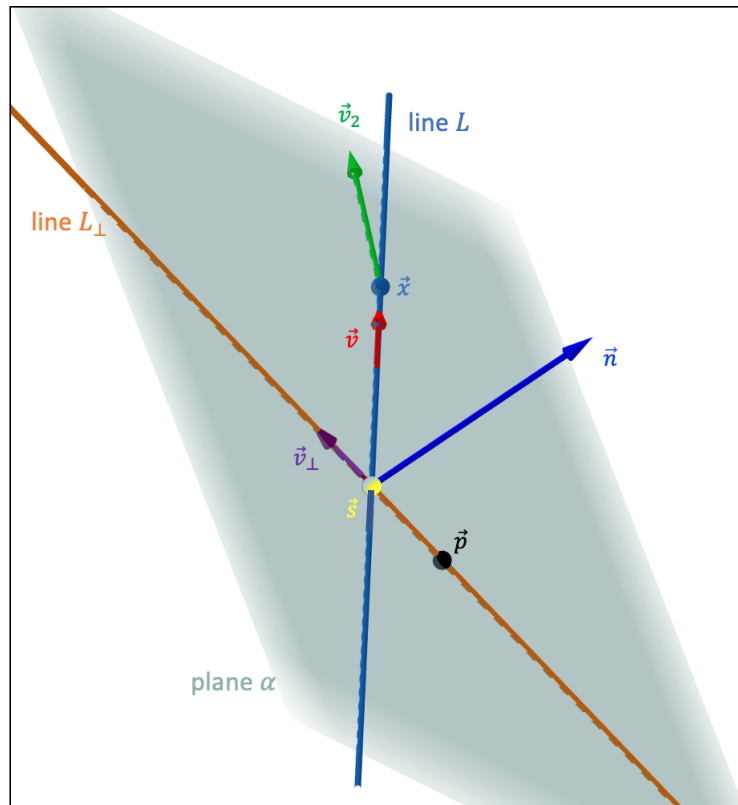


Figure 6.6: Calculating the distance from \vec{p} to line L

A directional vector of L is $\vec{v} = (-1, -3, 4)$. To construct the plane α , we need a second directional vector \vec{v}_2 that is linearly independent from \vec{v} . If we consider a random point $\vec{x} \in L$ with coordinates $\vec{x} = (0, -7, 17)$, we calculate \vec{v}_2 as $\vec{v}_2 = \vec{x} - \vec{p} = (0, -7, 17) - (-1, 5, -7) = (1, -12, 24)$. We can now construct the plane α :

$$\alpha : \begin{vmatrix} x+1 & y-5 & z+7 \\ -1 & -3 & 4 \\ 1 & -12 & 24 \end{vmatrix} = 0 \Leftrightarrow -24x + 28y + 15z - 59 = 0$$

A normal vector \vec{n} of α is equal to $\vec{n} = (-24, 28, 15)$. In a next step, we identify the directional vector \vec{v}_{\perp} . Based on the definition of the cross product, we find \vec{v}_{\perp} as $\vec{v}_{\perp} = \vec{n} \times \vec{v} = (-24, 28, 15) \times (-1, -3, 4) = (157, 81, 100)$. The line L_{\perp} is then described by the following set of Cartesian equations:

$$L_{\perp} : \frac{x+1}{157} = \frac{y-5}{81} = \frac{z+7}{100}$$

The point of intersection \vec{s} between L and L_{\perp} is equal to $\vec{s} = (\frac{131}{26}, \frac{211}{26}, -\frac{41}{13})$. The distance d between \vec{s} and \vec{p} is then calculated to be:

$$\begin{aligned} d &= \sqrt{(s_x - p_x)^2 + (s_y - p_y)^2 + (s_z - p_z)^2} \\ &= \sqrt{\left(\frac{131}{26} - (-1)\right)^2 + \left(\frac{211}{26} - 5\right)^2 + \left(-\frac{41}{13} - (-7)\right)^2} = \sqrt{\frac{1585}{26}} \end{aligned}$$

An **alternative and shorter way** to find the distance between a point and a straight line is based on the geometric interpretation of the norm of a cross product.

From Fig. 6.7 we can see that the vector $\vec{v} = \vec{b} - \vec{a} = \vec{c} - \vec{p}$, which is a directional vector of line L , and that $\vec{w} = \vec{a} - \vec{p} = \vec{b} - \vec{c}$. The area A of the parallelogram spanned by the vectors \vec{v} and \vec{w} is equal to $A = \|\vec{v} \times \vec{w}\|$.

The area A can also be calculated when taking the product of the length of the directional vector \vec{v} and the distance d , i.e., $A = \|\vec{v}\| \cdot d$.

Setting these two expressions for the area of A equal to each other, we find a new expression that allows us to calculate the distance d between a given point \vec{p} and the straight line L :

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot d \Leftrightarrow d = \frac{\|\vec{v} \times \vec{w}\|}{\|\vec{v}\|}$$

If we apply this to our previous example, whereby $\vec{a} = (4, 5, 1)$, $\vec{b} = (3, 2, 5)$, and $\vec{c} = (-2, 2, -3)$, $\vec{v} = \vec{b} - \vec{a} = (3, 2, 5) - (4, 5, 1) = (-1, -3, 4)$, and $\vec{w} = \vec{a} - \vec{p} = (4, 5, 1) - (-1, 5, -7) = (5, 0, 8)$, we find the distance d as follows:

$$d = \frac{\|\vec{v} \times \vec{w}\|}{\|\vec{v}\|} = \frac{\|(-1, -3, 4) \times (5, 0, 8)\|}{\|(-1, -3, 4)\|} = \frac{\|(-24, 28, 15)\|}{\|(-1, -3, 4)\|} = \sqrt{\frac{1585}{26}}$$

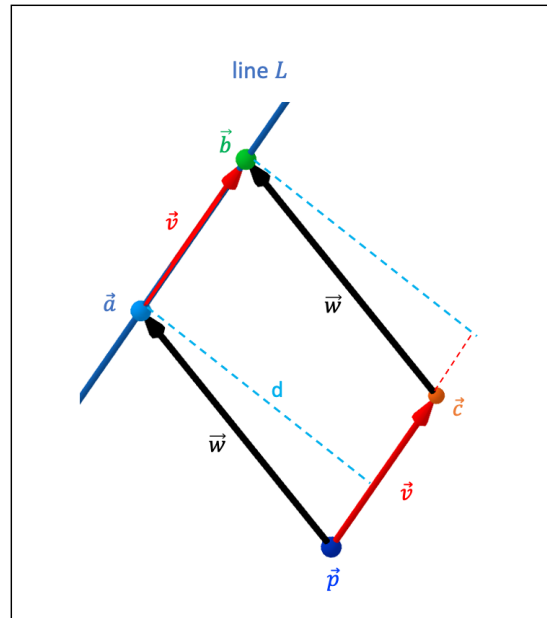


Figure 6.7: Calculating the distance from \vec{p} to line L

6.4.2 Distance between Two Skew Lines

The **distance between two skew lines** S_1 and S_2 is equal to the distance between the two points of intersection \vec{p}_1 and \vec{p}_2 that are found when the two skew lines intersect with their common perpendicular L_\perp , i.e., the line that is perpendicular to both skew lines.

We apply the following method. In a first instance, we identify a directional vector \vec{v}_\perp of L_\perp by looking for a vector that is perpendicular to both the directional vector \vec{v}_1 of S_1 and the directional vector \vec{v}_2 of S_2 . Next, we construct the plane α that contains both S_1 and L_\perp . In a next step, we identify the point of intersection \vec{p}_2 between α and S_2 . After writing down the set of equations that describes L_\perp , we can find the point of intersection \vec{p}_1 between the lines S_1 and L_\perp . In a final step, we calculate the distance between \vec{p}_1 and \vec{p}_2 .

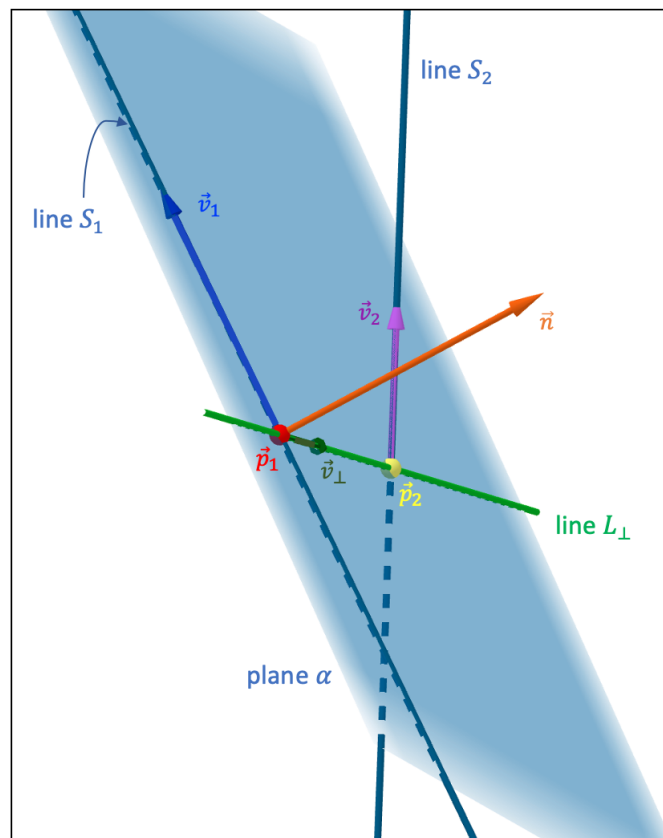


Figure 6.8: Calculating the distance between two skew lines S_1 and S_2

For example, let us calculate the distance between the skew lines S_1 and S_2 as given by the following parametric equations (see Fig. 6.8):

$$S_1 : \begin{cases} x = -2 + 9t \\ y = 6 - t \\ z = 11 + t \end{cases} \quad S_2 : \begin{cases} x = 3 + 3r \\ y = -2 - 2r \\ z = 1 + r \end{cases}$$

A directional vector of the lines S_1 and S_2 is equal to $\vec{v}_1 = (9, -1, 1)$ and $\vec{v}_2 = (3, -2, 1)$, respectively. The directional vector \vec{v}_\perp is found by taking the cross product of \vec{v}_1 and \vec{v}_2 : $\vec{v}_\perp = \vec{v}_1 \times \vec{v}_2 = (9, -1, 1) \times (3, -2, 1) = (1, -6, -15)$.

To construct the plane α , we first identify a normal vector \vec{n} of α by taking the cross product of \vec{v}_1 and \vec{v}_\perp , which span the plane α : $\vec{n} = \vec{v}_\perp \times \vec{v}_1 = (1, -6, -15) \times (9, -1, 1) = (-21, -136, 53)$. The plane α has then the form $\alpha : -21x - 136y + 53z + d = 0$. Given that S_1 lies within α , we find d by inserting, for instance, the point $(-2, 6, 11) \in S_1$ into the equation of α :

$$\alpha : -21x - 136y + 53z + d = 0 \Leftrightarrow -21 \cdot (-2) - 136 \cdot 6 + 53 \cdot 11 + d = 0 \Leftrightarrow d = 191$$

The equation of α is then $\alpha : -21x - 136y + 53z + 191 = 0$. Inserting the parametric equations of S_2 into the equation of α allows us to find the value of the parameter r that corresponds to the point of intersection \vec{p}_2 :

$$\begin{aligned} \alpha : -21x - 136y + 53z + 191 &= 0 \\ \Leftrightarrow -21(3 + 3r) - 136(-2 - 2r) + 53(1 + r) + 191 &= 0 \\ \Leftrightarrow r &= -\frac{453}{262} \\ \Rightarrow \vec{p}_2 &= \left(-\frac{573}{262}, \frac{191}{131}, -\frac{191}{262} \right) \end{aligned}$$

Given the directional vector $\vec{v}_\perp = (1, -6, -15)$ and the point \vec{p}_2 , we can write down the equation of the line L_\perp :

$$L_\perp : \begin{cases} x = -\frac{573}{262} + s \\ y = \frac{191}{131} - 6s \\ z = -\frac{191}{262} - 15s \end{cases}$$

The intersection between S_1 and L_\perp produces the parametric values $s = -\frac{203}{262}$ and $t = -\frac{14}{131}$, which correspond to the point of intersection $\vec{p}_1 = \left(-\frac{388}{131}, \frac{800}{131}, \frac{1427}{131} \right)$. Finally, the distance between \vec{p}_1 and \vec{p}_2 is equal to:

$$\begin{aligned} d &= \sqrt{(p_{1x} - p_{2x})^2 + (p_{1y} - p_{2y})^2 + (p_{1z} - p_{2z})^2} \\ &= \sqrt{\left[-\frac{388}{131} - \left(-\frac{573}{262} \right) \right]^2 + \left[\frac{800}{131} - \frac{191}{131} \right]^2 + \left[\frac{1427}{131} - \left(-\frac{191}{262} \right) \right]^2} = \sqrt{\frac{41209}{262}} \end{aligned}$$

6.4.3 Distance between a Point and a Plane

Consider a plane $\alpha : ax + by + cz + d_* = 0$, a point $\vec{p} = (p_x, p_y, p_z) \notin \alpha$ and a point $\vec{x} = (x, y, z) \in \alpha$, whereby \vec{x} is the orthogonal projection of \vec{p} onto the plane α and $\vec{n} = (a, b, c)$ is a normal vector of α . We wish to determine the **distance d between the point \vec{p} and the plane α** , which is equivalent to the distance between \vec{p} and \vec{x} , i.e., $d = \|\vec{p} - \vec{x}\|$.

Given that d is always defined as the shortest distance from \vec{p} to the plane α , we know that the line segment between \vec{p} and α must be parallel to any normal \vec{n} of the plane α . In other words, the vector $\vec{p} - \vec{x}$ must be parallel to \vec{n} , which means that the angle θ between $\vec{p} - \vec{x}$ and \vec{n} must be $\theta = 0^\circ$.

Based on the definition of the dot product, we can therefore write:

$$\begin{aligned}(\vec{p} - \vec{x}) \cdot \vec{n} &= \|\vec{p} - \vec{x}\| \|\vec{n}\| \cos \theta \\ &= \|\vec{p} - \vec{x}\| \|\vec{n}\| \cos(0^\circ) \\ &= \|\vec{p} - \vec{x}\| \|\vec{n}\|\end{aligned}$$

$$\begin{aligned}\Leftrightarrow d = \|\vec{p} - \vec{x}\| &= \frac{(\vec{p} - \vec{x}) \cdot \vec{n}}{\|\vec{n}\|} \\ &= \frac{[(p_x, p_y, p_z) - (x, y, z)] \cdot (a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{(ap_x + bp_y + cp_z) - (ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ap_x + bp_y + cp_z) + d_*|}{\sqrt{a^2 + b^2 + c^2}}\end{aligned}$$

In the above last step, we have introduced an absolute value sign, given that a distance must always be positive.

Let us, for example, consider the plane $\alpha : -14x + 21y - 3z + 44 = 0$ and the point $\vec{p} = (11, -12, 4)$. The distance d between \vec{p} and α is then calculated as follows:

$$\begin{aligned}d &= \frac{|(ap_x + bp_y + cp_z) + d_*|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|((-14) \cdot 11 + 21 \cdot (-12) + (-3) \cdot 4) + 44|}{\sqrt{(-14)^2 + 21^2 + (-3)^2}} \\ &= \frac{|-374|}{\sqrt{646}} = \frac{374}{\sqrt{646}}\end{aligned}$$

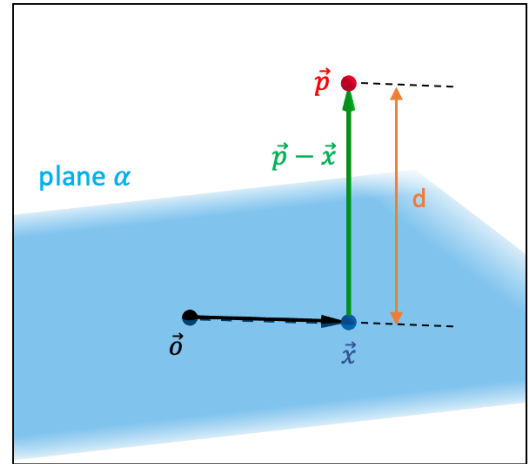


Figure 6.9: Distance between \vec{p} and α

6.4.4 Distance between Two Parallel Planes

To find the **distance between two parallel planes α and β** , it suffices to identify one point $\vec{p} \in \alpha$ ($\vec{p} \in \beta$) and then use the formula of the previous section 6.4.3 to calculate the distance from \vec{p} to the plane β (plane α).

Consider the planes $\alpha : 2x + 5y - 31z - 11 = 0$ and $\beta : -6x - 15y + 93z + 43 = 0$. If we for instance select the point $\vec{p} = (-1, \frac{47}{3}, 2) \in \beta$, we then find the distance between the two planes by calculating the distance between \vec{p} and the plane α :

$$\begin{aligned}d &= \frac{|(ap_x + bp_y + cp_z) + d|}{\sqrt{a^2 + b^2 + c^2}} \\&= \frac{\left| (2 \cdot (-1) + 5 \cdot \frac{47}{3} + (-31) \cdot 2) - 11 \right|}{\sqrt{2^2 + 5^2 + (-31)^2}} \\&= \frac{\frac{10}{3}}{\sqrt{990}} = \frac{\sqrt{110}}{99}\end{aligned}$$

7 Real Analysis

Within the field of mathematics, the subfield of **real analysis** deals with the study of real numbers, real functions, and sequences and series.

7.1 Functions: An Introduction

7.1.1 Definitions

Given two non-empty sets A and B of real numbers, a (real) **function** f describes a unique map between an element of A and an element of B , whereby each element of A can only be mapped to just one element of B .

Mathematically, this relation is written as $f : A \rightarrow B$, whereby A is called the **domain** of f and B the **codomain** of f .

If $x \in A$ and $y \in B$, f evaluated in x is equal to y , or $f(x) = y$. Therefore, the function f can also be mathematically displayed as $f : A \rightarrow B : x \mapsto f(x) = y$. The element x is referred to as the **argument or the variable** of f , whereas y represents the **value** of f .

The **image** of a function f is equal to the subset of B that consists of the elements $y \in B$ for which there exists at least one $x \in A$ whereby $y = f(x)$. So, the image of f is not necessarily equal to its codomain. For example, the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = x^2$ has the codomain $B = \mathbb{R}$ while its image is equal to \mathbb{R}^+ .

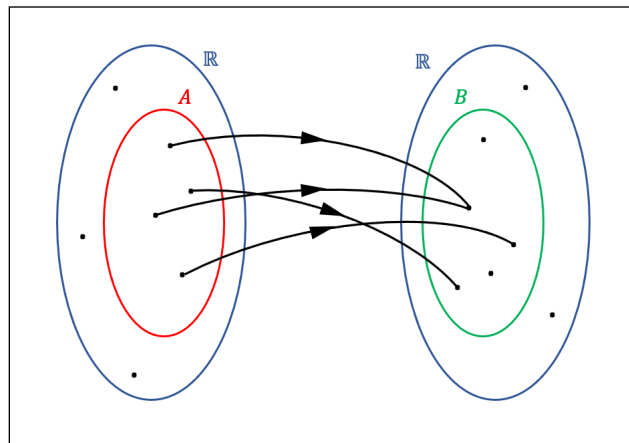


Figure 7.1: Graphic representation of a function

Note that the definition of a function allows a function to map *different* elements of A to the *same* element of B . For example, the function $f : \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto f(x) = |x|$ maps both the argument $x = -5$ and $x = 5$ to the value $y = 5$.

A counterexample is the horizontal parabola with equation $x = s \cdot (y - m_2)^2 + m_1$ (see section 4.3.4). If we rewrite this equation as $y = m_2 \pm \sqrt{\frac{x - m_1}{s}}$, we see that each argument x (except $x = m_1$) corresponds with two values y , which is in contradiction with the definition of a function. That is, a horizontal parabola is not a function.

To know whether a curve can be defined as a function, we apply the **vertical line test**. If the vertical line intersects the curve only once for every x -value, then the curve represents a function.

In the case of the horizontal parabola, the vertical line intersects the curve twice for each value of x (except $x = m_1$), so that, as we have seen, the horizontal parabola does not represent a function. Neither does a circle, for the same reasons.

A function is furthermore called **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$. Examples of even functions are $f(x) = |x|$, $f(x) = \cos x$, $f(x) = \sec x$, $f(x) = ax^2 + b$, and $f(x) = ax^2 + bx^4 + c$, whereas the functions $f(x) = \sin x$, $f(x) = \csc x$, $f(x) = \tan x$, $f(x) = ax^3 + bx$, and $f(x) = x$ are classified as odd functions.

Note that functions do not necessarily have to be even or odd. For instance, the function $f(x) = \cot x + x^2$ is neither even nor odd, because $f(-x) = -\cot x + x^2 \neq f(x)$ and $f(-x) = -\cot x + x^2 \neq -f(x) = -\cot x - x^2$.

If a function is described by two or more functions, it is called a **piecewise function**. For each individual function, a domain interval is defined, whereby the intervals cannot overlap. Consider for example the below piecewise function:

$$f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = \begin{cases} \sqrt{x-3} & \text{if } x \geq 3 \\ \frac{1}{x(x-3)} & \text{if } 0 < x < 3 \\ (x-1)\sqrt{-x} & \text{if } x \leq 0 \end{cases}$$

A function f that returns a scalar value is called a **scalar function**. A scalar function with two variables is written as $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y)$, and in the general case of n variables we write $f : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n)$.

An example of a scalar function f of three variables x , y , and z is $f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z) = 2xy^2 - 3y^3z + xyz - z$.

A function that returns a vector is known as a **vector function**. In the case of one parameter t , we can write $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^m : t \mapsto \vec{r}(t) = (f_1(t), f_2(t), \dots, f_m(t))$. The elements $f_i(t)$ are called the component functions of \vec{r} and are scalar functions. If we consider the general case whereby a vector function \vec{F} is described by n parameters, \vec{F} is defined as $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m : (t_1, t_2, \dots, t_n) \mapsto \vec{F}(t_1, t_2, \dots, t_n) = (f_1(t_1, t_2, \dots, t_n), f_2(t_1, t_2, \dots, t_n), \dots, f_m(t_1, t_2, \dots, t_n))$.

An example of a vector function with two parameters u and v and four component functions is $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^4 : (u, v) \mapsto \vec{F}(u, v) = (uv + 1, 4 - u, 4 - v, 2 + u^2v)$.

The **graph of a scalar function of one variable** is defined as the set of all the pairs $(x, f(x)) = (x, y)$ and represents a curve in the two-dimensional Euclidean space. The graph of f can also be denoted by $\{(x, f(x)) | x \in \mathbb{R}\} \subset \mathbb{R}^2$. Within a two-dimensional Euclidean plane, each point on the graph of f corresponds to one unique pair, whereby each pair represents Cartesian coordinates.

The **graph of a scalar function of two variables** x and y represents a two-dimensional surface in three-dimensional Euclidean space and can be written as $\{(x, y, f(x, y)) | (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. For more than two variables, we are unable to draw any graph. In the general case of n variables, the function f represents a n -dimensional hypersurface in the $(n + 1)$ -dimensional space.

The **graph of a vector function** with one and two parameters represents a curve and a surface in the m -dimensional space, respectively, and is denoted by $\{(f_1(t), f_2(t), \dots, f_m(t)) | t \in \mathbb{R}\} \subset \mathbb{R}^m$ and $\{(f_1(u, v), f_2(u, v), \dots, f_m(u, v)) | (u, v) \in \mathbb{R}^2\} \subset \mathbb{R}^m$, respectively. We are only able to draw a graph if $m = 2$ or $m = 3$.

As an example, Fig. 7.2 shows four graphs of different functions. Note that the domain of the vector function $\vec{F} = (12 \sin(u), -6 \cos(v), 6u^2v^2)$ is restricted to $[-1.5, 1.5] \times [0, 3]$, otherwise the equation of the surface would not represent a function (i.e., we would get different function values for the same couples of (u, v)).

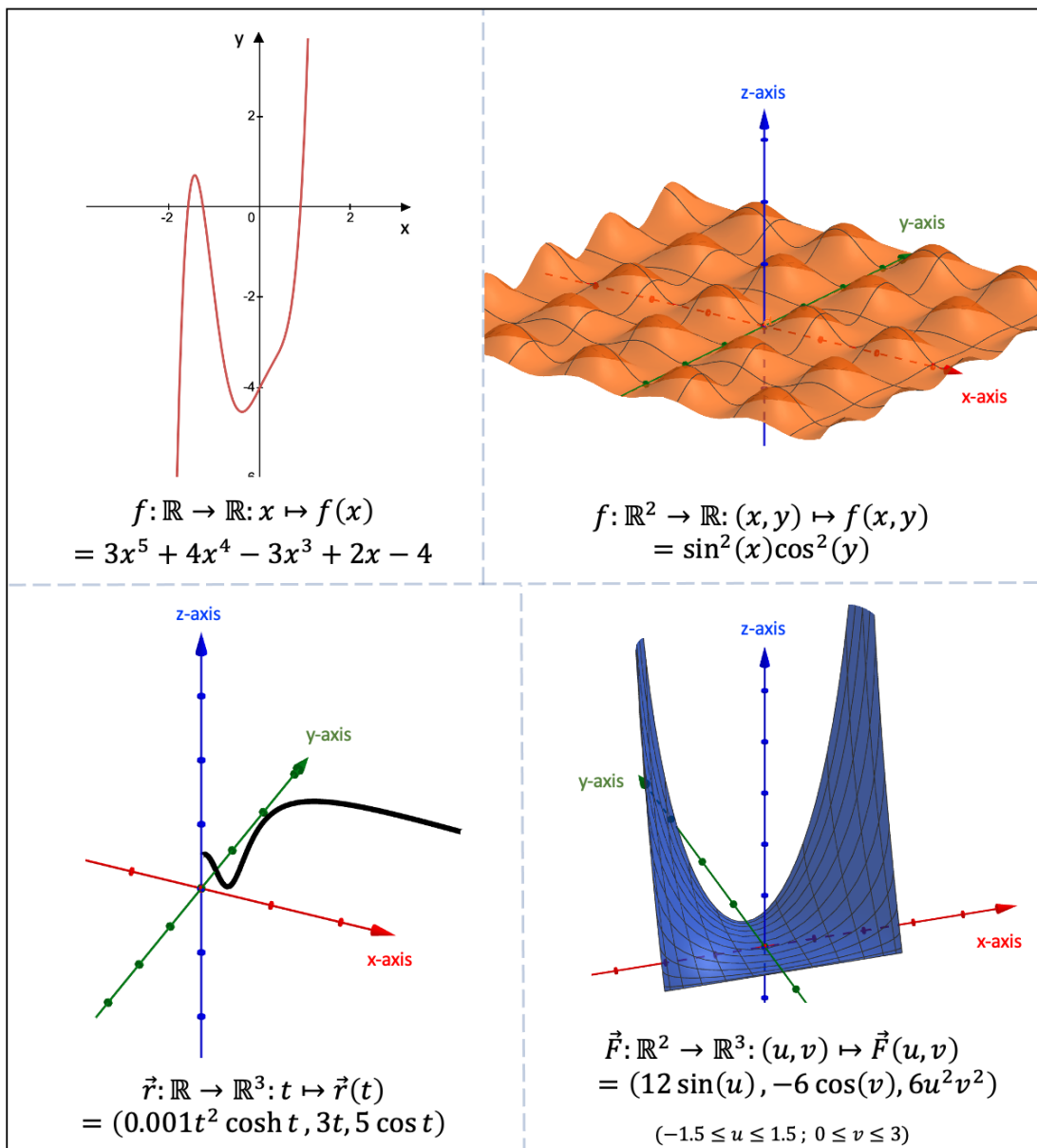


Figure 7.2: The graph of four different functions

The functions that are discussed in the remainder of section 7 are scalar functions of one variable, unless stated otherwise.

7.1.2 Roots

The **zeros or roots** of a function $f : A \rightarrow B : x \mapsto f(x)$ are equal to all the arguments $x \in A$ for which their function value is zero, i.e., $f(x) = 0$.

Graphically, a root $x = x_0$ is identified as the point of intersection between the curve of the function and the x -axis of the Cartesian coordinate system. The coordinates of the root are therefore of the form $(x_0, 0)$.

Consider, for instance, the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 2x^2 + \sqrt{17}x - 4$. The roots are found as follows:

$$\begin{aligned} f(x) = 0 &\Leftrightarrow 2x^2 + \sqrt{17}x - 4 = 0 \Leftrightarrow x = \frac{-\sqrt{17} \pm \sqrt{17 - 4 \cdot 2 \cdot (-4)}}{2 \cdot 2} \\ &= \frac{-\sqrt{17} \pm 7}{4} \end{aligned}$$

This function has thus two roots: $x = -\frac{\sqrt{17}+7}{4}$ and $x = \frac{-\sqrt{17}+7}{4}$.

A second example is the function $f : \mathbb{R} \setminus \{k\pi\} \rightarrow \mathbb{R} : \theta \mapsto f(\theta) = \cot \theta$ with $k \in \mathbb{Z}$. The roots of f are found when $f(\theta) = \cot \theta = 0$, i.e., $\theta = \frac{\pi}{2} + k\pi$.

7.1.3 Function Operations

If we consider the functions $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$, $g : A \subset \mathbb{R} \rightarrow \mathbb{R}$, and $h : A \subset \mathbb{R} \rightarrow \mathbb{R}$, whereby $\forall x \in A : h(x) \neq 0$, and the scalar $k \in \mathbb{R}$, Fig. 7.3 defines the scalar product as well as the sum, the product, and the quotient of two functions.

➤ Scalar product:	$k \cdot f : A \rightarrow \mathbb{R} : x \mapsto k \cdot f(x)$
➤ Sum:	$f + g : A \rightarrow \mathbb{R} : x \mapsto f(x) + g(x)$
➤ Product:	$f \cdot g : A \rightarrow \mathbb{R} : x \mapsto f(x) \cdot g(x)$
➤ Quotient:	$\frac{f}{h} : A \rightarrow \mathbb{R} : x \mapsto \frac{f(x)}{h(x)}$

Figure 7.3: Some basic operations of functions

For example, if we consider the functions $f : \mathbb{R} \rightarrow [2, +\infty[: x \mapsto f(x) = 5x^2 + 2$ and

$h : \mathbb{R} \setminus \{\frac{\pi}{2} + m\pi\} \rightarrow [-1, 1] : x \mapsto h(x) = \cos(x)$, with $m \in \mathbb{Z}$, we can define the quotient function $\frac{f}{h}$ as $\frac{f}{h} : \mathbb{R} \setminus \{\frac{\pi}{2} + m\pi\} \rightarrow \mathbb{R} : x \mapsto \frac{f}{h}(x) = \frac{5x^2+2}{\cos(x)}$.

The **composite function** is a function that uses another function as its argument. If $f : A \subset \mathbb{R} \rightarrow B \subset \mathbb{R}$ and $g : B \subset \mathbb{R} \rightarrow C \subset \mathbb{R}$ are two functions, we can define the composite function $g \circ f$ as $g \circ f : A \subset \mathbb{R} \rightarrow C \subset \mathbb{R} : x \mapsto (g \circ f)(x) = g(f(x))$, whereby $\forall x \in A$.

Consider, for instance, the functions $f : [-1, 1] \rightarrow [0, 1] : x \mapsto f(x) = \sqrt{1-x^2}$ and $g : [0, 1] \rightarrow [0, 1] : x \mapsto g(x) = \sqrt{1-x^2}$. The composite function $g \circ f$ is then defined as $g \circ f : [-1, 1] \rightarrow [0, 1] : x \mapsto g(f(x)) = \sqrt{1-(\sqrt{1-x^2})^2} = \sqrt{x^2} = |x|$.

If we wish to change the position of the graph of a function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x)$ within our Cartesian coordinate system, there are two basic operations we can perform.

For a **vertical translation** of the graph of f , we define a new function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = f(x) + a$, with $a \in \mathbb{R}$. If $a > 0$ ($a < 0$), the graph of f moves upwards (downwards) over a distance $|a|$.

For a **horizontal translation** of the graph of f , we define a new function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = f(x + a)$, with $a \in \mathbb{R}$. If $a > 0$ ($a < 0$), the graph of f moves to the left (right) over a distance $|a|$.

We might also be interested in rescaling the graph of the function f .

For a **vertical rescaling** of the graph of f , we define a new function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = af(x)$, with $a \in \mathbb{R}$. If $|a| > 1$, the graph of f is stretched in the direction away from the x -axis over a factor $|a|$, whereas if $|a| < 1$ the graph of f is compressed towards the x -axis. If $a < 0$, the graph of f will be reflected over the x -axis.

For a **horizontal rescaling** of the graph of f , we define a new function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = f(ax)$, with $a \in \mathbb{R}$. If $|a| > 1$, the graph of f is compressed towards the y -axis over a factor $|a|$, whereas if $|a| < 1$ the graph of f is stretched in the direction away from the y -axis. If $a < 0$, the graph of f will be reflected over the y -axis.

Fig. 7.4 demonstrates the effects of a linear rescaling of the graph of a function f . On the left, the gray dashed line represents the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = x^3 - x + 4$. The function of the red graph is equal to $g_1 : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g_1(x) = (0.5x)^3 - (0.5x) + 4$, which implies a horizontal stretch of the graph of f by a factor 0.5. The function of the green graph is equal to $g_2 : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g_2(x) = [0.5(x-4)]^3 - [0.5(x-4)] + 4$, which implies a horizontal translation to the right of the graph of g_1 over a distance of 4.

On the right, the gray dashed line represents the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = (0.5x)^4 - x^2 + 4$. The function of the red graph is equal to $g_1 : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g_1(x) = -0.5[(0.5x)^4 - x^2 + 4]$, which implies a vertical compression of the graph of f by a factor 0.5 and a reflection over the x -axis.

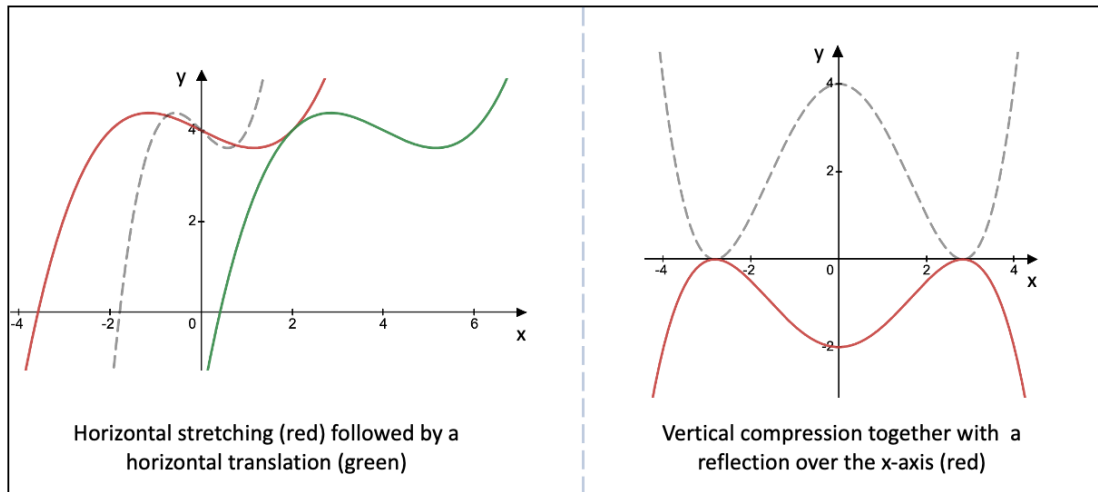


Figure 7.4: Linear rescaling of functions

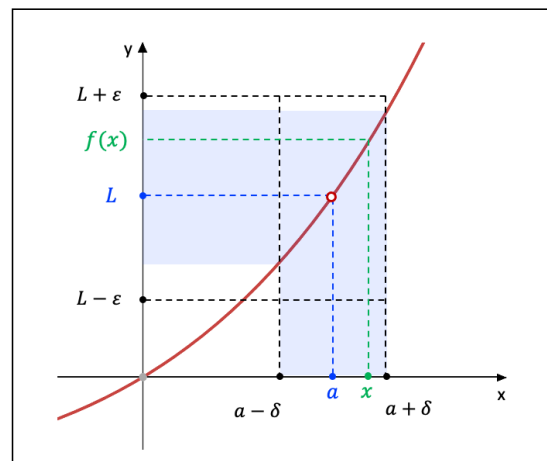
Additional examples of linear rescaling of functions is found in section 7.6.3.

7.2 Limit of Functions

7.2.1 Definitions

The **limit of a function** $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ for a number $a \in \mathbb{R}$ is equal to $L \in \mathbb{R}$, if the function value $f(x)$ moves increasingly closer to L as x moves increasingly closer to a . The limit L of a function f at an input value a is denoted by $\lim_{x \rightarrow a} f(x) = L$. Note that the function does not need to be defined in a .

We can reformulate the above definition of a limit L of a function f for an argument a as follows. If for any real number $\epsilon > 0$, which determines the interval $[L - \epsilon, L + \epsilon]$ on the y -axis, we can find a real number $\delta > 0$, which determines the interval $[a - \delta, a + \delta]$ on the x -axis, so that for any argument x belonging to the domain of f that lies at a smaller distance than δ from a (with $x \neq a$) we find that the function value $f(x)$ lies closer to L than ϵ , then we can say that the limit of f for the argument a is equal to L .



In mathematical language, this definition *Figure 7.5:* The definition of a limit of a function becomes the following:

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

The limit of a function f at a certain argument a need not take on a finite value L , but can

be infinite, either in the positive or the negative direction of the y -axis. We can define these **infinite limits** as follows:

- $\lim_{x \rightarrow a} f(x) = +\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : 0 < |x - a| < \delta \Rightarrow f(x) > \alpha$
- $\lim_{x \rightarrow a} f(x) = -\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : 0 < |x - a| < \delta \Rightarrow f(x) < \alpha$

Fig. 7.6 provides two examples of infinite limits. On the left-hand side, the graph in blue represents the function $f : \mathbb{R} \setminus \{0, 1\} \rightarrow \mathbb{R}^+ : x \mapsto f(x) = \frac{e^x}{|x^2 - x|}$. For the function f , we find that $\lim_{x \rightarrow 0} f(x) = +\infty$ and $\lim_{x \rightarrow 1} f(x) = +\infty$.

On the right-hand side, the graph in green represents the function $g : \left] \frac{\pi}{2}, \frac{3\pi}{2} \right[\rightarrow \mathbb{R} : x \mapsto g(x) = \tan(x)$. For the function g , we find that $\lim_{x \rightarrow \frac{\pi}{2}} g(x) = -\infty$ and $\lim_{x \rightarrow \frac{3\pi}{2}} g(x) = +\infty$.

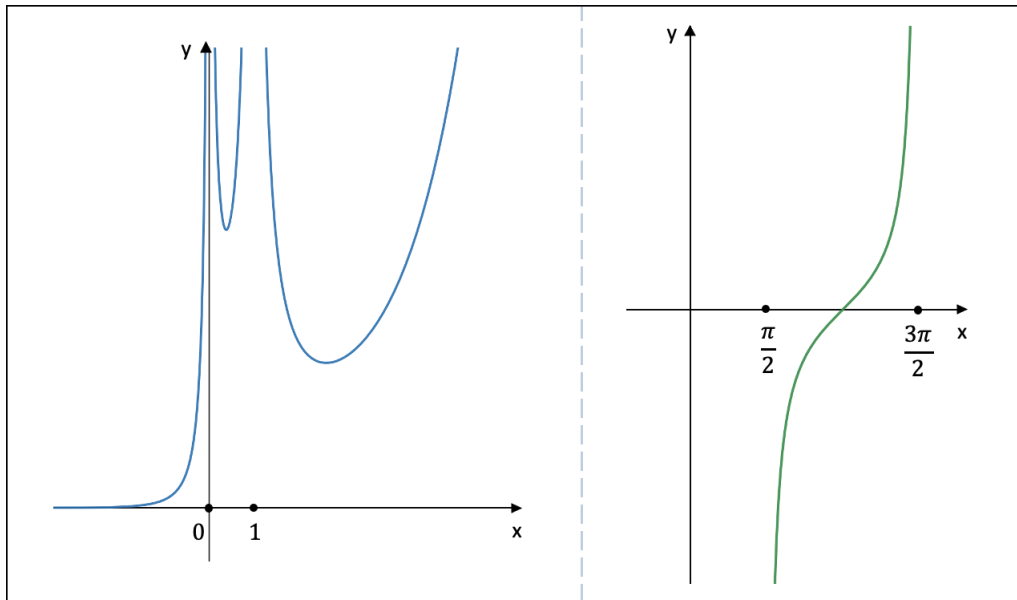


Figure 7.6: Two examples of infinite limits

A **limit at infinity** is a finite limit L of a function f that is obtained when we consider the limit of f at a region of infinity instead of a real number a . Since we can situate the region of infinity either at the negative or the positive side of the x -axis, we formulate the following two definitions of a limit at infinity:

- $\lim_{x \rightarrow +\infty} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \alpha \in \mathbb{R}, \forall x \in A : x > \alpha \Rightarrow |f(x) - L| < \epsilon$
- $\lim_{x \rightarrow -\infty} f(x) = L \Leftrightarrow \forall \epsilon > 0, \exists \alpha \in \mathbb{R}, \forall x \in A : x < \alpha \Rightarrow |f(x) - L| < \epsilon$

Two examples of limits at infinity are shown in Fig. 7.7. On the left, the graph in orange represents the function $f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{3x^3 - 2x + 5}{x^3 - 1}$. For the function f , we

find that $\lim_{x \rightarrow +\infty} f(x) = 3$ and $\lim_{x \rightarrow -\infty} f(x) = 3$.

On the right, the graph in purple represents the function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = \cos(xe^{-2x}) + 1$. For the function g , we find that $\lim_{x \rightarrow +\infty} g(x) = 2$.

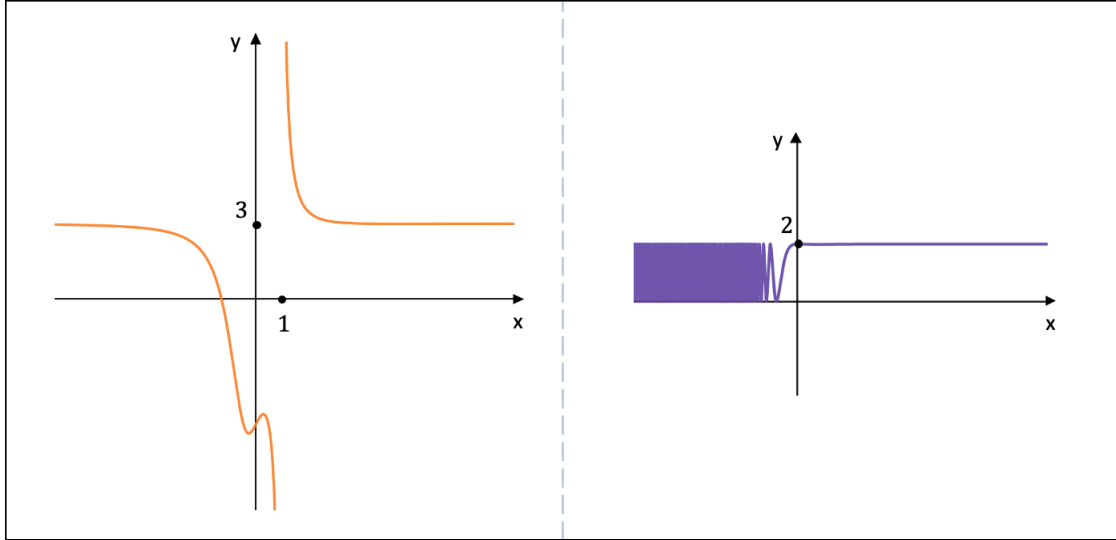


Figure 7.7: Two examples of limits at infinity

A **left-hand (right-hand) limit** of a function f evaluated at a real number a is equal to the limit of f when only considering arguments x of f that are strictly smaller (greater) than a . The left-hand (right-hand) limit L_1 (L_2) of f is denoted by $\lim_{x \rightarrow a^-} f(x) = L_1$ ($\lim_{x \rightarrow a^+} f(x) = L_2$).

We can write down their definitions as follows:

- $\lim_{x \rightarrow a^-} f(x) = L_1 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : a - \delta < x < a \Rightarrow |f(x) - L_1| < \epsilon$
- $\lim_{x \rightarrow a^+} f(x) = L_2 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : a < x < a + \delta \Rightarrow |f(x) - L_2| < \epsilon$

We can now also define the limit L of a function f at a real number a as follows: the limit L of f at a real number a is equal to L if the left-hand limit L_1 is equal to the right-hand limit L_2 at a , i.e., $L_1 = L_2 = L$.

Consider Fig. 7.8, which provides two examples of left-hand and right-hand limits. On the left-hand side, the graph in green represents the function $f :]-\frac{\pi}{2}, \frac{3\pi}{2}[\setminus \{\frac{\pi}{2}\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{|\cos(x)|}{\cos(x)}$. For the function f , we find that $L_1 = \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = 1$ and $L_2 = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = -1$. Given that $L_1 \neq L_2$, it follows that the limit $L = \lim_{x \rightarrow \frac{\pi}{2}} f(x)$ is not defined.

On the right-hand side, the graph in red represents the piecewise function g , which is defined as follows:

$$g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = \begin{cases} x^3 - 3x^2 - 2x + 7 & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ e^{0.5x} - 1 - e & \text{if } x > 2 \end{cases}$$

For the function g , we find that $L_1 = \lim_{x \nearrow 2} g(x) = -1$ and $L_2 = \lim_{x \searrow 2} g(x) = -1$. Given that $L_1 = L_2$, it follows that the limit $L = \lim_{x \rightarrow 2} g(x)$ is defined and equal to $L = -1$. Note that $g(2) = 0$ and did not come into play during the calculation of the limits.

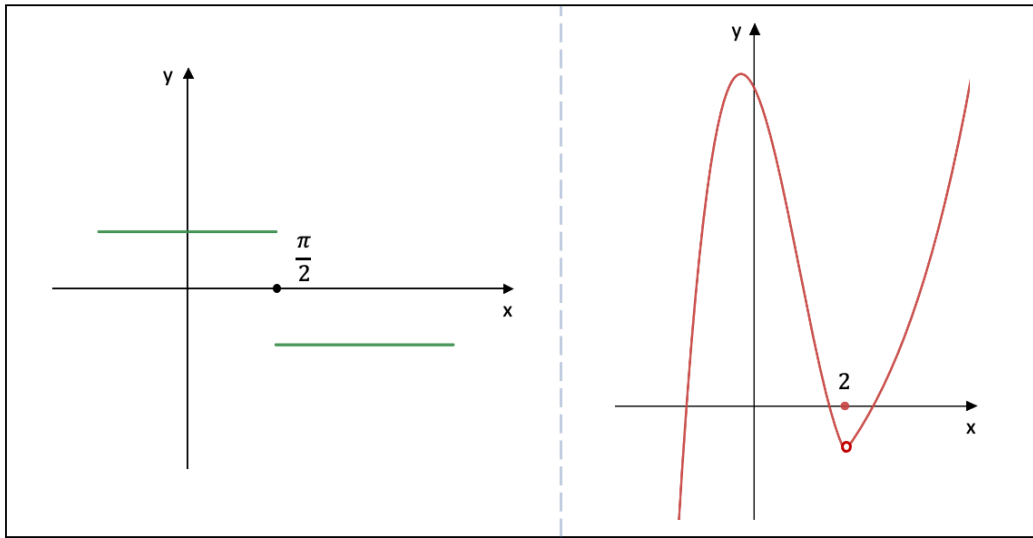


Figure 7.8: Two examples of left-hand/right-hand limits at a real number a

Left-hand and right-hand limits also exist in the context of infinite limits, which are defined as follows:

- $\lim_{x \nearrow a} f(x) = +\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : a - \delta < x < a \Rightarrow f(x) > \alpha$
- $\lim_{x \searrow a} f(x) = +\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : a < x < a + \delta \Rightarrow f(x) > \alpha$
- $\lim_{x \nearrow a} f(x) = -\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : a - \delta < x < a \Rightarrow f(x) < \alpha$
- $\lim_{x \searrow a} f(x) = -\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists \delta > 0, \forall x \in A : a < x < a + \delta \Rightarrow f(x) < \alpha$

Fig. 7.9 shows two examples of infinite left-hand and right-hand limits. On the left, the graph in red represents the function $f : \mathbb{R} \setminus \{-4\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{x^2 - x + 3}{x + 4}$. For the function f , we find that $\lim_{x \nearrow -4} f(x) = -\infty$ and $\lim_{x \searrow -4} f(x) = +\infty$. Given that the left- and right-hand limit are not equal, the limit $\lim_{x \rightarrow -4} f(x)$ is undefined.

On the right, the graph in blue represents the function $g : \mathbb{R} \setminus \{-\frac{\pi}{4} + k\pi\} \rightarrow \mathbb{R}^+ : x \mapsto g(x) = \frac{e^{-0.1x}}{|\sin(x) + \cos(x)|}$, with $k \in \mathbb{Z}$. For the function g , we find, for instance, that $\lim_{x \rightarrow -\frac{17\pi}{4}}^< g(x) = +\infty$ and $\lim_{x \rightarrow -\frac{17\pi}{4}}^> g(x) = +\infty$. Since the left-hand and right-hand limits are equal, the limit $\lim_{x \rightarrow -\frac{17\pi}{4}} g(x)$ is well-defined and equal to $+\infty$.

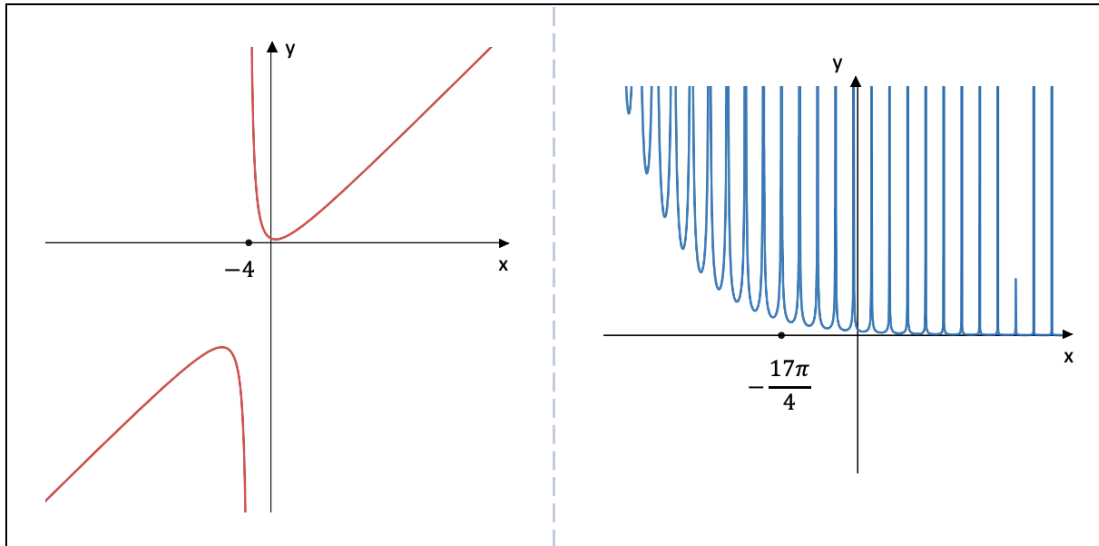


Figure 7.9: Two examples of infinite left-hand/right-hand limits

7.2.2 Limit Operations

Fig. 7.10 lists the basic operations of limits, whereby $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ with $L, M \in \mathbb{R}$ and a can be a real number, $+\infty$, or $-\infty$. Note furthermore that $k \in \mathbb{R}$.

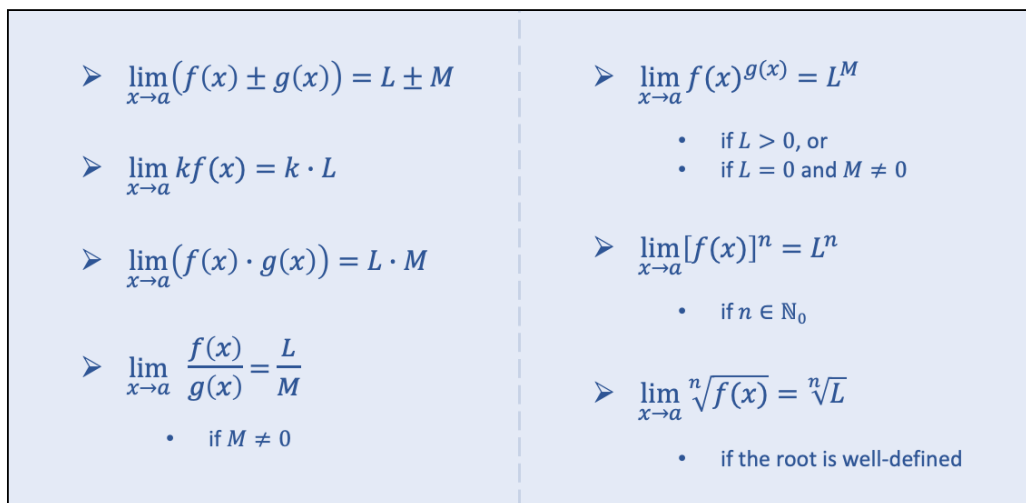


Figure 7.10: Basic operations of limits

For example, given a function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 2x^2 + 6$ and $\lim_{x \rightarrow 2} f(x) = 14$, we have that $\lim_{x \rightarrow 2} [f(x)]^2 = 14^2 = 196$ with $n = 2 \in \mathbb{N}_0$.

Consider a second example whereby $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 3x^2 - 9$ and $g : \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto g(x) = e^{3x+1}$. Since $\lim_{x \rightarrow -1} f(x) = -6$ and $\lim_{x \rightarrow -1} g(x) = e^{-2}$, we find that $\lim_{x \rightarrow -1} f(x) \cdot g(x) = -6 \cdot e^{-2}$.

A useful tool is the **squeeze theorem**. Given three functions f , g , and h that are well-defined in the vicinity of $a \in \mathbb{R}$, the squeeze theorem states that if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ and if $\exists \delta > 0$ so that for $0 < |x - a| < \delta$ we have that $f(x) \leq h(x) \leq g(x)$, it follows that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} g(x)$.

For instance, consider the functions $f : \mathbb{R} \rightarrow [0, 2] : x \mapsto f(x) = \sin(-x) + 1$, $g : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R} : x \mapsto g(x) = \frac{1}{x+1}$, and $h : \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto h(x) = e^{-x}$. Given that $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 1$ and that for, for instance, $\delta = 1$ we have that $0 < |x| < 1 : f(x) \leq h(x) \leq g(x)$, the squeeze theorem says that $\lim_{x \rightarrow 0} h(x) = 1$.

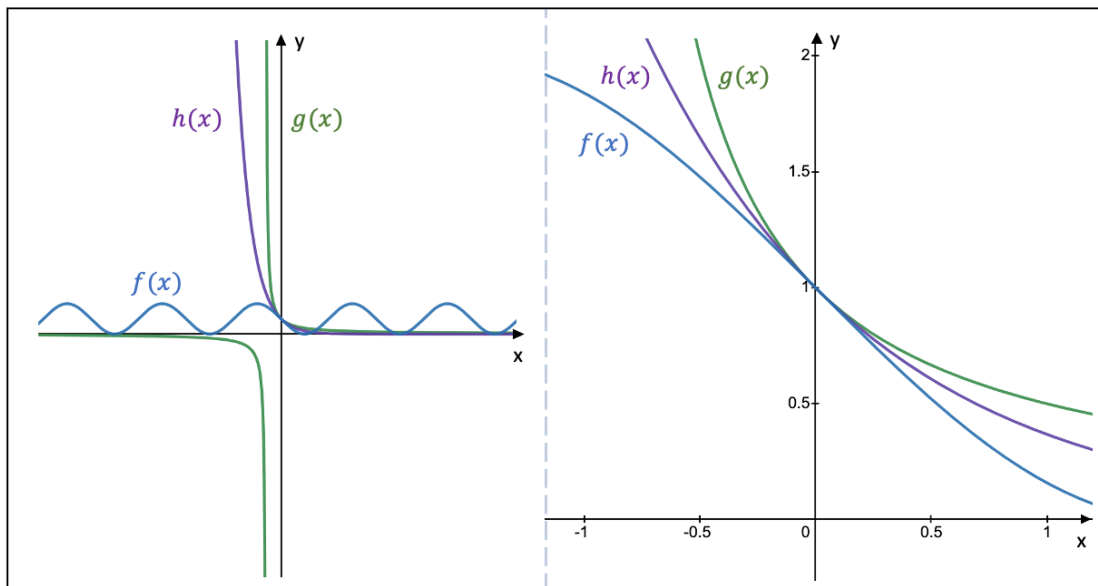


Figure 7.11: An example of the squeeze theorem

When calculating limits, it is also useful to keep in mind the following rules (with $k \in \mathbb{R}$):

- $(\pm\infty) + (\pm\infty) = \pm\infty$
- $k + (\pm\infty) = \pm\infty$
- $\left(\frac{k}{\pm\infty}\right) = 0$
- $(\pm\infty) \cdot (\mp\infty) = -\infty$
- $(\pm\infty) \cdot (\pm\infty) = +\infty$
- $k \cdot (\pm\infty) = \begin{cases} \pm\infty & \text{if } k > 0 \\ \mp\infty & \text{if } k < 0 \end{cases}$

Note hereby that the following expressions are undefined:

- $\frac{\pm\infty}{\pm\infty}$
- 0^0
- $0 \cdot (\pm\infty)$
- $\frac{\mp\infty}{\pm\infty}$
- $(+\infty) \pm (\mp\infty)$
- $\frac{\pm\infty}{\mp\infty}$
- $\frac{0}{0}$
- $1^{\pm\infty}$

7.2.3 Examples

Let put the basic calculation rules of the previous section into practice by looking at some examples.

- Example 1: $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)$

If we directly substitute 0 for x in the limit, we obtain $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{0}{0}$, which is undefined. In order to solve this limit, we will invoke the squeeze theorem.

Consider Fig. 7.12 and let us in the first place compare the area of the red triangle with the area of the purple circle sector (which lies underneath the red triangle) and with the area of the right triangle in green (which lies underneath both the purple circle sector and the red triangle). Given that $h = 1 \cdot \tan(x)$, we find that:

$$\frac{\sin(x) \cdot 1}{2} < \pi \cdot 1^2 \cdot \frac{x}{2\pi} < \frac{(1 \cdot \tan(x)) \cdot 1}{2}$$

$$\Leftrightarrow \sin(x) < x < \tan(x)$$

$$\Leftrightarrow 1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

$$\Leftrightarrow 1 > \frac{\sin(x)}{x} > \cos(x)$$

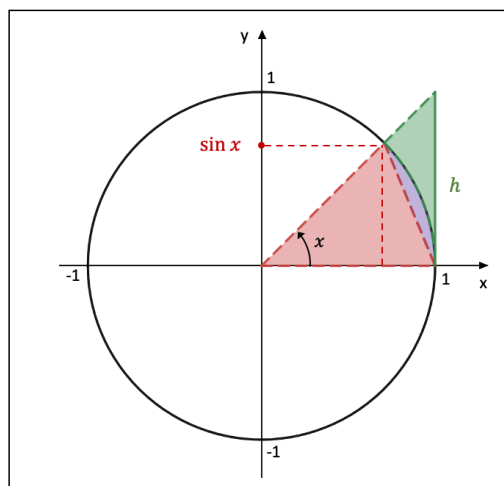


Figure 7.12: The unit circle

If we choose $\delta = 0.5$, we find for any x , whereby $0 < |x| < 0.5$, that $\cos(x) < \frac{\sin(x)}{x} < 1$. Given that $\lim_{x \rightarrow 0} 1 = \lim_{x \rightarrow 0} \cos(x) = 1$, the squeeze theorem tells us that $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$.

- Example 2: $\lim_{x \rightarrow \pm\infty} \left(\frac{\sin x}{x} \right)$

Directly substituting $\pm\infty$ for x in the limit does not produce any conclusive result—not even an undefined limit—because $\lim_{x \rightarrow \pm\infty} \sin(x)$ does not exist as $\sin(x)$ is a cyclical function. Again, we will rely on the squeeze theorem to find a solution.

We start with writing down the co-domain of the sine function and subsequently dividing that expression by $x \neq 0$:

$$-1 \leq \sin(x) \leq 1 \quad \Leftrightarrow \quad -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

Given that for any interval in the vicinity of $\pm\infty$, the above inequalities hold and that $\lim_{x \rightarrow \pm\infty} -\frac{1}{x} = \lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$, we find according to the squeeze theorem that $\lim_{x \rightarrow \pm\infty} \frac{\sin(x)}{x} = 0$.

- Example 3: $\lim_{x \rightarrow -2} \left(\frac{3 + \sqrt[3]{x-25}}{x+2} \right)$

If we directly substitute -2 for x , we find the undefined expression $\frac{0}{0}$. In order to solve this limit, we need to remember the special product $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ (see section 2.3.3). We can then see that:

$$\begin{aligned} 3^3 + (\sqrt[3]{x-25})^3 &= (3 + \sqrt[3]{x-25}) \left(3^2 - 3\sqrt[3]{x-25} + (\sqrt[3]{x-25})^2 \right) \\ \Leftrightarrow 3 + \sqrt[3]{x-25} &= \frac{3^3 + (\sqrt[3]{x-25})^3}{3^2 - 3\sqrt[3]{x-25} + (\sqrt[3]{x-25})^2} \\ &= \frac{x+2}{3^2 - 3\sqrt[3]{x-25} + (\sqrt[3]{x-25})^2} \end{aligned}$$

If we insert the above expression for $3 + \sqrt[3]{x-25}$ into our limit, we obtain the following result:

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{3 + \sqrt[3]{x-25}}{x+2} &= \lim_{x \rightarrow -2} \left[\frac{x+2}{3^2 - 3\sqrt[3]{x-25} + (\sqrt[3]{x-25})^2} \right] \cdot \frac{1}{x+2} \\ &= \lim_{x \rightarrow -2} \frac{1}{3^2 - 3\sqrt[3]{x-25} + (\sqrt[3]{x-25})^2} \\ &= \frac{1}{3^2 - 3\sqrt[3]{(-2)-25} + (\sqrt[3]{(-2)-25})^2} \\ &= \frac{1}{27} \end{aligned}$$

- Example 4: $\lim_{x \rightarrow +\infty} \left(\frac{4^{3x} + 2}{4 + 6^{2x}} \right)$

If we directly substitute $+\infty$ for x in the above limit, we stumble upon the undefined form $\frac{+\infty}{+\infty}$. Instead, we solve the limit as follows:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{4^{3x} + 2}{4 + 6^{2x}} &= \lim_{x \rightarrow +\infty} \frac{6^{2x} \cdot [4^{3x} \cdot 6^{-2x} + 2 \cdot 6^{-2x}]}{6^{2x} \cdot [4 \cdot 6^{-2x} + 1]} \\ &= \lim_{x \rightarrow +\infty} \frac{4^{3x} \cdot 6^{-2x} + 2 \cdot 6^{-2x}}{4 \cdot 6^{-2x} + 1} \\ &= \frac{(+\infty) + 0}{0 + 1} \\ &= +\infty \end{aligned}$$

- Example 5: $\lim_{x \rightarrow -\infty} (\sqrt{9x^2 - x - 5} + 3x)$

Directly substituting $-\infty$ for x in the limit produces the undefined form $(+\infty) + (-\infty)$. We solve this limit as follows:

$$\begin{aligned} \lim_{x \rightarrow -\infty} \sqrt{9x^2 - x - 5} + 3x &= \lim_{x \rightarrow -\infty} \left[(\sqrt{9x^2 - x - 5} + 3x) \cdot \left(\frac{\sqrt{9x^2 - x - 5} - 3x}{\sqrt{9x^2 - x - 5} - 3x} \right) \right] \\ &= \lim_{x \rightarrow -\infty} \frac{(\sqrt{9x^2 - x - 5})^2 - (3x)^2}{\sqrt{9x^2 - x - 5} - 3x} \\ &= \lim_{x \rightarrow -\infty} \frac{-x - 5}{\sqrt{9x^2 - x - 5} - 3x} \\ &= \lim_{x \rightarrow -\infty} \left[\left(\frac{-x - 5}{-3x} \right) \cdot \left(\frac{1}{1 - \frac{\sqrt{9x^2 - x - 5}}{3x}} \right) \right] \\ &= \left(\lim_{x \rightarrow -\infty} \frac{-x - 5}{-3x} \right) \cdot \left(\lim_{x \rightarrow -\infty} \frac{1}{1 - \frac{\sqrt{9x^2 - x - 5}}{3x}} \right) \\ &= \left(\lim_{x \rightarrow -\infty} \frac{-1 - \frac{5}{x}}{-3} \right) \cdot \left(\lim_{x \rightarrow -\infty} \frac{1}{1 - \frac{|x| \sqrt{9 - \frac{1}{x} - \frac{5}{x^2}}}{-3|x|}} \right) \\ &= \left(\frac{1}{3} \right) \cdot \left(\frac{1}{1 - \frac{\sqrt{9-0-0}}{-3}} \right) \\ &= \frac{1}{6} \end{aligned}$$

7.3 Asymptotes

7.3.1 Horizontal Asymptotes

A **horizontal asymptote** of a function f is equal to the finite limit L that f approaches when $x \rightarrow \pm\infty$. It is common practice to graphically represent the horizontal asymptote as a line with equation $y = L$.

Note that a function f can only have 0, 1, or 2 horizontal asymptotes. Moreover, with respect to rational functions (see section 7.6.5), the function only displays horizontal asymptotes if the degree of the numerator, i.e., the growth rate, is less than or equal to the degree of the denominator.

For example, consider the rational function $f : \mathbb{R} \setminus \{-5\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{(-x-3)}{|-x-5|}$. The limit for $x \rightarrow +\infty$ is equal to:

$$\lim_{x \rightarrow +\infty} \frac{(-x-3)}{|-x-5|} = \lim_{x \rightarrow +\infty} \frac{(-x-3)}{x+5} = \lim_{x \rightarrow +\infty} \frac{(-1 - \frac{3}{x})}{1 + \frac{5}{x}} = -1$$

The limit for $x \rightarrow -\infty$ is equal to:

$$\lim_{x \rightarrow -\infty} \frac{(-x-3)}{|-x-5|} = \lim_{x \rightarrow -\infty} \frac{(|x|-3)}{||x|-5|} = \lim_{x \rightarrow -\infty} \frac{(1 - \frac{3}{|x|})}{1 - \frac{5}{|x|}} = 1$$

In other words, the rational function f has two horizontal asymptotes, i.e., $y = -1$ and $y = 1$. Fig. 7.13 depicts this function together with the two horizontal asymptotes.

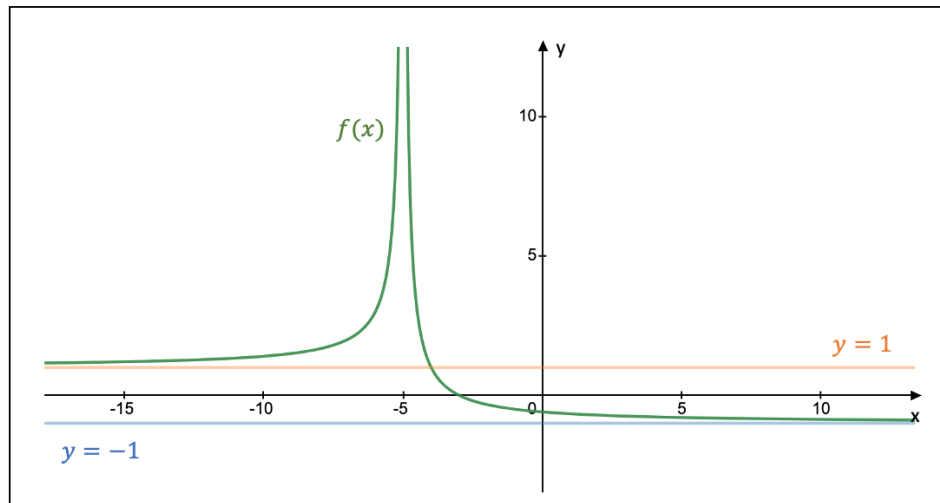


Figure 7.13: An example of a rational function with two horizontal asymptotes

7.3.2 Vertical Asymptotes

A **vertical asymptote** is equal to the real number a at which the limit of a function f approaches $\pm\infty$. In other words, a function f has vertical asymptotes if $\lim_{x \rightarrow a} f(x) = \pm\infty$,

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty, \text{ or } \lim_{x \rightarrow a^+} f(x) = \pm\infty.$$

Graphically, a vertical asymptote is typically represented by a vertical line with equation $x = a$. With respect to rational functions, a function f only has vertical asymptotes if the roots of the denominator are not shared with those of the numerator or if the multiplicity of any shared roots is larger in the denominator.

Consider the rational function $f : \mathbb{R} \setminus \{-4, 1, 7\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{-x^2 + 10x - 21}{x^4 - 11x^3 + 3x^2 + 203x - 196}$.

If we factor the polynomial in the numerator and the denominator, the function f obtains the following form:

$$f(x) = \frac{-x^2 + 10x - 21}{x^4 - 11x^3 + 3x^2 + 203x - 196} = \frac{(x - 7)(3 - x)}{(x + 4)(x - 7)^2(x - 1)} = \frac{(3 - x)}{(x + 4)(x - 7)(x - 1)}$$

We find that $\lim_{x \rightarrow -4^-} f(x) = -\infty$, $\lim_{x \rightarrow -4^+} f(x) = +\infty$, $\lim_{x \rightarrow 1^-} f(x) = +\infty$, $\lim_{x \rightarrow 1^+} f(x) = -\infty$, $\lim_{x \rightarrow 7^-} f(x) = +\infty$, $\lim_{x \rightarrow 7^+} f(x) = -\infty$, $\lim_{x \rightarrow -4^-} f(x) = +\infty$, and $\lim_{x \rightarrow 1^+} f(x) = -\infty$.

As a result, the function f has three vertical asymptotes, i.e., $x = -4$, $x = 1$, and $x = 7$, which are shown in Fig. 7.14.

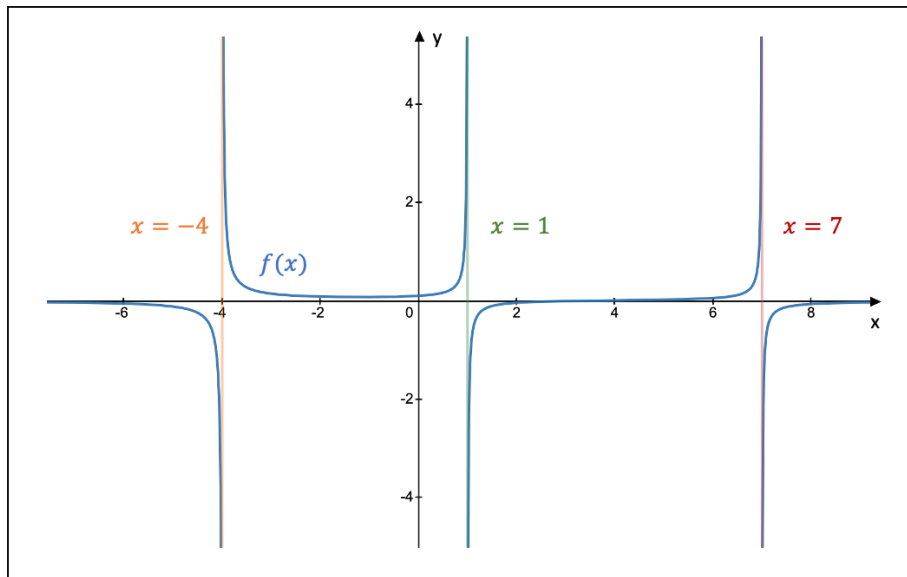


Figure 7.14: An example of a rational function with three vertical asymptotes

7.3.3 Oblique Asymptotes

An **oblique or slant asymptote** of a function f is represented by the line $y = ax + b$ (with $a \neq 0$) if $\lim_{x \rightarrow \pm\infty} [f(x) - (ax + b)] = 0$.

In the case of rational functions, a function f has an oblique asymptote if the degree of the numerator is equal to $n + 1$ (with $n \in \mathbb{N}$), given a degree n of the denominator.

To determine the equation of the oblique asymptote, we can follow two different procedures. The first procedure consists of first checking whether the limit $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}$ is equal to a non-zero real number. If so, we set that number equal to a and calculate the limit $b = \lim_{x \rightarrow \pm\infty} [f(x) - ax]$. The equation of the oblique asymptote is then equal to $y = ax + b$.

The second procedure is only valid for rational functions. If we apply polynomial long division to the given fraction of the function (see section 2.3.2), the equation of the oblique asymptote is equal to the quotient.

Let us consider two examples to demonstrate both procedures. In a first example, we are given the function $f : \mathbb{R} \setminus \{-2\sqrt{2}, 2\sqrt{2}\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{-3x^3 + 4x^2 - 5x + 1}{x^2 - 8}$. If we perform long division, we find that:

$$p(x) = q(x) \cdot s(x) + r(x) \Leftrightarrow -3x^3 + 4x^2 - 5x + 1 = (x^2 - 8)(-3x + 4) + (-29x + 33)$$

Since the quotient $q(x)$ is equal to $q(x) = -3x + 4$, the equation of the oblique asymptote is equal to $y = -3x + 4$.

As a second example, we consider the function $f : \mathbb{R} \setminus]-\sqrt{5}, \sqrt{5}[\rightarrow \mathbb{R} : x \mapsto f(x) = \sqrt{\frac{6}{5}(x^2 - 5)}$. We first calculate the coefficient a :

$$a = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{\frac{6}{5}(x^2 - 5)}}{x} = \lim_{x \rightarrow +\infty} \sqrt{\frac{6}{5} \left(1 - \frac{5}{x^2}\right)} = \sqrt{\frac{6}{5}}$$

In a next step, we calculate the following limit to obtain the coefficient b :

$$\begin{aligned} b &= \lim_{x \rightarrow +\infty} [f(x) - ax] = \lim_{x \rightarrow +\infty} \left[\sqrt{\frac{6}{5}(x^2 - 5)} - \sqrt{\frac{6}{5}} \cdot x \right] \\ &= \sqrt{\frac{6}{5}} \cdot \lim_{x \rightarrow +\infty} \left[\sqrt{(x^2 - 5)} - x \right] \\ &= \sqrt{\frac{6}{5}} \cdot \lim_{x \rightarrow +\infty} \left[\left(\sqrt{(x^2 - 5)} - x \right) \cdot \left(\frac{\sqrt{(x^2 - 5)} + x}{\sqrt{(x^2 - 5)} + x} \right) \right] \\ &= \sqrt{\frac{6}{5}} \cdot \lim_{x \rightarrow +\infty} \left[\frac{-5}{\sqrt{(x^2 - 5)} + x} \right] \\ &= \sqrt{\frac{6}{5}} \cdot \left[\frac{-5}{\sqrt{((+\infty)^2 - 5)} + (+\infty)} \right] \\ &= 0 \end{aligned}$$

If $x \rightarrow +\infty$, the equation of the oblique asymptote is equal to $y = ax + b = \sqrt{\frac{6}{5}} \cdot x$. If we perform a similar calculation for $x \rightarrow -\infty$, we find another oblique asymptote with equation $y = -\sqrt{\frac{6}{5}} \cdot x$.

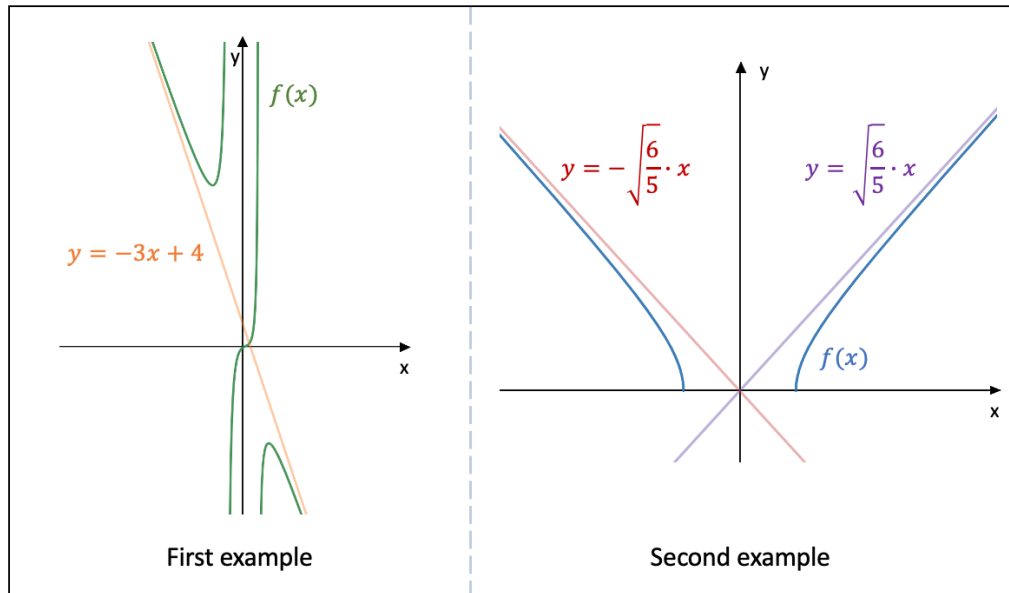


Figure 7.15: Two examples of a function with oblique asymptotes

7.4 Continuity

Consider the function $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ and the limit $\lim_{x \rightarrow a} f(x)$ with $a \in A$. The **function f is continuous in the point a** if the following three conditions are met:

1. The function f is well-defined in the point a
2. The limit $\lim_{x \rightarrow a} f(x)$ exists
3. The limit $\lim_{x \rightarrow a} f(x)$ is equal to the function value of a , i.e., $\lim_{x \rightarrow a} f(x) = f(a)$

In mathematical language, the condition of continuity of f in the point a is expressed as follows:

$$f \text{ is continuous in } a \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : x \in]a - \delta, a + \delta[\Rightarrow |f(x) - f(a)| < \epsilon$$

The **function f is continuous** if f is continuous in every point $a \in A$. The **function f is discontinuous** if f is not continuous in at least one point $a \in A$.

As an example, consider the piecewise function f :

$$f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = \begin{cases} \frac{\sin(5x + 2)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Even though the function is well-defined in the point $x = 0$, the function is not continuous in $x = 0$ because the limit $\lim_{x \rightarrow 0} f(x)$ does not exist, since $\lim_{x \rightarrow 0^-} f(x) = -\infty$ and $\lim_{x \rightarrow 0^+} f(x) = +\infty$.

Another example is the below piecewise function g :

$$g : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+ : x \mapsto g(x) = \begin{cases} \left| \frac{\ln x}{x-1} \right| & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Given that the function g is well-defined in $x = 1$, that the limit $\lim_{x \rightarrow 1} g(x)$ exists, and that $\lim_{x \rightarrow 1} g(x) = 1 = g(1)$ (the left-hand and right-hand limit are calculated with the assistance of the L'Hôpital Rule; see section 7.5.2), we can conclude that g is continuous in $x = 1$.

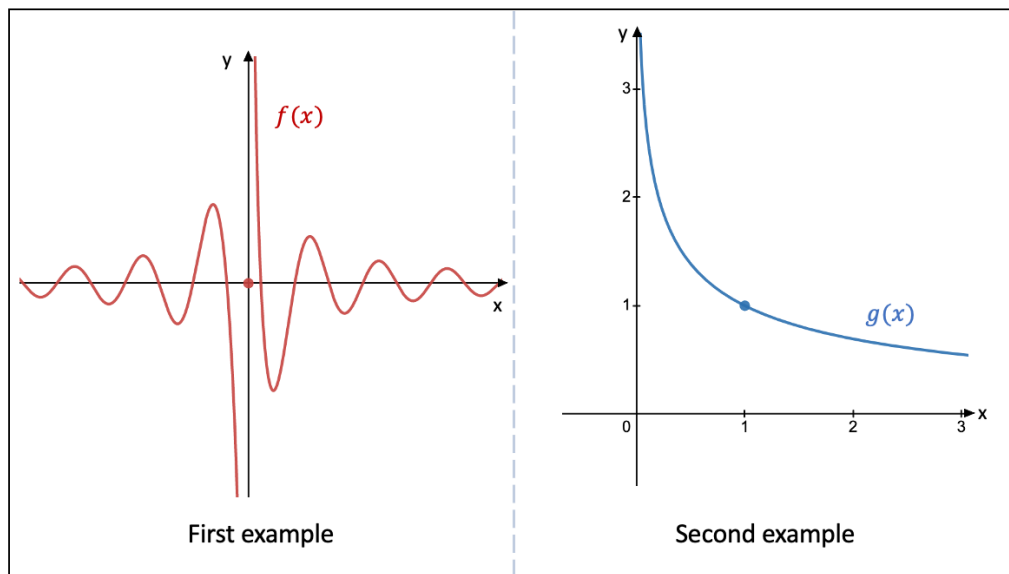


Figure 7.16: Functions and continuity

The definition of a function f being continuous in a point a can also be expressed in terms of **left-continuity** and **right-continuity**, which are defined as follows:

- f is left-continuous in $a \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : x \in]a - \delta, a] \Rightarrow |f(x) - f(a)| < \epsilon$
- f is right-continuous in $a \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0, \forall x \in A : x \in [a, a + \delta[\Rightarrow |f(x) - f(a)| < \epsilon$

A function f is then continuous in a point a , if f is both left-continuous and right-continuous in a , i.e., $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$.

A function f is **continuous over the closed interval** $[a, b]$ if f is continuous in each point of the open interval $]a, b[$, right-continuous in a , and left-continuous in b .

As an example, suppose we wish to examine the continuity over the closed interval $[3, 6]$ of the below piecewise function f :

$$f : \mathbb{R} \rightarrow [1, +\infty[: x \mapsto f(x) = \begin{cases} 0.3(x-3)^2 + 2 & \text{if } x < 3 \\ x - 2 & \text{if } 3 \leq x < 6 \\ 4e^{2(x-6)} & \text{if } x \leq 6 \end{cases}$$

Over the open interval $]3, 6[$, the function f represents the straight line $y = x - 2$, which is continuous in every point of that open interval. The function f is right-continuous in $a = 3$, if $\lim_{x \rightarrow 3^+} (x - 2) = f(3)$. This is indeed the case given that $\lim_{x \rightarrow 3^+} (x - 2) = 1 = f(3) = 3 - 2$. The function f is also left-continuous in $a = 6$, because $\lim_{x \rightarrow 6^-} (x - 2) = 4 = f(6) = 4e^{2(6-6)}$.

If f and g are continuous functions in point a , and $k \in \mathbb{R}$, the **scalar product** $k \cdot f$, the **sum** $f + g$, the **product** $f \cdot g$, and the **absolute value** $|f|$ are also continuous functions in a . If $g(a) \neq 0$, we have that the **quotient** $\frac{f}{g}$ is equally continuous in a . In case that g is continuous in $f(a)$, we find that the **composite function** $g \circ f$ is also continuous in a .

Consider as an example the function $g : \mathbb{R} \rightarrow [-1, +\infty[: x \mapsto g(x) = x^2 - 1$ and the below piecewise function f :

$$f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = \begin{cases} \sqrt{x^2 + x - 2} & \text{if } x \notin]-2, 1[\\ 0 & \text{if } x \in]-2, 1[\end{cases}$$

Given that f is continuous in $x = 1$ and that g is continuous in $f(1) = 0$, we know that $g \circ f$ is continuous in $x = 1$, whereby $g \circ f$ is defined as follows:

$$g \circ f : \mathbb{R} \rightarrow [-1, +\infty[: x \mapsto g(f(x)) = \begin{cases} x^2 + x - 3 & \text{if } x \notin]-2, 1[\\ -1 & \text{if } x \in]-2, 1[\end{cases}$$

Finally, a **theorem with respect to composite functions** states that if $\lim_{x \rightarrow a} f(x) = L$ and the function g is continuous in L , we can write the following:

$$\lim_{x \rightarrow a} g(f(x)) = g\left(\lim_{x \rightarrow a} f(x)\right) = g(L)$$

If we take the previous example, we find that $\lim_{x \rightarrow 3} f(x) = \sqrt{10}$ and that g is continuous in $x = \sqrt{10}$. Therefore, we can write that $\lim_{x \rightarrow 3} g(f(x)) = 9 = g(\sqrt{10})$.

7.5 Differential Calculus

7.5.1 Derivatives and Differentials

Consider the function f that is well-defined around the point $x = a$. If we now take the point $x = a + h$ in the near vicinity of $x = a$, the straight line through the points $\vec{a} = (a, f(a))$ and $\vec{b} = (a + h, f(a + h))$ represents approximately the graph of f around point $x = a$. The equation of this straight line is equal to:

$$\begin{aligned} y &= \left[\frac{f(a+h) - f(a)}{(a+h) - a} \right] (x - a) + f(a) = \left[\frac{f(a+h) - f(a)}{h} \right] (x - a) + f(a) \\ &= \frac{\Delta y}{\Delta x} (x - a) + f(a) \end{aligned}$$

The slope $\frac{\Delta y}{\Delta x}$ of the straight line is called the **difference quotient**. If we now let $\Delta x = h$ become infinitesimally small, we define the **derivative of the function f at the point a** , which is denoted by $f'(a)$, as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

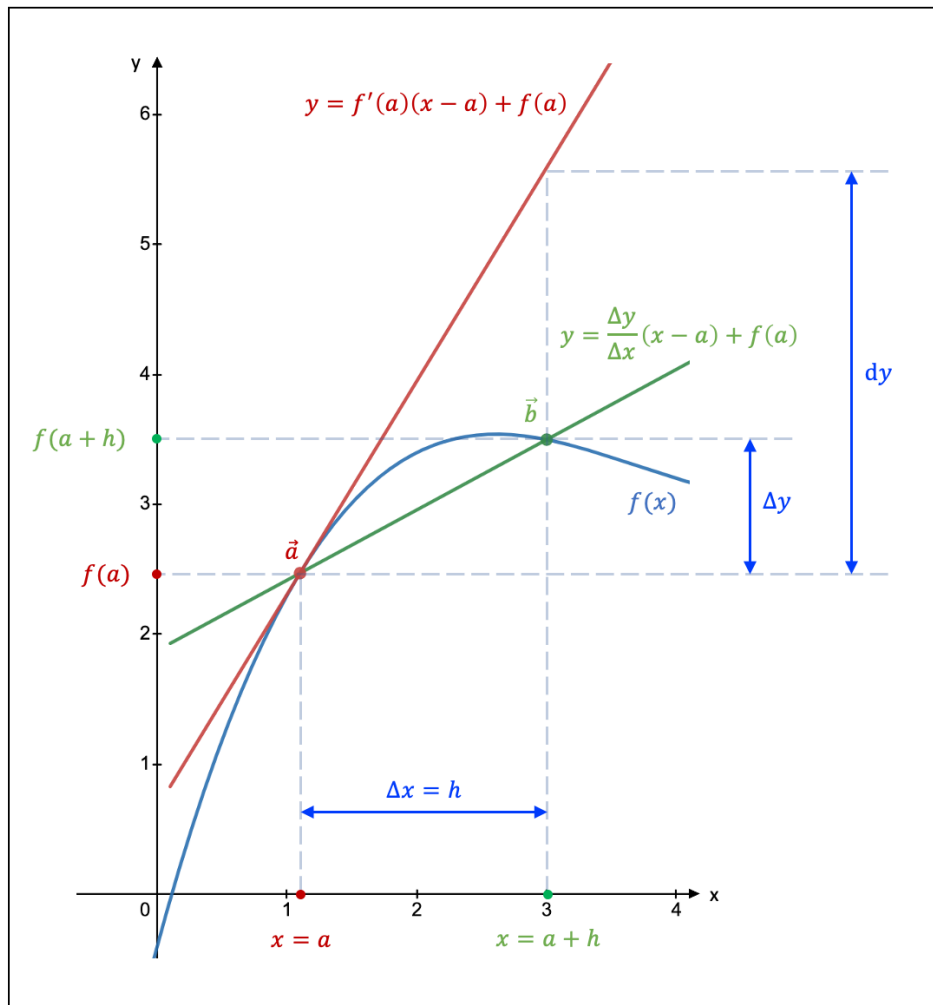


Figure 7.17: Graphic representation of the derivative and the differential

We call the function f **differentiable at point a** , if the following limit exists and is finite, i.e., the limit is equal to a real number:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \in \mathbb{R}$$

A function is furthermore called differentiable if it is differentiable at every point of its domain.

We define the **differential of the function f at the point a** as:

$$dy = f'(a)\Delta x$$

The differential dy of f in point a can thus be interpreted as the total amount of change of f given a certain change Δx of the input variable x and the rate of change $f'(a)$, which is defined as the derivative of f in a .

If we consider the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = x$, we find that $dy = d(x) = 1 \cdot \Delta x$, with $f'(a) = 1$. Therefore, we can see that $dx = \Delta x$ and we can write the derivative of f in point a as follows, which is known as the **Leibniz notation**:

$$f'(a) = \frac{dy}{dx} = \frac{df}{dx}(a)$$

If a function f is differentiable in point a , it follows that **f is continuous in a** , as shown below:

$$\begin{aligned} \lim_{x \rightarrow a} [f(x) - f(a)] &= \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \right] \cdot \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot \lim_{x \rightarrow a} (x - a) \\ &= 0 \end{aligned}$$

$$\Leftrightarrow \lim_{x \rightarrow a} f(x) = f(a)$$

We can also define the **left-hand and right-hand derivative of f in point a** , which are denoted by $f'_{<}(a)$ and $f'_{>}(a)$, respectively:

$$\left\{ \begin{array}{l} f'_{<}(a) = \lim_{h \underset{<}{\rightarrow} 0} \frac{f(a+h) - f(a)}{h} \\ f'_{>}(a) = \lim_{h \underset{>}{\rightarrow} 0} \frac{f(a+h) - f(a)}{h} \end{array} \right.$$

If the left-hand derivative of the function f in point a is equal to the right-hand derivative, we can conclude that f is differentiable in point a .

Let us now calculate the derivative of some functions based on the limit definition of the derivative. As a first example, consider the function $f : \mathbb{R} \rightarrow [-1, 1] : x \mapsto f(x) = \cos(x)$. To calculate its derivative, we will make use of the results of Example 1 of section 7.2.3, i.e., $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, and the trigonometric formulas $\cos(2x) = 1 - 2\sin^2 x$ and $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ from section 5.3:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\cos(x)\cos(h) - \sin(x)\sin(h)) - \cos(x)}{h} \\
 &= \lim_{h \rightarrow 0} \cos(x) \left(\frac{\cos(h) - 1}{h} \right) - \lim_{h \rightarrow 0} \sin(x) \frac{\sin(h)}{h} \\
 &= \cos(x) \cdot \lim_{h \rightarrow 0} \left(-\frac{\sin^2(\frac{h}{2})}{\frac{h}{2}} \right) - \sin(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \cos(x) \cdot \left(-\lim_{h \rightarrow 0} \sin\left(\frac{h}{2}\right) \right) \cdot \left(\lim_{h \rightarrow 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} \right) - \sin(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \cos(x) \cdot (-0) \cdot (1) - \sin(x) \cdot 1 \\
 &= -\sin(x)
 \end{aligned}$$

A second example is the function $f : \mathbb{R} \setminus \{-1\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{\sqrt{x^2+1}}{x+1}$. We calculate its derivative as follows:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\sqrt{(x+h)^2+1}}{h(x+h+1)} - \frac{\sqrt{x^2+1}}{h(x+1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{(x+1)\sqrt{(x+h)^2+1} - (x+h+1)\sqrt{x^2+1}}{h(x+1)(x+h+1)} \\
 &= \lim_{h \rightarrow 0} \frac{(x+1) \left[\sqrt{(x+h)^2+1} - \sqrt{x^2+1} \right] - h\sqrt{x^2+1}}{h(x+1)(x+h+1)} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h(x+h+1)} - \lim_{h \rightarrow 0} \frac{\sqrt{x^2+1}}{(x+1)(x+h+1)} \\
 &= \lim_{h \rightarrow 0} \left[\left(\frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h(x+h+1)} \right) \cdot \left(\frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \right) \right] - \frac{\sqrt{x^2+1}}{(x+1)^2}
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{2x + h}{(x + h + 1) \left(\sqrt{(x + h)^2 + 1} + \sqrt{x^2 + 1} \right)} - \frac{\sqrt{x^2 + 1}}{(x + 1)^2} \\
&= \frac{x}{(x + 1)\sqrt{x^2 + 1}} - \frac{\sqrt{x^2 + 1}}{(x + 1)^2} \\
&= \frac{x - 1}{(x + 1)^2 \sqrt{x^2 + 1}}
\end{aligned}$$

If we perform the same procedure as above for the most common functions, we obtain the list as shown in Fig. 7.18. Note hereby that $c, k, a \in \mathbb{R}$, with $a > 0$ and $a \neq 1$. Assume furthermore that each function is well-defined in x .

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\tan x$	$\sec^2 x$
x	1	$\cot x$	$-\csc^2 x$
x^k	kx^{k-1}	$\csc x$	$-\cot(x) \csc x$
\sqrt{x}	$\frac{1}{2\sqrt{x}}$	$\sec x$	$\tan(x) \sec x$
e^x	e^x	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
a^x	$a^x \ln a$	$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$
$\ln x$	$\frac{1}{x}$	$\arctan x$	$\frac{1}{1+x^2}$
$\log_a x$	$\frac{1}{x \ln a}$	$\operatorname{arccot} x$	$\frac{-1}{1+x^2}$
$\sin x$	$\cos x$	$\operatorname{arccsc} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\cos x$	$-\sin x$	$\operatorname{arcsec} x$	$\frac{1}{x\sqrt{x^2-1}}$

Figure 7.18: Common derivatives

7.5.2 Calculation Rules and Applications

If we define two functions $f : A \subset \mathbb{R} \rightarrow B \subset \mathbb{R} : x \mapsto f(x)$ and $g : C \subset \mathbb{R} \rightarrow D \subset \mathbb{R} : x \mapsto g(x)$ that are differentiable in a point $a \in \mathbb{R}$, Fig. 7.19 shows how to calculate the derivative of a linear combination, of a product, of a quotient, of a composite function, and of an inverse function.

Note that $\alpha, \beta \in \mathbb{R}$. With respect to the derivative of a quotient, we need $g(a) \neq 0$. Regarding the composite function, an extra condition applies: g must be differentiable in $f(a)$. Finally, when it comes to the derivative of an inverse function f , we need f to be invertible (see section 7.6.7), continuous over A , differentiable in $f^{-1}(b) = a \in A$, and $f'(a) \neq 0$.

<ul style="list-style-type: none"> ➤ $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$ ➤ $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$ ➤ $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$ 	<ul style="list-style-type: none"> ➤ $(g \circ f)'(a) = g'(f(a))f'(a)$ ➤ $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$
---	---

Figure 7.19: Basic operations with derivatives

The calculation rule to determine the derivative of the product of two functions is also known as the **chain rule**.

We can express the above rules also in terms of differentials, whereby we use $z = g(f(x))$ and $y = f(x)$ for the derivative of the composite function and $x = f^{-1}(y)$ for the inverse function:

- $d(\alpha f + \beta g) = \alpha df + \beta dg$
- $d(fg) = df \cdot g + f \cdot dg$
- $d\left(\frac{f}{g}\right) = \frac{df \cdot g - f \cdot dg}{g^2}$
- $dz = \frac{dz}{dy} dy = \frac{dz}{dy} \frac{dy}{dx} dx$
- $dx = \frac{1}{\frac{dy}{dx}} dy$

Let us look at four examples. Consider the function $f : \mathbb{R}^+ \setminus \{(\frac{\pi}{2} + k\pi)^2\} \rightarrow \mathbb{R} : x \mapsto f(x) = 5^x \sec(\sqrt{x})$, with $k \in \mathbb{R}$. The derivative of f is calculated as follows:

$$\begin{aligned}
 f'(x) &= (5^x \sec(\sqrt{x}))' = (5^x)' \cdot \sec(\sqrt{x}) + 5^x \cdot (\sec(\sqrt{x}))' \\
 &= [5^x \ln(5)] \sec(\sqrt{x}) + 5^x \cdot \left[\tan(\sqrt{x}) \sec(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \right] \\
 &= 5^x \sec(\sqrt{x}) \left[\ln(5) + \frac{\tan(\sqrt{x})}{2\sqrt{x}} \right]
 \end{aligned}$$

In a second example, we will use the rule for inverse functions to calculate the derivative of the function $f :]-\infty, -1] \cup [1, +\infty[\rightarrow]-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\} : x \mapsto f(x) = \operatorname{arccsc}(x)$. Given that $y = \operatorname{arccsc}(x) \Leftrightarrow x = \csc(y)$ and that $\cot(x) = \sqrt{\csc^2(x) - 1}$ (see section 5.3), we can write:

$$\begin{aligned} f'(x) &= \frac{1}{(\csc)'(\operatorname{arccsc}(x))} = \frac{1}{-\cot(\operatorname{arccsc}(x)) \csc(\operatorname{arccsc}(x))} \\ &= \frac{1}{-\left(\sqrt{\csc^2(\operatorname{arccsc}(x)) - 1}\right) \csc(\operatorname{arccsc}(x))} \\ &= \frac{-1}{x\sqrt{x^2 - 1}} \end{aligned}$$

A third example is the function $f : \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto f(x) = x^x$, whose derivative is equal to:

$$\begin{aligned} f'(x) &= (x^x)' = \left(e^{\ln(x^x)}\right)' = \left(e^{x \ln x}\right)' = e^{x \ln x} \cdot (x \ln x)' \\ &= x^x \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \\ &= x^x \cdot (\ln x + 1) \end{aligned}$$

As a final example, consider the function $f :]k\pi, \frac{\pi}{2} + k\pi[\rightarrow \mathbb{R} : x \mapsto f(x) = \frac{\log_7(\sin(2x))}{\sec(2x)}$, with $k \in \mathbb{Z}$. Its derivative is calculated as follows:

$$\begin{aligned} f'(x) &= \left(\frac{\log_7(\sin(2x))}{\sec(2x)}\right)' = \frac{\frac{2 \cos(2x)}{\sin(2x) \ln(7)} \cdot \sec(2x) - \log_7(\sin(2x)) \cdot \tan(2x) \sec(2x) \cdot 2}{\sec^2(2x)} \\ &= \frac{2 \cot(2x) \cos(2x) - 2 \ln(\sin(2x)) \sin(2x)}{\ln(7)} \\ &= \frac{2}{\ln(7)} [\cot(2x) \cos(2x) - \ln(\sin(2x)) \sin(2x)] \end{aligned}$$

If $f :]x_0, x_1[\subset \mathbb{R} \rightarrow \mathbb{R} : x \mapsto y = f(x)$ is differentiable over $]x_0, x_1[$ and $f'(x)$ is again differentiable, we can formulate the **second differential**, which is denoted by d^2y , as follows, bearing hereby in mind that we take the change in the variable x , i.e., $dx = \Delta x$, to be constant at every differentiation step:

$$\begin{aligned} d^2y &= d(dy) = d(f'(x)dx) = (f'(x)dx)' dx = (f''(x)dx + f'(x)d(dx)) dx \\ &= (f''(x)dx + f'(x) \cdot 0) dx \\ &= f''(x)dx^2 \end{aligned}$$

The expression $f''(x)$ is called the **second derivative** and takes the following form:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}(x)$$

Take, for instance, the function $f : [-1, 1] \rightarrow \mathbb{R} : x \mapsto f(x) = \sqrt{1-x^2}$. The first derivative is equal to:

$$f'(x) = \left(\sqrt{1-x^2}\right)' = \frac{-x}{\sqrt{1-x^2}}$$

The function $f' :]-1, 1[\rightarrow \mathbb{R} : x \mapsto f'(x)$ is differentiable in every point of the interval $] -1, 1[$ (you can verify this with the help of the limit definition of the derivative) so that the second derivative becomes:

$$f''(x) = \left(\frac{-x}{\sqrt{1-x^2}}\right)' = \frac{-\sqrt{1-x^2} - \frac{(-x)(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)} = \frac{-1}{(1-x^2)\sqrt{1-x^2}}$$

Generally, the n^{th} **derivative** of a function f exists if f is $(n-1)$ times differentiable. We write the n^{th} derivative as follows:

$$f^{(n)}(x) = \frac{d^{(n)}y}{dx^{(n)}} = \frac{d^{(n)}f}{dx^{(n)}}(x)$$

If we wish to calculate the n^{th} derivative of the product of two functions f and g , we can use the **Leibniz formula**:

$$(f(x)g(x))^{(n)} = \sum_{i=0}^n \binom{n}{i} f^{(i)}(x)g^{(n-i)}(x) \quad \text{with} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Let us for example calculate the third derivative of the product of the functions $f : \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto f(x) = \log_9(x)$ and $g : \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto g(x) = 9^x$:

$$\begin{aligned} (\log_9(x) \cdot 9^x)^{(3)} &= \sum_{i=0}^3 \binom{3}{i} (\log_9 x)^{(i)} (9^x)^{(3-i)} \\ &= 1 \cdot \log_9 x \cdot 9^x (\ln 9)^3 + 3 \cdot \frac{1}{x \ln 9} \cdot 9^x (\ln 9)^2 + 3 \cdot \frac{(-1)}{x^2 \ln 9} \cdot 9^x \ln 9 + 1 \cdot \frac{2}{x^3 \ln 9} \cdot 9^x \\ &= 9^x \left[4(\ln 3)^2 \ln x + \frac{6 \ln 3}{x} - \frac{3}{x^2} + \frac{1}{x^3 \ln 3} \right] \end{aligned}$$

L'Hôpital's rule is a useful tool that can be used when we encounter undefined expressions during the calculation of limits (see section 7.2.2). If two functions f and g are differentiable in a vicinity of a (but not necessarily at point a), and $g'(x) \neq 0$ in that interval, and if the limit $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists, we know that:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Consider, for example, the following function h for which we wish to examine the limit $\lim_{x \rightarrow \frac{\pi}{4}} h(x)$:

$$h : \mathbb{R} \setminus \left\{ -\frac{7\pi}{4} + 2k\pi, -\frac{\pi}{4} + 2m\pi \right\} \rightarrow \mathbb{R} : x \mapsto h(x) = \frac{-\cot x + e^{x-\frac{\pi}{4}}}{\sin(2x) - 2 \tan x + \frac{\sqrt{2}}{2} \csc x}$$

whereby $k, m \in \mathbb{Z}$ and $k \neq 1$. Directly substituting $\frac{\pi}{4}$ for x in the limit produces the undefined expression $\frac{0}{0}$. We find the limit by applying L'Hôpital's rule:

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{-\cot x + e^{x-\frac{\pi}{4}}}{\sin(2x) - 2 \tan x + \frac{\sqrt{2}}{2} \csc x} \right) &= \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{\csc^2 x + e^{x-\frac{\pi}{4}}}{2 \cos(2x) - 2 \sec^2 x - \frac{\sqrt{2}}{2} \cot(x) \csc(x)} \right) \\ &= \frac{2 + 1}{2 \cdot 0 - 2 \cdot 2 - \frac{\sqrt{2}}{2} \cdot 1 \cdot \sqrt{2}} \\ &= -\frac{3}{5} \end{aligned}$$

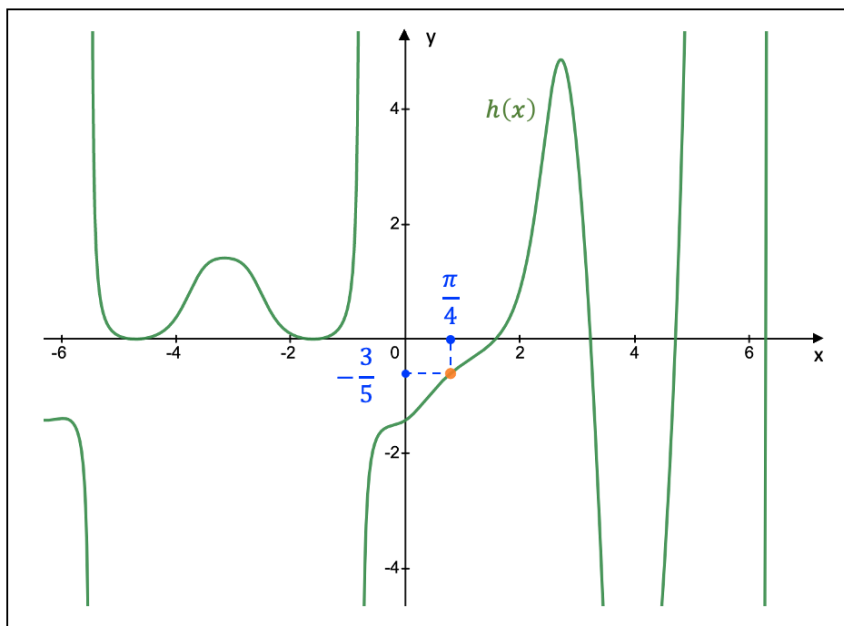


Figure 7.20: The graphical representation of $\lim_{x \rightarrow \frac{\pi}{4}} h(x)$

Finally, when solving limits, it can be useful to know that **Euler's number** $e = 2.718281828\dots$ can be written as the following limit:

$$e = \lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x} \right)^x \quad \text{or} \quad e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}}$$

If we take, for instance, the following function f , for which we wish to calculate the limit $\lim_{x \rightarrow 1} f(x)$:

$$f :]-\infty, 2] \setminus \{1\} \rightarrow \mathbb{R} : x \mapsto f(x) = (1 + \ln 2 - x \ln 2)^{\frac{1}{1-x}}$$

If we make the substitution $s = (1 - x) \ln 2$, we solve the limit as follows:

$$\begin{aligned} \lim_{x \rightarrow 1} (1 + \ln 2 - x \ln 2)^{\frac{1}{1-x}} &= \lim_{s \rightarrow 0} (1 + s)^{\frac{\ln 2}{s}} \\ &= \left(\lim_{s \rightarrow 0} (1 + s)^{\frac{1}{s}} \right)^{\ln 2} \\ &= e^{\ln 2} \\ &= 2 \end{aligned}$$

7.5.3 Taylor's Theorem

If $p : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto p(x)$ is a polynomial of degree n (see section 2.3), **Taylor's formula** rewrites $p(x)$ in terms of derivatives in the following way:

$$p(x) = \sum_{i=0}^n \frac{p^{(i)}(a)}{i!} (x - a)^i$$

whereby $a \in \mathbb{R}$, $p^{(i)}$ represents the i^{th} derivative of $p(x)$, and $i! = 1 \cdot 2 \cdot \dots \cdot (i - 1) \cdot i$ refers to the factorial of i .

Based on Taylor's formula, we can state **Taylor's theorem**, which provides an approximation of a function f around a certain point a by expressing f as a polynomial and a remainder term. Because a function is rewritten as a sum of many terms, applying Taylor's theorem is also referred to as a **Taylor series expansion**.

Taylor's theorem says that if f is n times differentiable over an interval $]x_0, x_1[$, whereby $a, x \in]x_0, x_1[$, there exists a real number $c \in]a, x[$ so that f can be written as:

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x - a)^i + r_n(x) \quad \text{with } r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - a)^{n+1}$$

The remainder term $r_n(x)$ formulated in this way is known as the **Lagrange form**.

Consider for example the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : x \mapsto f(x) = \sqrt{x}$. If we wish to write down the Taylor series expansion of f for $a = 8$, we obtain:

$$\begin{aligned} f(x) = \sqrt{x} &= \sqrt{8} + \frac{1}{2\sqrt{8}}(x - 8) - \frac{1}{64\sqrt{8}}(x - 8)^2 + \frac{1}{1024\sqrt{8}}(x - 8)^3 + \dots + \\ &\left[\frac{(-1)^{n-1}}{2^n} \left(\prod_{s=1}^n |2s - 3| \right) 8^{-\frac{(2n-1)}{2}} \right] \frac{(x - 8)^n}{n!} + r_n(x) \end{aligned}$$

whereby $\prod_{s=1}^n |2s - 3| = (|2 \cdot 1 - 3|) \cdot (|2 \cdot 2 - 3|) \cdot (|2 \cdot 3 - 3|) \cdot \dots \cdot (|2 \cdot n - 3|)$ and $n \geq 1$. Fig. 7.21 shows how the 3rd-order Taylor polynomial (in green), which is represented by the function $g : \mathbb{R} \rightarrow \mathbb{R}$, approximates the function f (in purple) around the point $a = 8$.

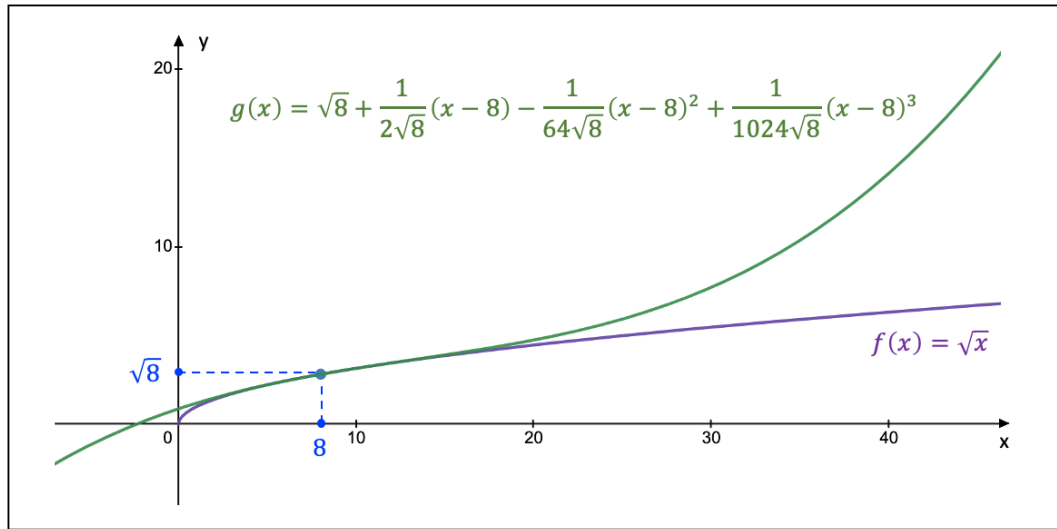


Figure 7.21: The 3rd-order Taylor expansion of the function f around $a = 8$

If $a = 0$ in Taylor's series expansion, the expression of Taylor's theorem is often referred to as the **Maclaurin formula**, which is written as follows:

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(0)}{i!} x^i + r_n(x) \quad \text{with } r_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} \text{ and } 0 < \theta < 1$$

Let us, for instance, write down the Maclaurin formula for the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = \cos(\sin(x))$ to the 4th order. We first calculate the first four derivatives of f in point $a = 0$:

- $f'(x) = -\sin(\sin(x)) \cos(x)$
 $\Rightarrow f'(0) = 0$
- $f''(x) = -\cos(\sin(x)) \cos^2(x) + \sin(\sin(x)) \sin(x)$
 $\Rightarrow f''(0) = -1$
- $f'''(x) = \sin(\sin(x)) \cos(x) [\cos^2(x) + 1] + 3 \cos(\sin(x)) \sin(x) \cos(x)$
 $\Rightarrow f'''(0) = 0$
- $f^{(4)}(x) = \cos(\sin(x)) [\cos^4(x) + \cos^2(x) + 3 \cos(2x)] -$
 $\sin(\sin(x)) \sin(x) [6 \cos^2(x) + 1]$
 $\Rightarrow f^{(4)}(0) = 5$

The Maclaurin formula of the function f to the 4th order, whereby $f(0) = \cos(\sin(0)) = 1$, is then equal to:

$$f(x) = \cos(\sin(x)) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{f^{(6)}(\theta x)}{720}x^6$$

Note that the remainder term is of order 6 and not 5, because the Taylor polynomial of this particular function f contains only terms of even degrees. Fig. 7.22 demonstrates how the Taylor polynomial (in green) approximates the function f (in purple) at the point $a = 0$.

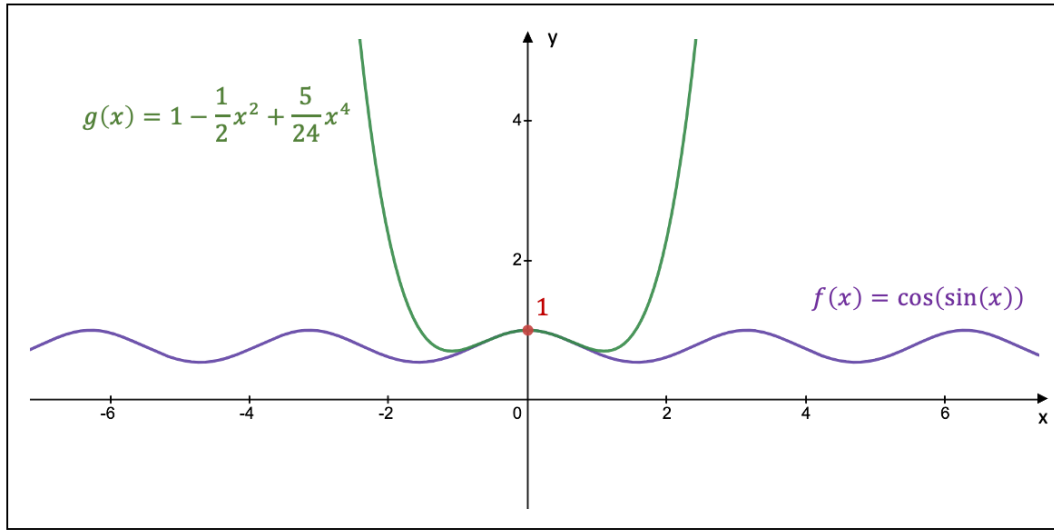


Figure 7.22: The 4th-order Taylor expansion of the function f around $a = 0$

If the remainder term in Taylor's theorem takes on the **Liouville form** instead of the Lagrange form, $r_n(x)$ is expressed in the following manner:

$$f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + r_n(x) \quad \text{with } r_n(x) = \frac{\lambda(x)}{n!} (x-a)^n \text{ and } \lim_{x \rightarrow a} \lambda(x) = 0$$

For example, consider the function $f : \mathbb{R} \rightarrow \mathbb{R}^+ : x \mapsto f(x) = 5^{2x}$. The Taylor expansion of f at the point $a = -1$ with a Liouville remainder term has the following form:

$$f(x) = 5^{2x} = \frac{1}{25} + \frac{2 \ln 5}{25} (x+1) + \frac{2 (\ln 5)^2}{25} (x+1)^2 + \dots + \frac{2^n (\ln 5)^n}{25 \cdot n!} (x+1)^n + \frac{\lambda(x)}{n!} (x+1)^n$$

Fig. 7.23 shows how the 3rd-order Taylor polynomial (in green) approximates the function f (in purple) at the point $a = -1$.

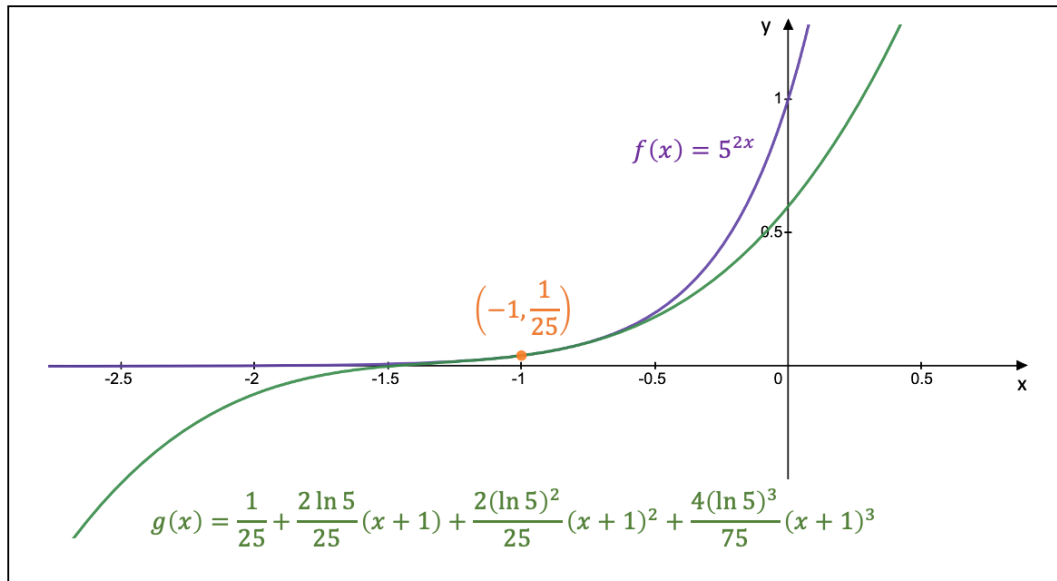


Figure 7.23: The 3rd-order Taylor expansion of the function f around $a = -1$

Instead of using L'Hôpital's rule to calculate limits of undefined expressions, we can also rely on Taylor's theorem (with a Liouville remainder). Consider, for example, the following function f (see Fig. 7.24), whereby $k \in \mathbb{Z}$:

$$f : \mathbb{R} \setminus \{-4 + 16k, -\frac{3}{2}, 2, 4 + 16k\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{\sin\left(\frac{3\pi}{4}x\right) + \tan\left(\frac{\pi}{8}x\right)}{e^{(x^2-4)} - e^{\left(\frac{x}{2}-1\right)}}$$

If we would like to find the limit $\lim_{x \rightarrow 2} f(x)$, directly substituting 2 for x produces the undefined expression $\frac{0}{0}$. Instead, we replace every term, both in the numerator and the denominator, with its corresponding, say, 2nd-order Taylor polynomial expanded around the point $a = 2$. The limit $\lim_{x \rightarrow 2} f(x)$ is then found as follows:

$$\begin{aligned} & \lim_{x \rightarrow 2} \frac{\sin\left(\frac{3\pi}{4}x\right) + \tan\left(\frac{\pi}{8}x\right)}{e^{(x^2-4)} - e^{\left(\frac{x}{2}-1\right)}} \\ &= \lim_{x \rightarrow 2} \frac{\left[-1 + \frac{9\pi^2}{32}(x-2)^2 + \frac{\lambda(x)}{2}(x-2)^2\right] + \left[1 + \frac{\pi}{4}(x-2) + \frac{\pi^2}{32}(x-2)^2 + \frac{\lambda(x)}{2}(x-2)^2\right]}{\left[1 + 4(x-2) + 9(x-2)^2 + \frac{\lambda(x)}{2}(x-2)^2\right] - \left[1 + \frac{1}{2}(x-2) + \frac{1}{8}(x-2)^2 + \frac{\lambda(x)}{2}(x-2)^2\right]} \\ &= \lim_{x \rightarrow 2} \frac{\frac{\pi}{4}(x-2) + \frac{5\pi^2}{16}(x-2)^2 + \lambda(x)(x-2)^2}{\frac{7}{2}(x-2) + \frac{71}{8}(x-2)^2} \\ &= \lim_{x \rightarrow 2} \frac{\frac{\pi}{4} + \frac{5\pi^2}{16}(x-2) + \lambda(x)(x-2)}{\frac{7}{2} + \frac{71}{8}(x-2)} \\ &= \frac{\pi}{14} \end{aligned}$$

Fig. 7.24 depicts the function f , whereby we can see that f approaches the value of $\frac{\pi}{14}$ when

x gets ever closer to $x = 2$.

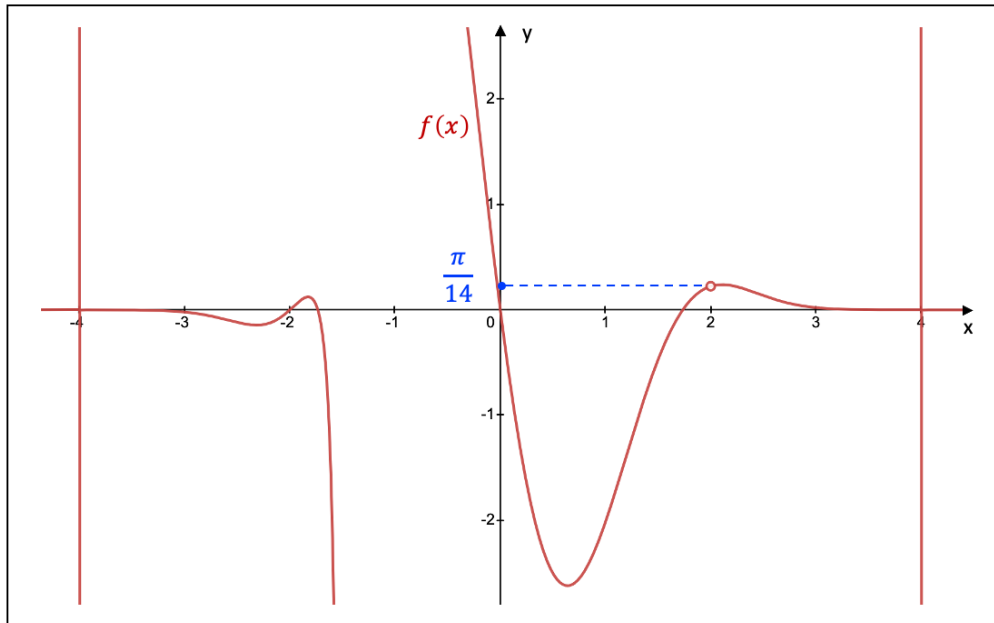


Figure 7.24: The graphical representation of $\lim_{x \rightarrow 2} f(x)$

7.5.4 Extremum Points, Tangent Lines, and Inflection Points

Extremum points are points of a function $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ at which f attains a (local) minimum or (local) maximum. If $a \in A$ and we can identify an interval $O_a =]a - \delta, a + \delta[$, we can say that, $\forall x \in O_a$:

$$\begin{cases} \text{if } f(x) \leq f(a) & \Rightarrow f \text{ attains a (local) maximum in } a \\ \text{if } f(x) \geq f(a) & \Rightarrow f \text{ attains a (local) minimum in } a \end{cases}$$

If f attains an extremum in point a and f is differentiable in a , it follows that $f'(a) = 0$. The points a for which $f'(a) = 0$ are called **stationary points**.

But the existence of stationary points does not necessarily imply that these stationary points are extremum points. Consider, for example, the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 5(x - 1)^5 + 1$. We find that $f'(x) = 25(x - 1)^4$, so that $f'(1) = 0$, but Fig. 7.25 shows clearly that f does not attain an extremum in the stationary point $x = 1$.

Therefore, we need the following condition to make sure we are dealing with an extremum point.

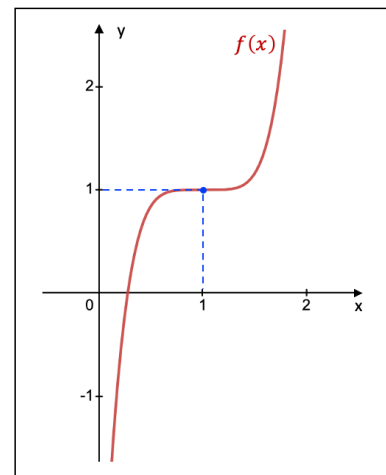


Figure 7.25: The graph of $f(x)$

$$\text{if } \begin{cases} f'(x) > 0 & \text{for } x < a \\ f'(x) < 0 & \text{for } x > a \end{cases} \Rightarrow f \text{ attains a (local) maximum in } a$$

$$\text{if } \begin{cases} f'(x) < 0 & \text{for } x < a \\ f'(x) > 0 & \text{for } x > a \end{cases} \Rightarrow f \text{ attains a (local) minimum in } a$$

For the above example, we can see that the sign of $f'(x)$ does not change for any chosen interval around the stationary point $x = 1$. For instance, for $\delta = 0.5$, we find that $f'(0.5) = f'(1.5) = 1.5625 > 0$, so that no extremum is found at $x = 1$.

From Fig. 7.17 (see section 7.5.1) we can see that the derivative of f in point a represents the slope of the **tangent**, which is the line tangent to f at point a .

Since the derivative of f at an extremum point a is equal to zero, the tangent through a is a horizontal line, i.e., its slope is zero. However, if the derivative of f at a certain point $x = c$ is equal to $\pm\infty$, the tangent is a vertical line. Moreover, from section 7.3.2 we know that if $\lim_{x \rightarrow c} f'(x) = \pm\infty$, $x = c$ represents a vertical asymptote of the function $f'(x)$.

As an example, let us consider the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = \sqrt[3]{-x^3 + 6x^2 - 11x + 6}$, whose graph is shown in orange in Fig. 7.26.

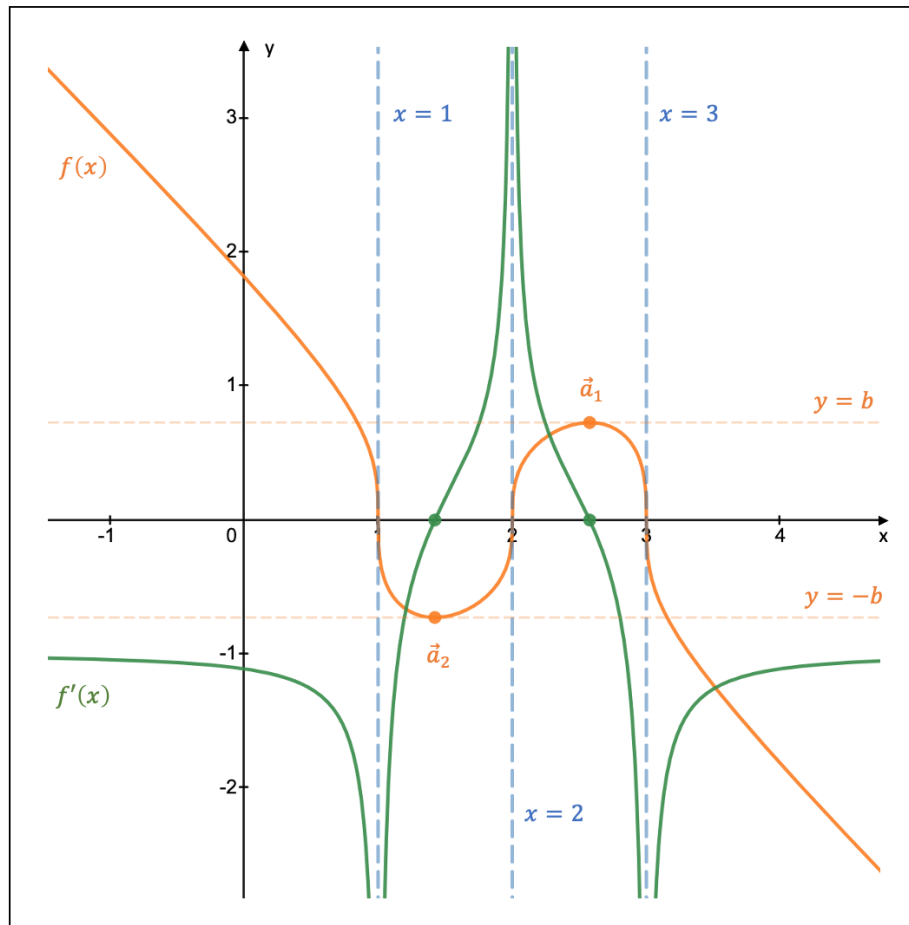


Figure 7.26: An example of horizontal and vertical tangent lines

The derivative of f is equal to:

$$f'(x) = \frac{-3x^2 + 12x - 11}{3\sqrt[3]{(-x^3 + 6x^2 - 11x + 6)^2}}$$

The graph of $f'(x)$ is shown in green in Fig. 7.26. The polynomial in the numerator of $f'(x)$ becomes zero if $x = a_1 = 2 + \frac{\sqrt{3}}{3}$ or $x = a_2 = 2 - \frac{\sqrt{3}}{3}$. The corresponding function values are $y = b = 0.727$ and $y = -b = -0.727$.

If we choose $\delta = 0.1$, we find that $f'(a_1 - 0.1) > 0$ and $f'(a_1 + 0.1) < 0$, so that the point $\vec{a}_1 = (a_1, b)$ corresponds with a local maximum. Similarly, we find that $f'(a_2 - 0.1) < 0$ and $f'(a_2 + 0.1) > 0$, so that $\vec{a}_2 = (a_2, -b)$ represents a local minimum.

The horizontal orange lines $y = b$ and $y = -b$ in Fig. 7.26 are the tangents of f at the extremum points \vec{a}_1 and \vec{a}_2 , respectively.

The function f is not differentiable at the points $x = 1$, $x = 2$, and $x = 3$. This means that at these points the function $f'(x)$ has vertical asymptotes (the blue dashed lines in Fig. 7.26), which at the same time represent vertical tangents of the function $f(x)$.

In a second example, we will address a different kind of problem. Given is the function $f : \mathbb{R} \setminus \{4\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{x^2 - 3}{x + 4}$ and we wish to find the equation of the line(s) tangent to f and going through the external point $\vec{x} = (-12, -5)$.

If we define the point $\vec{x}_0 = (x_0, y_0)$ as the point at which the line is tangent to f , we know that \vec{x}_0 belongs to f , so that we can write:

$$y_0 = \frac{x_0^2 - 3}{x_0 + 4}$$

Next, we calculate the derivative of f :

$$f'(x) = \frac{2x(x + 4) - (x^2 - 3)(1)}{(x + 4)^2} = \frac{x^2 + 8x + 3}{(x + 4)^2}$$

The slope of f at $x = x_0$ is then equal to $f'(x_0) = \frac{x_0^2 + 8x_0 + 3}{(x_0 + 4)^2}$. By writing down the equation of the tangent with slope $f'(x_0)$ and going through the point $\vec{x} = (-12, -5)$, we find the value(s) of x_0 :

$$\begin{aligned} y - y_0 &= f'(x_0)(x - x_0) \Leftrightarrow (-5) - \left(\frac{x_0^2 - 3}{x_0 + 4}\right) = \left[\frac{x_0^2 + 8x_0 + 3}{(x_0 + 4)^2}\right](-12 - x_0) \\ &\Leftrightarrow -5(x_0 + 4)^2 - (x_0^2 - 3)(x_0 + 4) = (x_0^2 + 8x_0 + 3)(-12 - x_0) \\ &\Leftrightarrow 11x_0^2 + 62x_0 - 32 = 0 \end{aligned}$$

The solutions to this quadratic equation are $x_{01} = \frac{-31 + \sqrt{1313}}{11}$ and $x_{02} = \frac{-31 - \sqrt{1313}}{11}$. We therefore have two tangents L_1 and L_2 with a slope equal to $f'(x_{01}) = 0.351$ and $f'(x_{02}) = -1.914$, respectively.

In a final step, we construct the equation of the tangents L_1 and L_2 : $L_1 : y + 5 = 0.351(x + 12)$ and $L_2 : y + 5 = -1.914(x + 12)$. Fig. 7.27 depicts the graph of $f(x)$ (in blue) as well as the two tangents (in green).

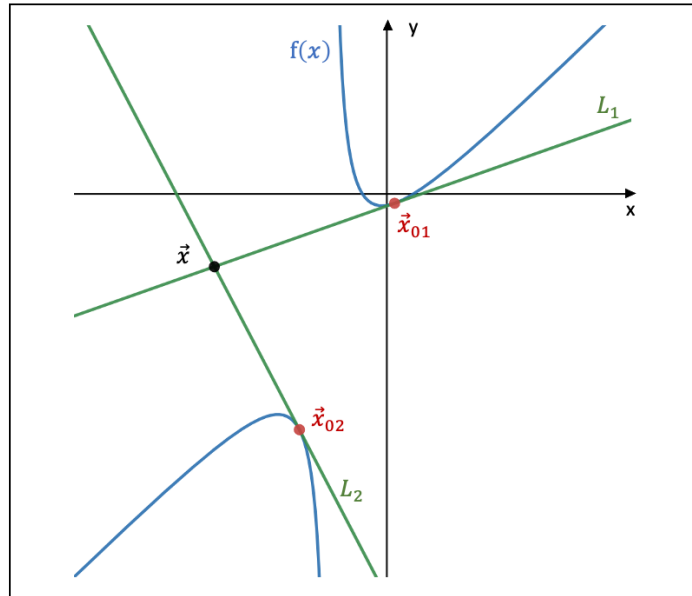


Figure 7.27: Drawing tangents from an external point

Consider the interval $A =]x_1, x_2[$. We call a function f **strictly increasing** (**strictly decreasing**) over A if $f'(x) > 0$ ($f'(x) < 0$) for every point $x \in A$.

If $f'(x)$ is strictly increasing over A , i.e., $\forall x \in A : f''(x) > 0$, we call the function f **convex**. Similarly, if $f'(x)$ is strictly decreasing over A , i.e., $\forall x \in A : f''(x) < 0$, we call the function f **concave**.

An **inflection point** x_i of a function f is a point that satisfies two conditions:

- f either is differentiable in x_i or possesses a vertical tangent ($\lim_{x \rightarrow x_i} f'(x) = \pm\infty$)
- f is convex (concave) to the left of x_i and concave (convex) to the right of x_i

If x_i is an inflection point, it follows that $f''(x_i) = 0$. However, $f''(a) = 0$ does not necessarily imply that a is an inflection point.

For example, consider the function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{6(x+1)^6}{x^4} - 1$. The second derivative of f is equal to

$$f''(x) = \frac{12(x+1)^4}{x^6}(x^2 - 4x + 10)$$

Even though $f''(-1) = 0$, $x = -1$ is not an inflection point, since $f''(x) > 0$ for both $x < -1$ and $x > -1$, which contradicts the definition.

Fig. 7.28 shows the graph of both $f(x)$ (in red) and $f''(x)$ (in blue), whereby $\vec{a} = (-1, -1)$ and $\vec{b} = (-1, 0)$.

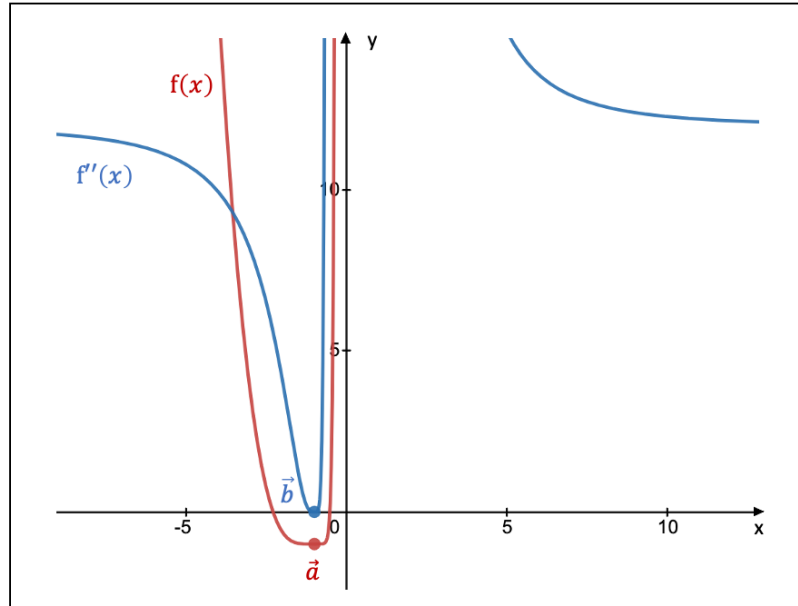


Figure 7.28: The graph of $f(x)$ and $f''(x)$

Note furthermore that if $\lim_{x \rightarrow a^-} f'(x) = +\infty$ and $\lim_{x \rightarrow a^+} f'(x) = -\infty$, or vice versa, the function $f'(x)$ displays a vertical asymptote $x = a$ and the vertical line in the graph of the function $f(x)$ is referred to as a **vertical cusp** (and not a vertical tangent).

Let us consider the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = (x+6)\sqrt[3]{(x^2-1)^2}$. Its first and second derivative are equal to:

$$f'(x) = \frac{7x^2 + 24x - 3}{3\sqrt[3]{x^2 - 1}} \quad \text{and} \quad f''(x) = \frac{4(7x^3 + 6x^2 - 9x - 18)}{9(x^2 - 1)\sqrt[3]{x^2 - 1}}$$

Given that $\lim_{x \rightarrow -1^-} f'(x) = -\infty$ and $\lim_{x \rightarrow -1^+} f'(x) = +\infty$ and also that $\lim_{x \rightarrow 1^-} f'(x) = -\infty$ and $\lim_{x \rightarrow 1^+} f'(x) = +\infty$, we find that $x = -1$ and $x = 1$ represent vertical cusps in the graph of f (the green dashed lines in Fig. 7.29).

Moreover, the function f has an inflection point at $\vec{x} = (1.392, 7.088)$ because f is differentiable at $x = 1.392$ and because f is concave to the left ($f''(x) < 0$) and convex to the right ($f''(x) > 0$) of this point. We have found this point by setting $f''(x) = 0$.

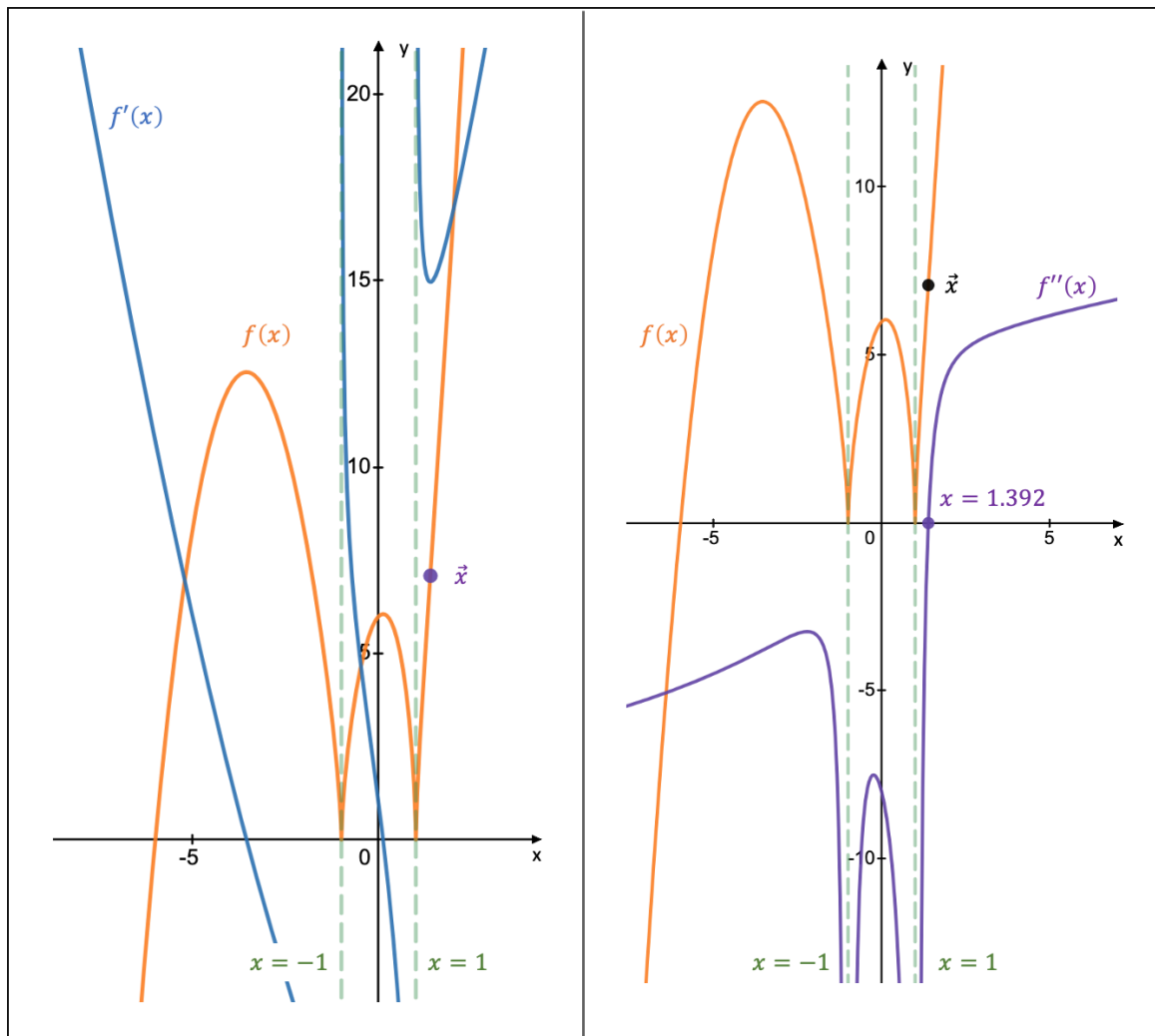


Figure 7.29: Vertical cusps and an inflection point

7.6 Different Types of Functions

7.6.1 Zero-Degree Polynomial Functions

A **zero-degree polynomial function** is an even function with the general form $f(x) = a$ whereby a represents any real number. Graphically, this function corresponds to a horizontal line that intersects the vertical axis at the point a .

The **root** of such function is equal to all real numbers if $a = 0$. In the case that $a \neq 0$, the function has no roots as it does not intersect the horizontal axis.

Fig. 7.30 demonstrates the example of $f(x) = 4$.

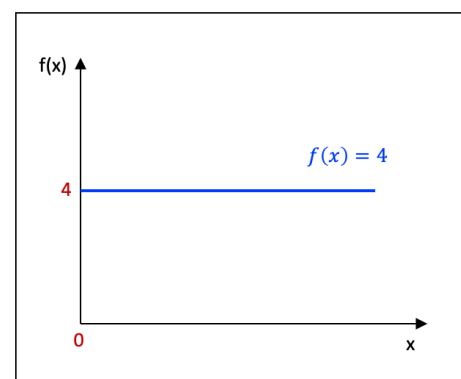


Figure 7.30: A graphical representation of the function $f(x) = 4$

7.6.2 First-Degree Polynomial Functions

A **first-degree polynomial function** has the general form $f(x) = ax + b$ (with a and b real numbers and $a \neq 0$) and is graphically displayed as a non-vertical **straight line**. The coefficient a is called the gradient or slope and b the intercept. The **gradient** measures the rate of change of the function f and is calculated by taking the derivative of f . The **intercept** indicates at what value the function f intersects the vertical coordinate axis $x = 0$.

Let us consider two examples (see Fig. 7.31). In a first example, we look at the polynomial function $f(t) = v(t) = 0.78t + 0.5$, whereby $v(t)$ represents the speed of an object (in $\text{m} \cdot \text{s}^{-1}$) and t the time variable. The slope and the intercept are equal to $a = 0.78$ and $b = 0.5$, respectively.

In a speed-time graph, the slope of the function reflects the acceleration a_c of the object, so that in our example the object is accelerating at a constant rate of $a_c = a = 0.78 \text{ m} \cdot \text{s}^{-2}$. At for instance $t = 10 \text{ s}$, the speed equals $v = 0.78t + 0.5 = 0.78 \cdot 10 + 0.5 = 8.3 \text{ m} \cdot \text{s}^{-1}$. The area S under the graph and above the horizontal time axis, constrained by the time interval $[0, 10]$, represents the total distance d covered by the object. That is, $S = d = \frac{1}{2}(v - 0.5)t + 0.5t = \frac{1}{2} \cdot (8.3 - 0.5) \cdot 10 + 0.5 \cdot 10 = 44 \text{ m}$.

As a second example, we take Hooke's Law, which describes the amount of force F that is required to compress or extend a spring over a displacement Δx . This physical law is represented by the polynomial function $f(x) = F(x) = k \Delta x$, whereby the slope $a = k$ reflects the stiffness of the spring. Since the intercept b is equal to zero, we know that the straight line passes through the origin of our coordinate system.

To extend the spring by a distance of, for instance, $\Delta x = 0.35 \text{ m}$ —and assuming a spring constant of $k = 12.3 \text{ N} \cdot \text{m}^{-1}$ —we need to apply a force of $F = k \Delta x = 12.3 \cdot 0.35 = 4.3 \text{ N}$. In a graph that displays Hooke's Law, the area under the graph represents the work done by the force F . In our example, the work done by F is equal to $A = \frac{1}{2} F \Delta x = \frac{1}{2} (k \Delta x) \Delta x = \frac{1}{2} (12.3 \cdot 0.35) \cdot 0.35 = 0.75 \text{ N} \cdot \text{m}$. (Note that we have neglected the vector notation of the force F).

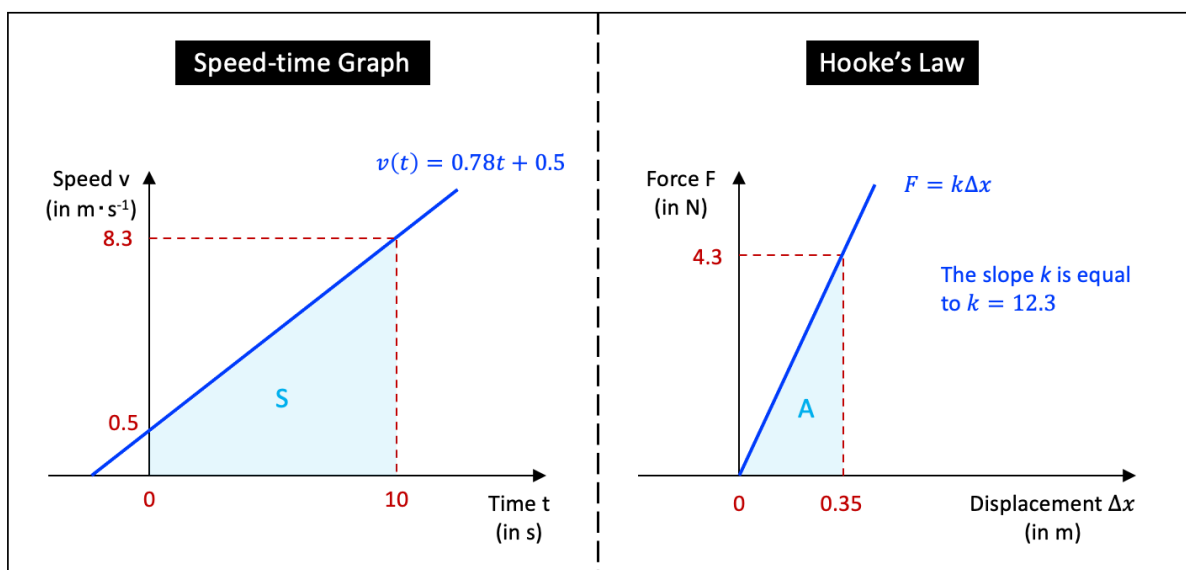


Figure 7.31: Two examples of a first-degree polynomial function

The **root** of a first-degree polynomial function $f(x) = ax + b$ can be found by solving the equation $f(x) = 0$. The general answer is equal to $x = -\frac{b}{a}$. For instance, the root of the above equation $v(t) = 0.78t + 0.5$ is calculated as $t = -\frac{b}{a} = -\frac{0.5}{0.78} = -0.64$ s.

7.6.3 Quadratic Functions

A **second-degree polynomial function a.k.a. a quadratic function** can be generally written as $f(x) = ax^2 + bx + c$ (with a , b , and c real numbers and $a \neq 0$), whereby the corresponding graph has the form of a **parabola**. If $a > 0$ ($a < 0$), the parabola will be open upwards (downwards).

If we wish to identify the **roots** of a quadratic function, it is easier if we first factor the polynomial function (see Section 2.3.3). For instance, factoring the quadratic function $f(x) = 6x^2 - 5x - 6$ results in the altered form $f(x) = (2x - 3)(3x + 2)$, so that the roots are equal to $x = \frac{3}{2}$ and $x = -\frac{2}{3}$.

To construct the **tangent** of the quadratic function at a certain point (x_1, y_1) , we first calculate the gradient by taking the derivative $f'(x)$ of the function. In the general case, we find that $f'(x) = 2ax + b$. The slope of the tangent at that point is then equal to $f'(x_1) = 2ax_1 + b$. The equation of the tangent y_t can then be written as $y_t - y_1 = f'(x_1)(x - x_1) \Leftrightarrow y_t = (2ax_1 + b)(x - x_1) + y_1$.

To continue with our previous example, the gradient is equal to $f'(x) = 12x - 5$ and the slope of the tangent at the point $(0, -6)$ is calculated as $f'(0) = -5$. As a result, the equation of the tangent y_t has the form $y_t - (-6) = (-5)(x - 0) \Leftrightarrow y_t = -5x - 6$.

The tangent at the **top or bottom point** of the parabola is a horizontal line, which has a slope equal to zero. Therefore, the corresponding x -coordinate is calculated as $f'(x) = 0 = 2ax + b \Leftrightarrow x = -\frac{b}{2a}$. Inserting this value for x in the general form of the quadratic function gives us the y -coordinate $y = c - \frac{b^2}{4a}$.

In the above example, $a = 6 > 0$ which means that the parabola is open upwards. We then know that the parabola has a bottom point, which is located at the position $(\frac{5}{12}, -\frac{169}{24})$.

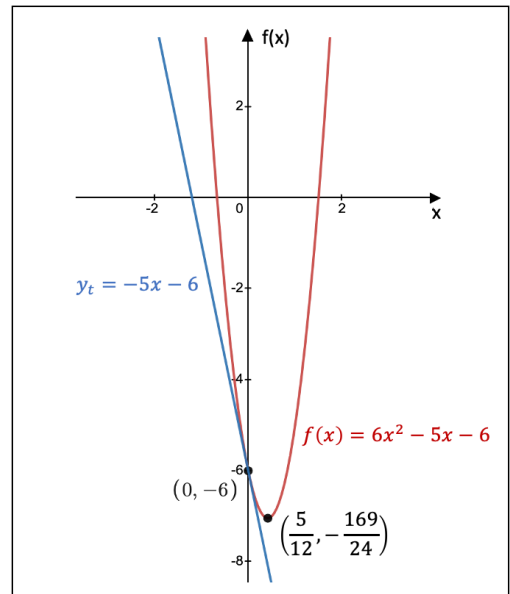


Figure 7.32: A graphical representation of the function $f(x) = 6x^2 - 5x - 6$

The **shape and the position** of the parabola can be altered in various ways. To move the parabola horizontally, the argument x must be replaced by the argument $x + h$ with h any real number, whereby the parabola shifts to the right (left) if $h < 0$ ($h > 0$). To change its location in the vertical direction, the function's intercept c has to be substituted by $c + v$ with v any real number. The parabola moves upwards if $v > 0$ and downwards if $v < 0$.

To modify the parabola's width, we have to multiply the argument x with a real number s . To obtain a narrower parabola, s has to satisfy the inequality $s > 1$, whereas the parabola grows wider if $0 < s < 1$. If we wish to flip the parabola upside down, we change $f(x)$ into $-f(x)$, which is called a reflection of the function f over the x -axis.

As an example, Fig. 7.33 showcases different changes in the shape and position of the polynomial function $f(x) = 3x^2 - 4x + 1$.

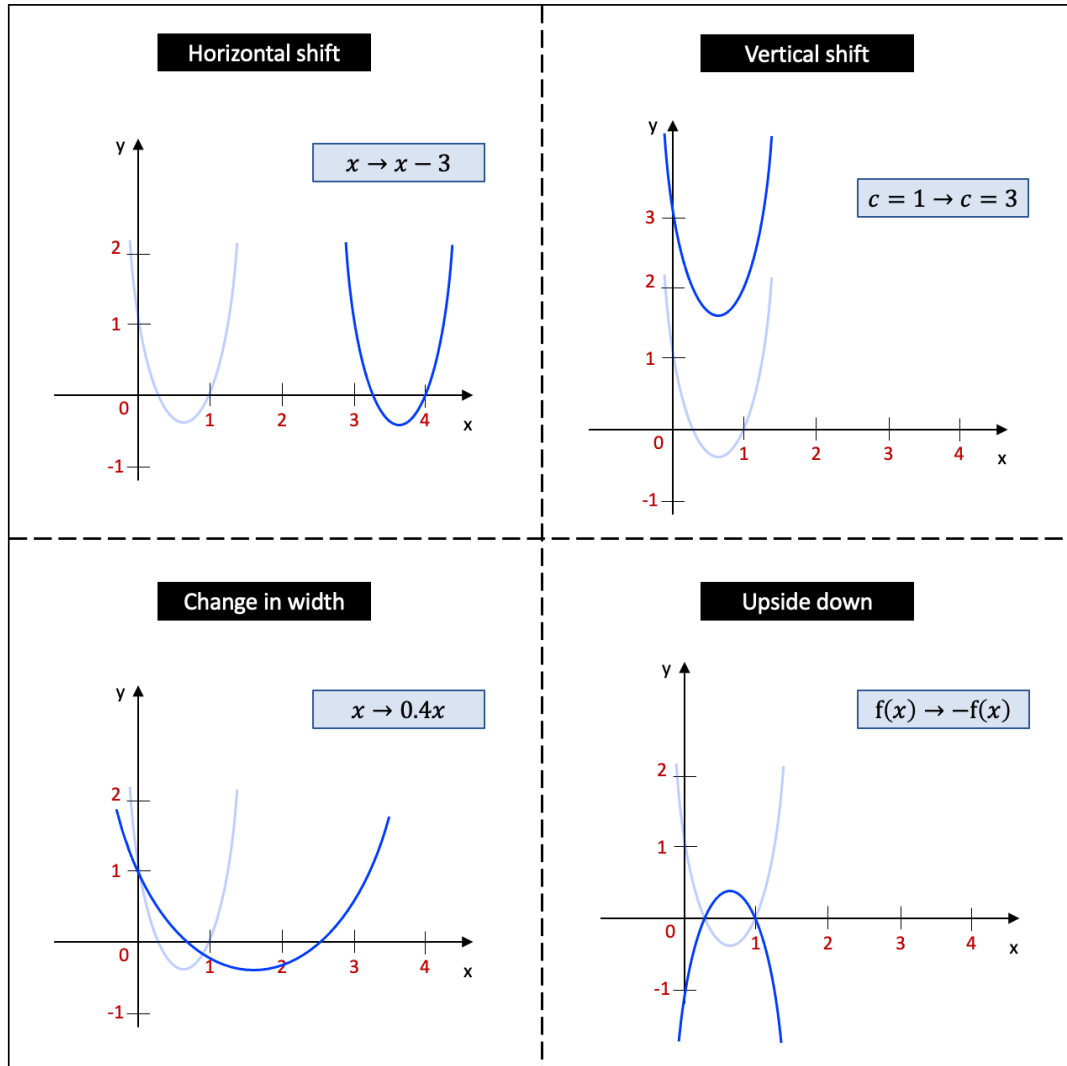


Figure 7.33: Modifications of the shape and position of the quadratic function $f(x) = 3x^2 - 4x + 1$

7.6.4 Higher-Degree Polynomial Functions

An n^{th} -degree polynomial function is a polynomial function whereby the highest power of the unknown variable is equal to n .

A polynomial function of degree n has a maximum of n roots and if n is an odd real number, the function has at least one root.

Fig. 7.34 shows the graph of two higher-degree polynomial functions. The one on the left is

a fifth-degree polynomial function and has 3 roots, whereas the function on the right is a fourth-degree polynomial function with 4 roots.

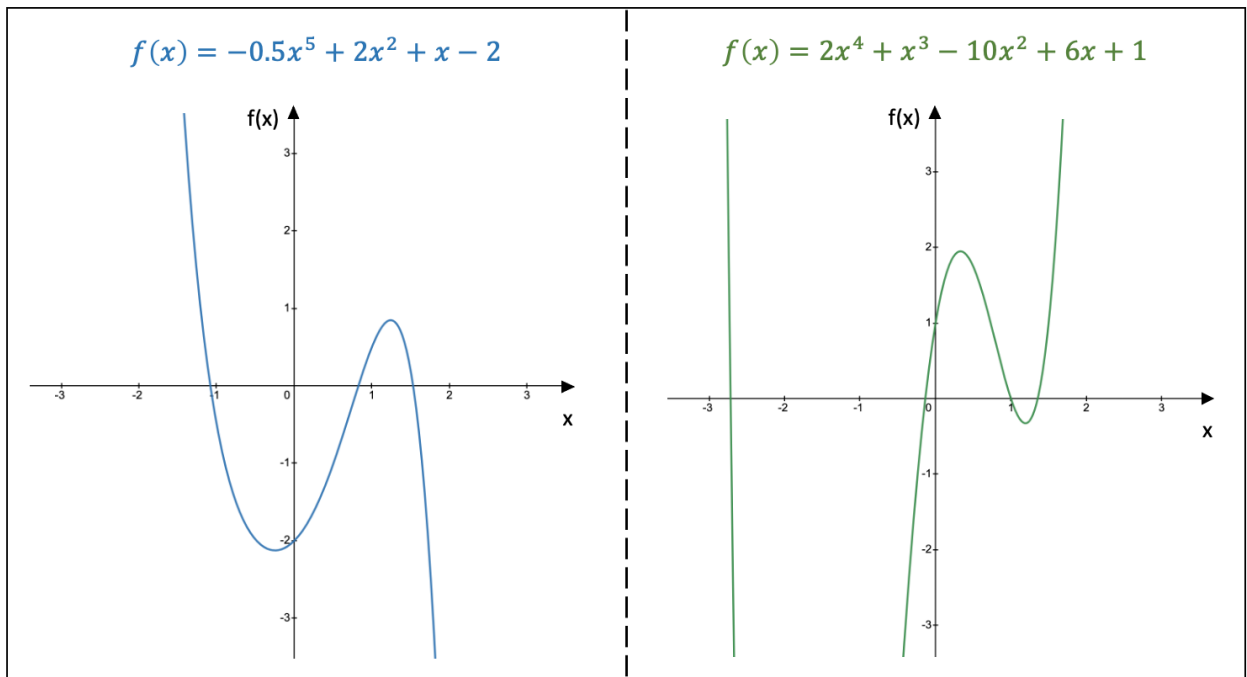


Figure 7.34: A graphical representation of the function $f(x) = -0.5x^5 + 2x^2 + x - 2$ (left) and the function $f(x) = 2x^4 + x^3 - 10x^2 + 6x + 1$ (right)

7.6.5 Rational and Irrational Functions

A **rational function** is a real function of the general form $f(x) = \frac{p(x)}{q(x)}$, whereby $p(x)$ and $q(x)$ are polynomial functions. The domain of a rational function is defined as the set of real numbers excluding the roots of the function $q(x)$.

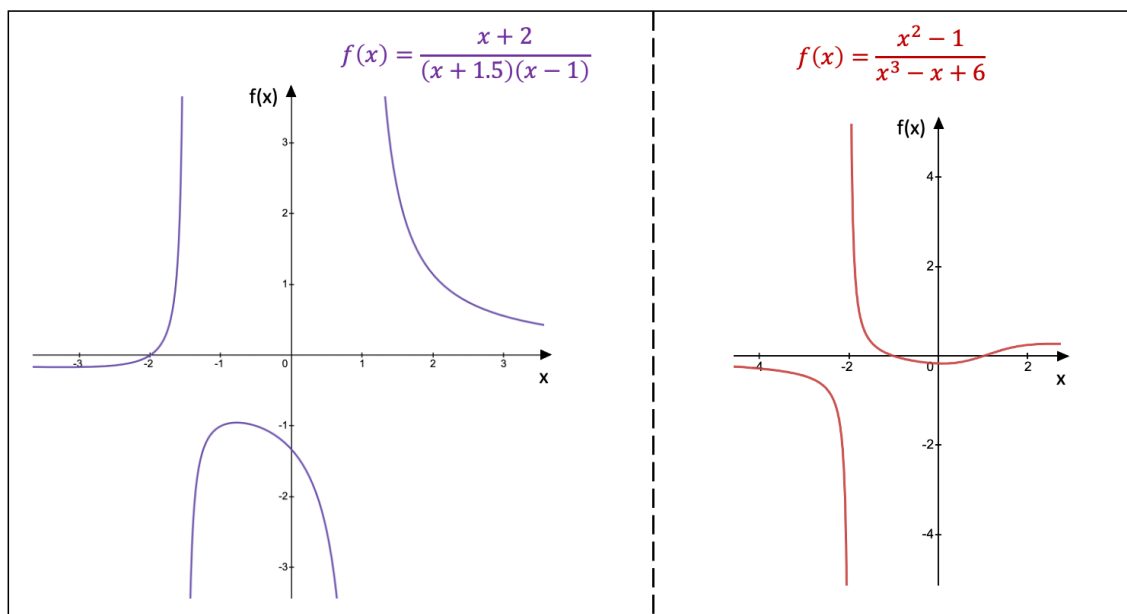


Figure 7.35: A graphical representation of two rational functions

Fig. 7.35 represents two rational functions. On the left, we have the rational function $f(x) = \frac{x+2}{(x+1.5)(x-1)}$, which has the domain $\mathbb{R} \setminus \{-1.5, 1\}$ and a root equal to $x = -2$. The two vertical asymptotes are equal to $x = -1.5$ and $x = 1$. On the right, the rational function $f(x) = \frac{x^2-1}{x^3-x+6}$ has the domain $\mathbb{R} \setminus \{-2\}$ and two roots equal to $x = -1$ and $x = 1$. The function has one vertical asymptote $x = -2$.

An **irrational function** is generally referred to as a function whereby the power of the variable in at least one term is a rational number. In other words, an irrational function contains at least one square root, cube root, fourth root, etc., of the variable.

Fig. 7.36 shows two examples of irrational functions. On the left, we have the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = (x^2 - 1)x^{\frac{3}{5}} = (x^2 - 1)\sqrt[5]{x^3}$ and on the right the function $g : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto g(x) = (3^x)\sqrt[3]{x^2}$.

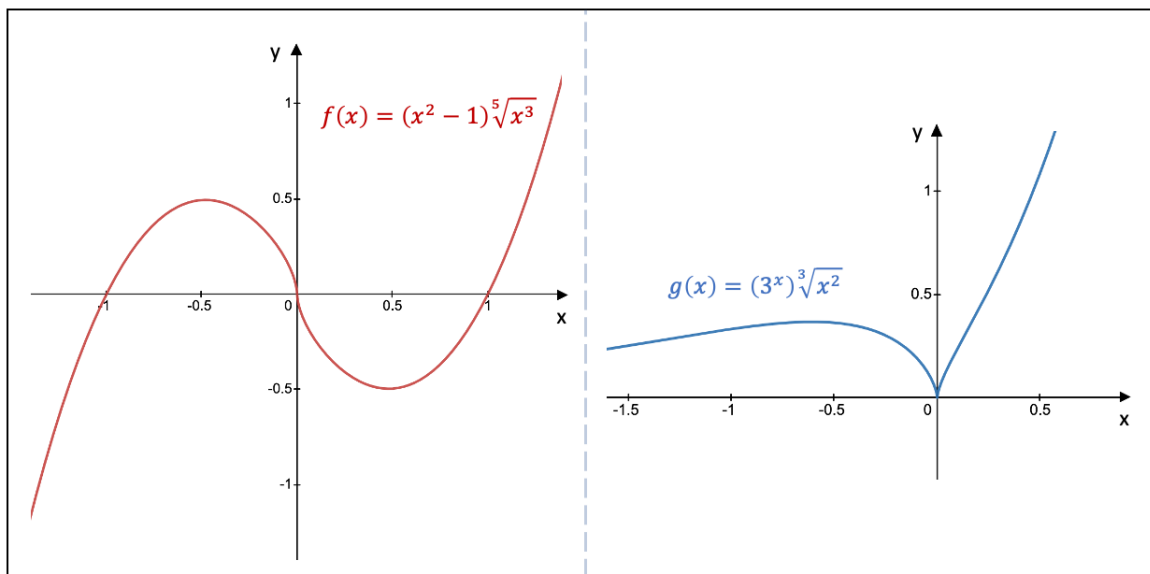


Figure 7.36: A graphical representation of two irrational functions

7.6.6 Exponential and Logarithmic Functions

The **exponential function** is defined as $f : \mathbb{R} \rightarrow \mathbb{R}_0^+ : x \mapsto f(x) = a^x$ and the **logarithmic function** as $f : \mathbb{R}_0^+ \rightarrow \mathbb{R} : x \mapsto f(x) = \log_a x$, whereby $a > 0$ and $a \neq 1$.

If a is equal to Euler's number $e = 2.71828\dots$, then $\log_e x$ is called the **natural logarithm** and written as $\log_e x = \ln x$.

All exponential functions pass through the point $(0, 1)$ and have no roots. Neither do they possess any extremum points. Also, they all share $y = 0$ as a horizontal asymptote.

Similarly, all logarithmic functions go through the point $(1, 0)$ and $x = 1$ is their only root. As with the exponential functions, they do not have any extremum points. The logarithmic functions all have $x = 0$ as a vertical asymptote.

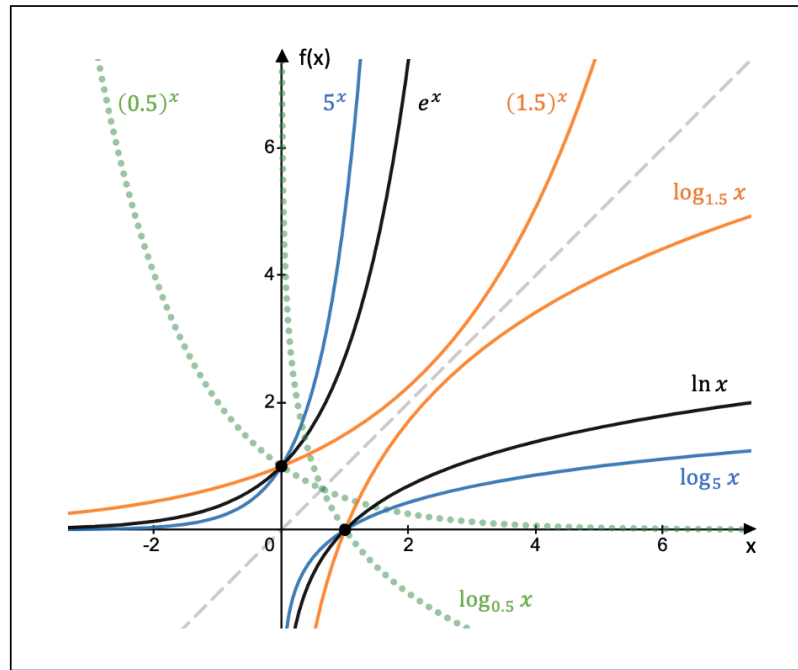


Figure 7.37: A graphical representation of different exponential and logarithmic functions

7.6.7 Inverse Functions

Let us consider the function $f : A \rightarrow B : x \mapsto f(x) = y$. If the element $y \in B$ is the function value of just one element $x \in A$, then we can define the **inverse function** as $f^{-1} : B \rightarrow A : y \mapsto f^{-1}(y) = x$.

We can obtain the inverse function f^{-1} by switching the x and y coordinates of the original function f , which corresponds graphically with a reflection of the function f around the straight line $y = x$.

For instance, if we reverse the role of x and y of the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = x + 5$, we obtain $x = y + 5$ so that the inverse function becomes $f^{-1} : \mathbb{R} \rightarrow \mathbb{R} : y \mapsto f^{-1}(y) = y + 5$. In terms of the variable x , this can be written as $y = x - 5$.

Another example is the function $f : \mathbb{R}_0 \rightarrow \mathbb{R}_0 : x \mapsto f(x) = \frac{1}{x}$, which is its own inverse function since switching the coordinates gives the same function.

As a third example, let us look at the function $f : \mathbb{R}^+ \rightarrow [2, +\infty) : x \mapsto f(x) = x^2 + 2$. The inverse function (in terms of the variable x) is formulated as $f^{-1} : [2, +\infty) \rightarrow \mathbb{R}^+ : x \mapsto f^{-1}(x) = \sqrt{x - 2}$, because $x = y^2 + 2$ when switching the roles of x and y in the original function f .

Fig. 7.38 shows graphically how the inverse of the function f in each of our three examples is obtained by reflecting the respective function around the bisector $y = x$ of the first quadrant (this line is represented by the grey dashed line).

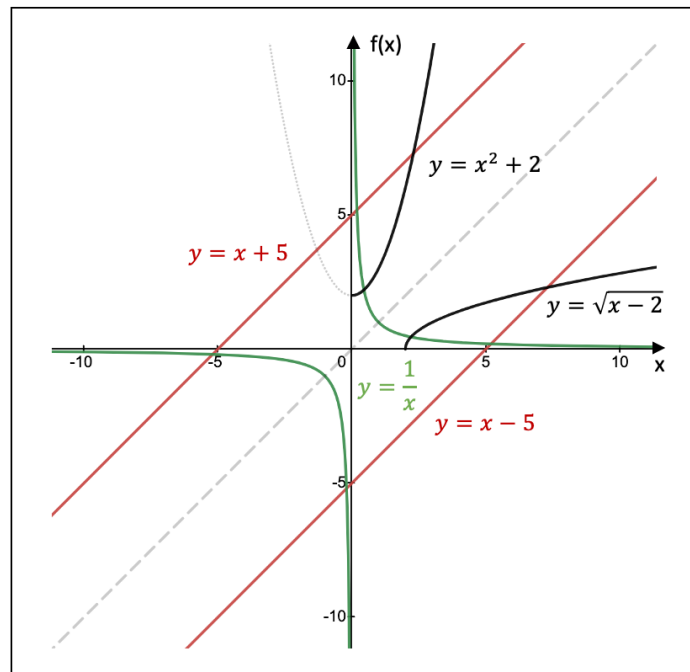


Figure 7.38: A graphical representation of different functions and their inverse

7.6.8 Goniometric and Cyclometric Functions

Fig. 7.39 lists the **goniometric functions**. The cosine and the secant are even functions whereas the sine, the cosecant, the tangent, and the cotangent are odd functions. Note that the argument of these functions is expressed in radians ($2\pi \text{ rad} = 360^\circ$).

➤ sine function	$\sin: \mathbb{R} \rightarrow [-1,1]: x \mapsto \sin x$
➤ cosecant function	$\csc: \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}: x \mapsto \csc x = \frac{1}{\sin x}$
➤ cosine function	$\cos: \mathbb{R} \rightarrow [-1,1]: x \mapsto \cos x$
➤ secant function	$\sec: \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\right\} \rightarrow \mathbb{R}: x \mapsto \sec x = \frac{1}{\cos x}$
➤ tangent function	$\tan: \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi; k \in \mathbb{Z}\right\} \rightarrow \mathbb{R}: x \mapsto \tan x = \frac{\sin x}{\cos x}$
➤ cotangent function	$\cot: \mathbb{R} \setminus \{k\pi; k \in \mathbb{Z}\} \rightarrow \mathbb{R}: x \mapsto \cot x = \frac{\cos x}{\sin x}$

Figure 7.39: The goniometric functions

The **roots** of the sine and tangent function are equal to $x = k\pi$, while those of the cosine

and cotangent function are equal to $x = \frac{\pi}{2} + k\pi$. The cosecant and secant function do not have any roots.

Fig. 7.40 shows that these functions reflect a certain **periodicity**. The sine, cosine, secant, and cosecant functions generate the same goniometric number for each time the argument is increased (or decreased) by a multiple of 2π . In contrast, the tangent and cotangent functions exhibit such periodicity when the argument is changed by a multiple of π . The numbers 2π and π are referred to as the **period** of the respective functions. We can therefore write (with $k \in \mathbb{Z}$):

$$\left\{ \begin{array}{l} \sin(x + 2k\pi) = \sin x \\ \csc(x + 2k\pi) = \csc x \\ \cos(x + 2k\pi) = \cos x \end{array} \right. \quad \left\{ \begin{array}{l} \sec(x + 2k\pi) = \sec x \\ \tan(x + k\pi) = \tan x \\ \cot(x + k\pi) = \cot x \end{array} \right.$$

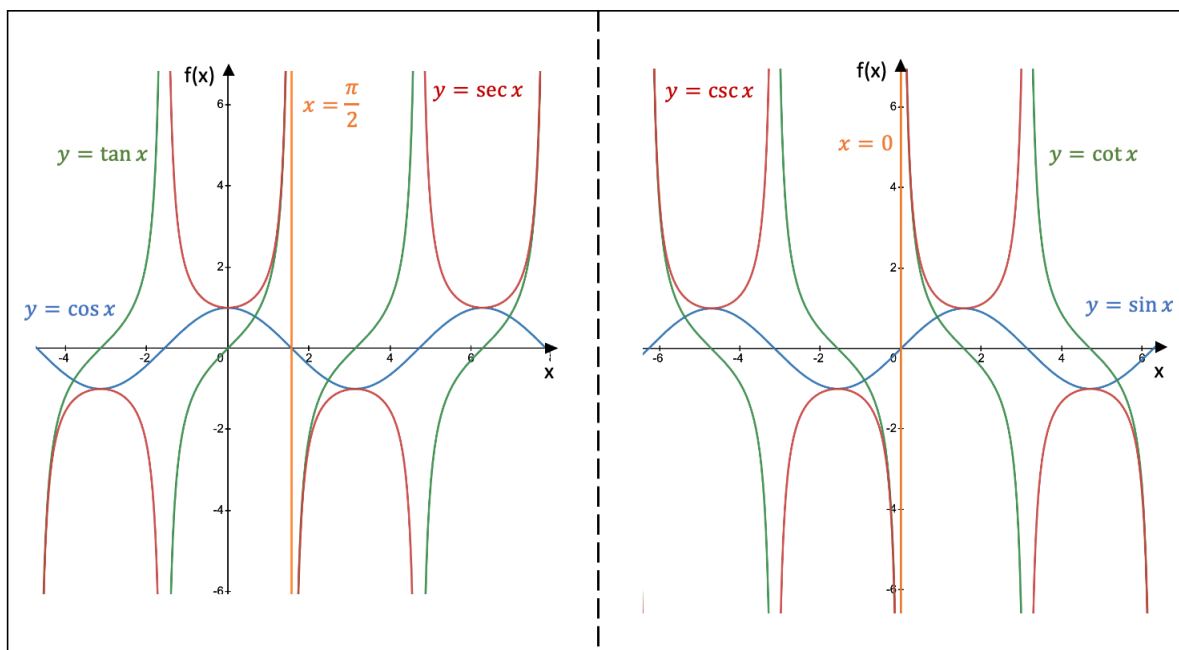


Figure 7.40: The goniometric functions

Fig. 7.40 furthermore tells us that the secant and tangent function share the vertical asymptotes $x = \frac{\pi}{2} + k\pi$, whereas the cosecant and the cotangent function have the same vertical asymptotes $x = k\pi$.

Finally, we can draw some conclusions about the extremum points. The cosine function reaches a maximum in the points $x = 2k\pi$ and a minimum in the points $x = \pi + 2k\pi$. The sine function finds a maximum and minimum in the points $x = \frac{\pi}{2} + 2k\pi$ and $x = -\frac{\pi}{2} + 2k\pi$, respectively. Every maximum (minimum) of the cosine function corresponds with a minimum (maximum) of the secant function. The same relationship exists for the sine and the cosecant function.

The **cyclometric functions** are defined as the inverse of the goniometric functions, provided that their domain and codomain are properly restricted. For instance, the inverse of the secant function is written as $\sec^{-1} x = \operatorname{arcsec} x$. Fig. 7.41 provides the description of the cyclometric functions with their appropriate domains and codomains, while Fig. 7.42 shows their graphical representation. Note that only a certain section of the domain of the arc cosecant and arc secant functions are displayed.

➤ arc sine function	$\operatorname{arcsin}: [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$: x \mapsto \operatorname{arcsin} x$
➤ arc cosecant function	$\operatorname{arccsc}: [1, +\infty[\rightarrow \left]0, \frac{\pi}{2}\right[$	$: x \mapsto \operatorname{arccsc} x$
➤ arc cosine function	$\operatorname{arccos}: [-1,1] \rightarrow [0, \pi]$	$: x \mapsto \operatorname{arccos} x$
➤ arc secant function	$\operatorname{arcsec}: [1, +\infty[\rightarrow \left[0, \frac{\pi}{2}\right[$	$: x \mapsto \operatorname{arcsec} x$
➤ arc tangent function	$\operatorname{arctan}: \mathbb{R} \rightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$	$: x \mapsto \operatorname{arctan} x$
➤ arc cotangent function	$\operatorname{arccot}: \mathbb{R} \rightarrow \left]0, \pi\right[$	$: x \mapsto \operatorname{arccot} x$

Figure 7.41: The cyclometric functions

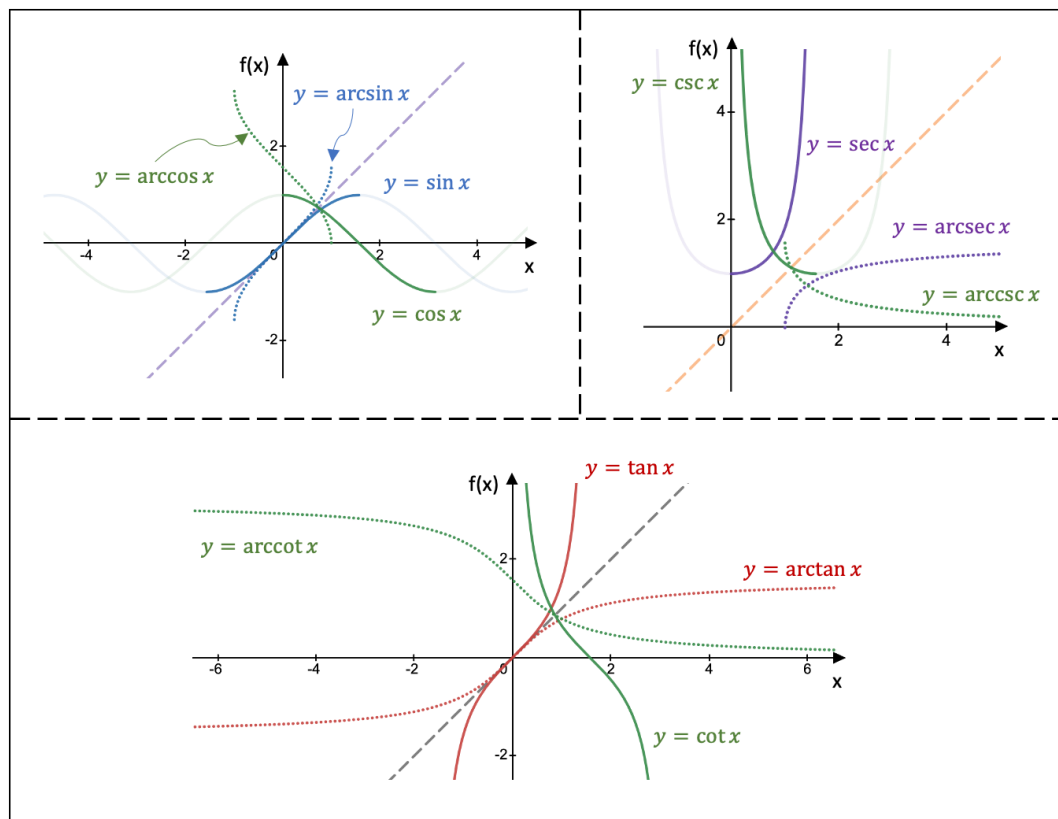


Figure 7.42: The graphical representation of the cyclometric functions

7.6.9 Hyperbolic Functions

The **hyperbolic functions** are similar to the trigonometric functions, but instead of being defined on the unit circle, the hyperbolic functions are defined on the right half of the unit hyperbola with equation $x^2 - y^2 = 1$. While $(\cos x, \sin x)$ describes the unit circle for $0 \leq x < 2\pi$, $(\cosh x, \sinh x)$ describes the right half of the unit hyperbola with \cosh (sinh) the hyperbolic cosine (sine) and $x \in \mathbb{R}$. Note that $x = 2A$, with A the red area in Fig. 7.43.

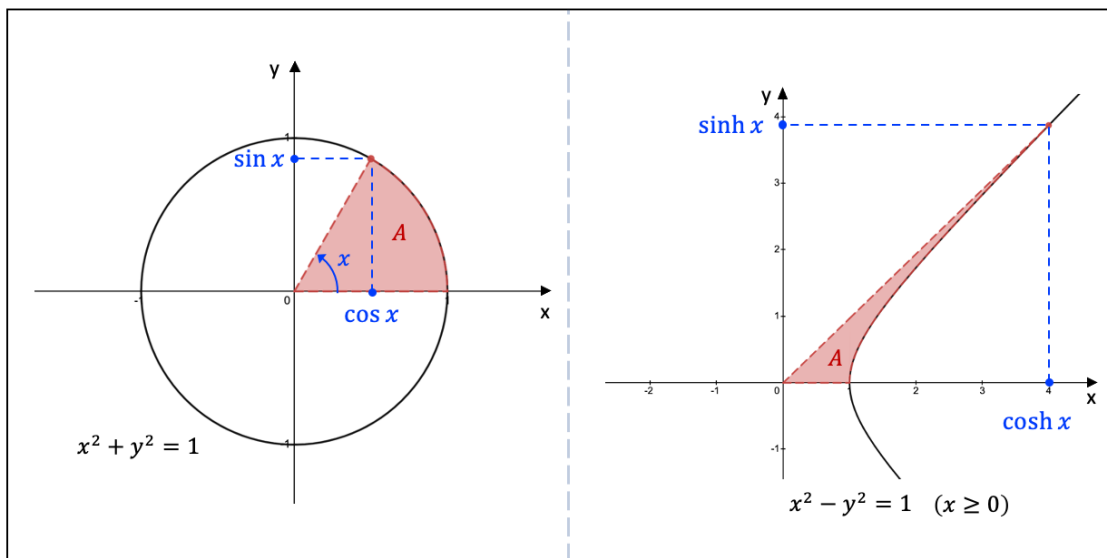


Figure 7.43: The unit circle and the right half of the unit hyperbola

Fig. 7.44 lists the definitions of the six hyperbolic functions. Note that only the hyperbolic cosine and secant functions are even functions, whereas the other remaining four functions are odd functions. Their graph is shown in Fig. 7.46.

➤ hyperbolic sine function	$\sinh: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \sinh x = \frac{e^x - e^{-x}}{2}$
➤ hyperbolic cosecant function	$\operatorname{csch}: \mathbb{R}_0 \rightarrow \mathbb{R}_0: x \mapsto \operatorname{csch} x = \frac{1}{\sinh x}$
➤ hyperbolic cosine function	$\cosh: \mathbb{R} \rightarrow [1, +\infty[: x \mapsto \cosh x = \frac{e^x + e^{-x}}{2}$
➤ hyperbolic secant function	$\operatorname{sech}: \mathbb{R} \rightarrow]0, 1]: x \mapsto \operatorname{sech} x = \frac{1}{\cosh x}$
➤ hyperbolic tangent function	$\tanh: \mathbb{R} \rightarrow]-1, 1[: x \mapsto \tanh x = \frac{\sinh x}{\cosh x}$
➤ hyperbolic cotangent function	$\operatorname{coth}: \mathbb{R}_0 \rightarrow \mathbb{R} \setminus [-1, 1]: x \mapsto \operatorname{coth} x = \frac{\cosh x}{\sinh x}$

Figure 7.44: The six hyperbolic functions

If we appropriately restrict the domain of the hyperbolic functions, we can define the six

inverse hyperbolic functions, which are listed in Fig. 7.45 and displayed in Fig. 7.46.

➤ arc hyperbolic sine function	$\operatorname{arcsinh}: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$
➤ arc hyperbolic cosecant function	$\operatorname{arccsch}: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+: x \mapsto \operatorname{arccsch} x = \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right)$
➤ arc hyperbolic cosine function	$\operatorname{arcosh}: [1, +\infty[\rightarrow \mathbb{R}^+: x \mapsto \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1})$
➤ arc hyperbolic secant function	$\operatorname{arcsech}:]0, 1] \rightarrow \mathbb{R}^+: x \mapsto \operatorname{arcsech} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$
➤ arc hyperbolic tangent function	$\operatorname{artanh}:]-1, 1[\rightarrow \mathbb{R}: x \mapsto \operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$
➤ arc hyperbolic cotangent function	$\operatorname{arcoth}: \mathbb{R} \setminus [-1, 1] \rightarrow \mathbb{R}_0: x \mapsto \operatorname{arcoth} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$

Figure 7.45: The six inverse hyperbolic functions

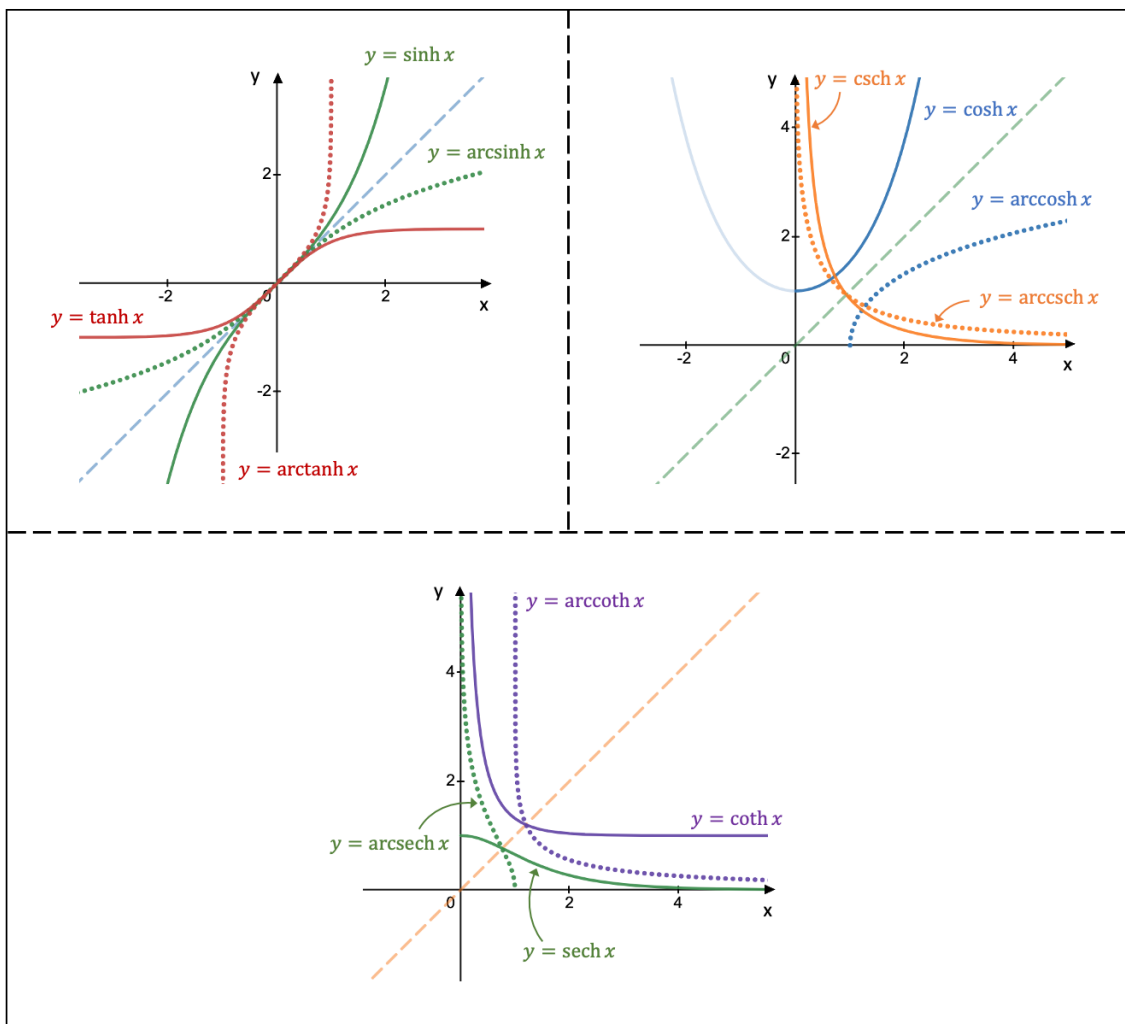


Figure 7.46: The graph of the six (inverse) hyperbolic functions

Similar to the trigonometric formulas (see section 5.3), we can formulate various relationships between the hyperbolic functions. To start with, the **hyperbolic identity relation** is equal to $\cosh^2 x - \sinh^2 x = 1$. From there, we can derive two more relationships (whereby $\sinh x \neq 0$):

$$\begin{cases} 1 - \tanh^2 x = \operatorname{sech}^2 x \\ \coth^2 x - 1 = \operatorname{csch}^2 x \end{cases}$$

The **hyperbolic double-angle formulas** are shown in Fig. 7.47.

$\sinh 2x = 2 \sinh(x) \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2 \cosh^2 x - 1$ $= 1 + 2 \sinh^2 x$	$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
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Figure 7.47: Hyperbolic double-angle formulas

The **hyperbolic Carnot's formulas** and the **half-angle formulas** are derived from the double-angle formulas and are listed in Fig 7.48.

<u>Carnot's Formulas</u>	<u>Half-angle Formulas</u>
$\sinh^2 x = \frac{\cosh 2x - 1}{2}$	$\sinh \frac{x}{2} = \pm \sqrt{\frac{\cosh x - 1}{2}}$
$\cosh^2 x = \frac{\cosh 2x + 1}{2}$	$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$
	$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x}$

Figure 7.48: Hyperbolic half-angle formulas

The **hyperbolic angle addition and subtraction formulas** together with the **hyperbolic Simpson's formulas** are given in below Fig. 7.49.

<u>Angle addition formulas</u>	<u>Angle subtraction formulas</u>
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$	$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$
$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh(x) \tanh y}$	$\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh(x) \tanh y}$
<u>Simpson's formulas</u>	
$\sinh x + \sinh y = 2 \sinh\left(\frac{x + y}{2}\right) \cosh\left(\frac{x - y}{2}\right)$	$2 \sinh x \cosh y = \sinh(x - y) + \sinh(x + y)$
$\sinh x - \sinh y = 2 \sinh\left(\frac{x - y}{2}\right) \cosh\left(\frac{x + y}{2}\right)$	$2 \sinh y \cosh x = \sinh(y - x) + \sinh(x + y)$
$\cosh x + \cosh y = 2 \cosh\left(\frac{x - y}{2}\right) \cosh\left(\frac{x + y}{2}\right)$	$2 \sinh x \sinh y = \cosh(x + y) - \cosh(x - y)$
$\cosh x - \cosh y = 2 \sinh\left(\frac{x - y}{2}\right) \sinh\left(\frac{x + y}{2}\right)$	$2 \cosh x \cosh y = \cosh(x - y) + \cosh(x + y)$

Figure 7.49: Hyperbolic angle addition, angle subtraction and Simpson's formulas

Finally, Fig. 7.50 lists the **hyperbolic tangent half-angle substitution formulas**.

$$t = \tanh \frac{x}{2} \quad \Rightarrow \quad \begin{cases} \cosh x = \frac{1 + t^2}{1 - t^2} \\ \sinh x = \frac{2t}{1 - t^2} \\ \tan \theta = \frac{2t}{1 + t^2} \end{cases}$$

Figure 7.50: The hyperbolic tangent half-angle substitution formulas

7.7 Sequences and Series

7.7.1 Definitions

A **sequence** is a list of numbers that displays a particular pattern and whereby the order of the numbers is important. The individual elements of a sequence are called **terms** and can be either positive or negative. A sequence can furthermore be finite or infinite, ascending or descending, or alternating with respect to the sign of the terms.

A sequence is denoted by (u_n) with u_n the n^{th} term of the sequence. We will assume that the position of the first term of a sequence corresponds to the number 1, so that $n \in \mathbb{N}_0$.

In the next few paragraphs, we will look at different kinds of sequences. Note that a sequence does not necessarily have to belong to any of these categories.

- An **arithmetic sequence** is a sequence whereby the difference between two successive terms is a constant called the common difference d . For instance, the infinite sequence $(u_n) = 3, 8, 13, 18, 23, \dots$ has a common difference equal to $d = 5$.
- If the difference itself increases or decreases by a constant number with every successive term, the sequence is called a **quadratic sequence**. This constant number is referred to as the second difference d_s . For example, the descending sequence $(u_n) = 5, 2, -2, -7, -13, -20, \dots$ has a second difference equal to $d_s = -1$. The sequence of the differences between every two consecutive terms is therefore equal to $(u_n) = -3, -4, -5, -6, -7, \dots$
- A **geometric sequence** is a sequence whereby the ratio of a term to its preceding term is a constant number, which is called the common ratio r . In other words, a geometric sequence is a sequence whereby every term (except the first term) is formed by multiplying the preceding term by a constant number. For instance, the alternating sequence $(u_n) = 6, -18, 54, -162, 486, \dots$ has a common ratio equal to $r = -3$.
- A **harmonic sequence** is a sequence whereby every term is formed by taking the reciprocal of the corresponding term of an arithmetic sequence, as long as the number zero is not part of the arithmetic sequence. For example, given the arithmetic sequence $(u_n) = 1, 5, 9, 13, 17, \dots$, the corresponding harmonic sequence is equal to $(u_n) = 1, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{17}, \dots$
- A **Fibonacci sequence** is a sequence whereby every term (except the first two terms) is made of the sum of the two preceding terms. An example of a Fibonacci sequence is the sequence $(u_n) = 1, -1, 0, -1, -1, -2, -3, -5, \dots$
- A **triangle number sequence** is a sequence whereby every term is able to form an equilateral triangle made out of dots, if the number of every term corresponds to the number of dots. For instance, the sequence $(u_n) = 1, 10, 28, 55, 91, \dots$ is a triangle number sequence and also happens to be a quadratic sequence with $d_s = 9$.
- A **square number sequence** is a sequence whereby the number of dots represented by the number of the term can be arranged to form a square. An example is the sequence $(u_n) = 1, 4, 16, 64, 256, \dots$, which also happens to be a geometric sequence. Note that a square number sequence does not necessarily also have to be a quadratic sequence, which is demonstrated by the example.

- A **cube number sequence** is a sequence whereby the number of dots represented by the number of the term can be arranged into the shape of a cube. The sequence $(u_n) = 1, 27, 125, 343, 729, \dots$ is an example of a cube number sequence.

If we add together all the terms of a sequence, the resulting sum is called a **series** and is denoted by $\sum_{n=1}^{+\infty} u_n$, with u_n the n^{th} term of the corresponding sequence. If we only wish to calculate the sum of the first n terms of a sequence, we define the **partial sum** s_n as $s_n = \sum_{i=1}^n u_i$.

For example, the partial sum of the first seven terms of the Fibonacci sequence $(u_n) = -2, 3, 1, 4, 5, 9, 14, 23, \dots$ is equal to $\sum_{i=1}^7 u_i = -2 + 3 + 1 + 4 + 5 + 9 + 14 = 34$.

As a second example, let us design a general formula for the arithmetic series with common difference $d = 1$, i.e., the sum of the natural numbers. If we consider the first n terms, we can write the partial sum s_n as $s_n = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$. Reversing the order of the elements, we can write s_n also as $s_n = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$. If we now add these two equations together, we obtain $2s_n = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1) = n(n+1)$, so that the partial sum s_n is equal to $s_n = \frac{n(n+1)}{2}$.

7.7.2 Explicit and Recursive Formulas of Sequences

In order to identify a random term of a given sequence, it is useful to formulate a general expression for the n^{th} term of the sequence. If the formula is written in terms of the position n , it is called an **explicit formula**. If the formula is expressed by using only one or more preceding terms, the formula is referred to as a **recursive formula**.

- For an **arithmetic sequence**, the formulas are equal to:

$$\begin{cases} \text{Explicit formula:} & u_n = u_1 + d(n-1) \\ \text{Recursive formula:} & u_n = u_{n-1} + d \quad \text{with } n \geq 2 \end{cases}$$

For example, if we know for a certain arithmetic sequence that $u_1 = -3$ and $d = 4$, the 12^{th} term is then equal to $u_{12} = -3 + 4(12-1) = 41$.

- In the case of a **quadratic sequence**, the formulas are the following, whereby the coefficients a , b , and c can be found by solving the below set of three equations (note that for the recursive formula $n \geq 3$):

$$\begin{cases} \text{Explicit formula:} & u_n = an^2 + bn + c \\ \text{Recursive formula:} & u_n = 2u_{n-1} - u_{n-2} + d_s \end{cases} \quad \text{with} \quad \begin{cases} u_1 = a + b + c \\ u_2 - u_1 = 3a + b \\ d_s = 2a \end{cases}$$

Consider the sequence $(u_n) = -4, 2, 12, 26, 44, \dots$. Given that the second difference is equal to $d_s = 4$, we find from the third equation that $a = 2$. The second equation tells us that $6 = 3 \cdot 2 + b \Leftrightarrow b = 0$, and from the first equation we know that

$-4 = 2 + 0 + c \Leftrightarrow c = -6$. The explicit formula is then equal to $u_n = 2n^2 - 6$.

- For a **geometric sequence**, the explicit and recursive formulas are equal to:

$$\begin{cases} \text{Explicit formula:} & u_n = u_1 \cdot r^{n-1} \\ \text{Recursive formula:} & u_n = u_{n-1} \cdot r \quad \text{with } n \geq 2 \end{cases}$$

For instance, if we know that $u_3 = 6$ and $u_6 = 162$ for a given sequence, we can see that $u_6 = 162 = u_1 r^5 = (u_1 r^2) r^3 = u_3 r^3 = 6r^3 \Leftrightarrow r = 3$. We therefore find that $u_3 = 6 = u_1 r^2 = u_1 3^2 \Leftrightarrow u_1 = \frac{2}{3}$, so that the explicit formula reads $u_n = 2 \cdot 3^{n-2}$.

- If we are dealing with a **harmonic sequence**, the formulas are equal to:

$$\begin{cases} \text{Explicit formula:} & u_n = \frac{1}{u_1 + d(n-1)} \\ \text{Recursive formula:} & u_n = \frac{u_{n-1}}{1 + d \cdot u_{n-1}} \quad \text{with } n \geq 2 \end{cases}$$

Consider an arithmetic sequence with $u_1 = 3$ and $d = -2$. The 5th term of the harmonic sequence is then equal to $u_5 = \frac{1}{3 + (-2)(5-1)} = -\frac{1}{5}$.

- In the case of a **Fibonacci sequence**, the formulas are the following, whereby $a = u_1$ and $b = u_2$:

$$\begin{cases} \text{Explicit formula:} & u_n = \left(\frac{b-a}{2} + \frac{3a-b}{2\sqrt{5}} \right) \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{b-a}{2} - \frac{3a-b}{2\sqrt{5}} \right) \left(\frac{1-\sqrt{5}}{2} \right)^n \\ \text{Recursive formula:} & u_n = u_{n-1} + u_{n-2} \quad \text{with } n \geq 3 \end{cases}$$

If we know, for instance, that for a given sequence $u_3 = 21$ and $u_8 = 128$, we can set $u_4 = c$ and use the recursive formula to find an expression for u_8 in terms of c . We find that $u_8 = 63 + 5c = 128 \Leftrightarrow c = 13$. If we now set $u_2 = b$, we have that $u_2 + u_3 = u_4 \Leftrightarrow b + 21 = 13 \Leftrightarrow b = -8$. By the same reasoning, we find that $u_1 = a = 29$. If we now wish to calculate the 14th term, we use the explicit formula with $n = 14$, $a = 29$, and $b = -8$. We find that $u_{14} = 2312$.

- If a pattern can be identified in a **triangle number sequence**, the following formulas are in place, whereby s in the summation of the explicit formula depends on the specific sequence (also note that the recursive formula is equal to that of the quadratic sequence):

$$\begin{cases} \text{Explicit formula:} & u_n = \sum_{k=1}^s k \\ \text{Recursive formula:} & u_n = 2u_{n-1} - u_{n-2} + d_s \quad \text{with } n \geq 3 \end{cases}$$

Consider, for example, the triangle number sequence $(u_n) = 3, 21, 55, 105, 171, 253, \dots$.

We find that $s = 4n - 2$ in the explicit formula and that the second difference is equal to $d_s = 16$. If we wish to calculate u_7 , we can use the recursive formula, so that $u_7 = 2u_6 - u_5 + d_s = 2 \cdot 253 - 171 + 16 = 351$. If we wish to calculate u_{22} , we can use the explicit formula, so that $u_{22} = \sum_{k=1}^{4 \cdot 22 - 2} k = \frac{s(s+1)}{2} = \frac{86(86+1)}{2} = 3741$, whereby we made use of the formula for the partial sum of the natural numbers (see section 7.7.1).

- For a **square number sequence**, the formulas are equal to:

$$\begin{cases} \text{Explicit formula:} & u_n = (an + b)^2 \quad \text{with } a, b \in \mathbb{Z} \\ \text{Recursive formula:} & u_n = 2u_{n-1} - u_{n-2} + 2a^2 \quad \text{with } n \geq 3 \end{cases}$$

For example, if $u_3 - u_2 = 39$ and $u_5 - u_4 = 75$ for a given sequence, we find that $u_3 - u_2 = (3a + b)^2 - (2a + b)^2 = 5a^2 + 2ab = 39$ and $u_5 - u_4 = (5a + b)^2 - (4a + b)^2 = 9a^2 + 2ab = 75$. These two equations give us two solutions, i.e., $a = -3$ and $b = 1$ and $a = 3$ and $b = -1$, which describe the same sequence. If we wish to calculate u_7 , we find $u_7 = (-3 \cdot 7 + 1)^2 = (3 \cdot 7 - 1)^2 = 400$.

- Finally, in the case of a **cube number sequence**, the formulas are the following:

$$\begin{cases} \text{Explicit formula:} & u_n = (an + b)^3 \quad \text{with } a, b \in \mathbb{Z} \\ \text{Recursive formula:} & u_n = 3(u_{n-1} - u_{n-2}) + u_{n-3} + 6a^3 \quad \text{with } n \geq 4 \end{cases}$$

For instance, if $u_6 - u_5 = -296$, $u_5 - u_4 = -152$, and $u_4 - u_3 = -56$ for a given sequence, we find via the recursive formula $u_6 = 3(u_5 - u_4) + u_3 + 6a^3$ that $a = -2$. If we then take the explicit formula to look at the difference $u_4 - u_3 = [(-2) \cdot 4 + b]^3 - [(-2) \cdot 3 + b]^3 = -56$, we obtain two solutions: $b = 4$ and $b = 10$. Only $b = 4$ produces the given differences, so that the explicit formula of this sequence is equal to $u_n = (-2n + 4)^3$.

7.7.3 Convergence and Divergence of Sequences

We might be interested to know whether a given sequence approaches a real number L when we consider the n^{th} term with n a large number. In other words, we wish to determine the **limit of the sequence** when n goes to infinity. If the limit of a sequence approaches a real number L , the sequence is said to be **convergent**, whereas it is called **divergent** if the limit goes to infinity. In mathematical language, this becomes:

$$\left\{ \begin{array}{l} \text{Convergent:} \\ \lim_{n \rightarrow +\infty} u_n = L \Leftrightarrow \forall \epsilon > 0, \exists N : n > N \Rightarrow |u_n - L| < \epsilon \\ \\ \text{Divergent:} \\ \bullet \lim_{n \rightarrow +\infty} u_n = +\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists N : n > N \Rightarrow u_n > \alpha \\ \bullet \lim_{n \rightarrow +\infty} u_n = -\infty \Leftrightarrow \forall \alpha \in \mathbb{R}, \exists N : n > N \Rightarrow u_n < \alpha \end{array} \right.$$

Suppose we have two sequences (u_n) and (r_n) , whereby $\lim_{n \rightarrow +\infty} u_n = L_1$ and $\lim_{n \rightarrow +\infty} r_n = L_2$. We can then perform the following **operations of sequences**:

- $\lim_{n \rightarrow +\infty} (u_n + r_n) = L_1 + L_2$
- $\lim_{n \rightarrow +\infty} |u_n| = |L_1|$
- $\lim_{n \rightarrow +\infty} (u_n \cdot r_n) = L_1 \cdot L_2$
- $\lim_{n \rightarrow +\infty} \left(\frac{u_n}{r_n} \right) = \frac{L_1}{L_2}$

whereby $r_n \neq 0$ and $L_2 \neq 0$ in case of the quotient of u_n and r_n . In case that $L_1 = L_2 = L$ and there exists a third sequence (w_n) so that $u_n \leq w_n \leq r_n$, the **squeeze theorem** then states that $\lim_{n \rightarrow +\infty} w_n = L$.

Let us consider some examples.

- Example 1

Let us have a closer look at the Fibonacci sequence:

$$u_n = \left(\frac{b-a}{2} + \frac{3a-b}{2\sqrt{5}} \right) \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{b-a}{2} - \frac{3a-b}{2\sqrt{5}} \right) \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Since $|1 - \sqrt{5}| < 2$, we know that $\lim_{n \rightarrow +\infty} \left(\frac{1-\sqrt{5}}{2} \right)^n = 0$. Similarly, given that $|1 + \sqrt{5}| > 2$, we find that $\lim_{n \rightarrow +\infty} \left(\frac{1+\sqrt{5}}{2} \right)^n = +\infty$. This means that the Fibonacci sequence is always divergent. Whether it diverges to $+\infty$ or $-\infty$ depends on the value of $a = u_1$ and $b = u_2$. For instance, if $a = \frac{1}{2}$ and $b = -\frac{1}{3}$, we can see that $\left(\frac{b-a}{2} + \frac{3a-b}{2\sqrt{5}} \right) = \frac{11-5\sqrt{5}}{12\sqrt{5}} < 0$, so that the sequence is divergent to $-\infty$.

- Example 2

Consider the sequence (u_n) with the following n^{th} term:

$$u_n = \frac{-n^2 + 2n + 5}{(n-1 + \sqrt{6})(5n+3)}$$

To understand the behaviour of the sequence when n grows very large, we calculate the limit:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{-n^2 + 2n + 5}{(n-1 + \sqrt{6})(5n+3)} &= \lim_{n \rightarrow +\infty} \frac{-(n-1 + \sqrt{6})(n-1 - \sqrt{6})}{(n-1 + \sqrt{6})(5n+3)} \\ &= \lim_{n \rightarrow +\infty} \frac{-(n-1 - \sqrt{6})}{5n+3} \\ &= \lim_{n \rightarrow +\infty} \frac{-(1 - \frac{1}{n} - \frac{\sqrt{6}}{n})}{5 + \frac{3}{n}} \\ &= -\frac{1}{5} \end{aligned}$$

In other words, the sequence $(u_n) = \frac{\sqrt{6}}{8}, \frac{-1+\sqrt{6}}{13}, \frac{-2+\sqrt{6}}{18}, \frac{-3+\sqrt{6}}{23}, \dots$ converges to the real number $-\frac{1}{5}$.

- Example 3

We wish to determine the limit of the sequence that consists of the product of the sequences (u_n) and (r_n) , whereby (u_n) is defined in the above second example and the n^{th} term of (r_n) is equal to:

$$r_n = \frac{10n^2 + n - 3}{(n - 1 + \sqrt{6})(3n + 5)}$$

Let us first calculate the limit of (r_n) :

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{10n^2 + n - 3}{(n - 1 + \sqrt{6})(3n + 5)} &= \lim_{n \rightarrow +\infty} \frac{(5n + 3)(2n - 1)}{(n - 1 + \sqrt{6})(3n + 5)} \\ &= \lim_{n \rightarrow +\infty} \frac{(5 + \frac{3}{n})(2 - \frac{1}{n})}{(1 - \frac{1}{n} + \frac{\sqrt{6}}{n})(3 + \frac{5}{n})} \\ &= \frac{10}{3} \end{aligned}$$

The limit of the product of (u_n) and (r_n) is then equal to $\lim_{n \rightarrow +\infty} (u_n \cdot r_n) = \left(-\frac{1}{5}\right) \left(\frac{10}{3}\right) = -\frac{2}{3}$.

- Example 4

Consider the following n^{th} term of the sequence (u_n) :

$$u_n = \left(\sec \frac{1}{\sqrt{n+2}}\right)^n$$

Taking the limit $n \rightarrow \infty$ of u_n results in the undefined expression $1^{+\infty}$. Let us rewrite u_n and make the substitution $r = \frac{1}{\sqrt{n}}$:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\sec \frac{1}{\sqrt{n+2}}\right)^n &= \lim_{n \rightarrow +\infty} e^{\ln\left[\left(\sec \frac{1}{\sqrt{n+2}}\right)^n\right]} = \lim_{n \rightarrow +\infty} e^{\frac{\ln\left(\sec \frac{1}{\sqrt{n+2}}\right)}{\frac{1}{n}}} = \lim_{r \rightarrow 0} e^{\frac{\ln\left(\sec \frac{r}{\sqrt{1+2r^2}}\right)}{r^2}} \\ &= e^{\left[\lim_{r \rightarrow 0} \frac{\ln\left(\sec \frac{r}{\sqrt{1+2r^2}}\right)}{r^2}\right]} \end{aligned}$$

Let us now calculate the limit for which $r \rightarrow 0$. Since directly substituting 0 for r in the limit produces the undefined expression $\frac{0}{0}$, we Taylor expand the numerator of the quotient around the point $a = 0$ to the second order (with a Liouville remainder term):

$$\lim_{r \rightarrow 0} \frac{\ln\left(\sec \frac{r}{\sqrt{1+2r^2}}\right)}{r^2} = \lim_{r \rightarrow 0} \frac{\frac{r^2}{2} + \frac{\lambda(r)}{2}r^2}{r^2} = \lim_{r \rightarrow 0} \frac{\frac{1}{2} + \frac{\lambda(r)}{2}}{1} = \frac{1}{2}$$

This means that the sequence (u_n) converges to the real number \sqrt{e} .

- Example 5

Consider the sequence $(u_n) = \frac{7}{3}, \left(\frac{9}{5}\right)^2, \left(\frac{11}{7}\right)^3, \left(\frac{13}{9}\right)^4, \dots$. The n^{th} term is then equal to:

$$u_n = \left(\frac{2n+5}{2n+1}\right)^n$$

To determine the behaviour of the sequence for $n \rightarrow +\infty$, we calculate its limit as follows:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\frac{2n+5}{2n+1}\right)^n &= \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{2n+1}\right)^n \\ &= \lim_{s \rightarrow +\infty} \left(1 + \frac{1}{s}\right)^{2s-\frac{1}{2}} \\ &= \left[\lim_{s \rightarrow +\infty} \left(1 + \frac{1}{s}\right)^s \right]^2 \cdot \lim_{s \rightarrow +\infty} \left(1 + \frac{1}{s}\right)^{-\frac{1}{2}} \\ &= e^2 \end{aligned}$$

Note that in the second line, we introduced the substitution $n = \frac{4s-1}{2}$, and in the last line, we used the expression for Euler's number e in terms of limits as discussed in section 7.5.2.

7.7.4 Convergence and Divergence of Series

Given is the series $\sum_{i=1}^{+\infty} u_i$ of the sequence (u_n) . As the partial sum s_n is defined as $s_n = \sum_{i=1}^n u_i$, we can consider the sequence (s_n) of the partial sums of the sequence (u_n) . We say that the series $\sum_{i=1}^{+\infty} u_i$ is **convergent** if:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \sum_{i=1}^n u_i = s \in \mathbb{R}$$

The series is said to be **divergent** if $\lim_{n \rightarrow +\infty} s_n = \pm\infty$ or if the sequence (s_n) is alternating, i.e., the plus and minus sign in the series alternate.

Let us consider two examples.

- Example 1

Consider the sequence $(u_n) = \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \frac{1}{56}, \dots$. If we first look at the sequence of the denominators, i.e., $(u_n) = 12, 20, 30, 42, 56, \dots$, we see that it is a quadratic sequence with a second difference equal to $d_s = 2$. We therefore find that the n^{th} term is equal to

$n^2 + 5n + 6 = (n + 2)(n + 3)$, as discussed in section 7.7.2. We now want to examine the following limit:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \sum_{i=1}^n \frac{1}{(i+2)(i+3)}$$

To find that limit, we first rewrite the partial sum s_n as follows:

$$\begin{aligned} & \sum_{i=1}^n \frac{1}{(i+2)(i+3)} \\ &= \sum_{i=1}^n \left(\frac{1}{i+2} - \frac{1}{i+3} \right) \\ &= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \left(\frac{1}{n+2} - \frac{1}{n+3} \right) \\ &= \frac{1}{3} - \frac{1}{n+3} \end{aligned}$$

We can now calculate the limit:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \sum_{i=1}^n \frac{1}{(i+2)(i+3)} = \lim_{n \rightarrow +\infty} \left(\frac{1}{3} - \frac{1}{n+3} \right) = \frac{1}{3}$$

The series $\sum_{i=1}^{+\infty} u_i$ thus converges to the real number $\frac{1}{3}$.

- Example 2

Given is the sequence $(u_n) = a, a^2, a^3, a^4, \dots$, with $a \neq 0$ and $a \neq 1$, and we wish to examine the behaviour of the series $\sum_{i=1}^{+\infty} a^i$.

First, let us find an expression for the partial sum $s_i = \sum_{k=1}^i a^k = a + a^2 + \dots + a^{i-1} + a^i$ by looking at the generalization of the special product $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$, which we have studied in section 2.3.3 (see Fig. 2.5). In below step 2, we set $b = 1$, and in step 3, we use the definition of the partial sum s_i :

$$\begin{aligned} a^{i+1} - b^{i+1} &= (a - b)(a^i + a^{i-1}b + a^{i-2}b^2 + \dots + a^2b^{i-2} + ab^{i-1} + b^i) \\ \Leftrightarrow a^{i+1} - 1 &= (a - 1)(a^i + a^{i-1} + a^{i-2} + \dots + a^2 + a + 1) \\ \Leftrightarrow a^{i+1} - 1 &= (a - 1)(s_i + 1) \\ \Leftrightarrow s_i &= \frac{a^{i+1} - 1}{a - 1} - 1 \end{aligned}$$

We now consider three separate cases: $|a| < 1$, $a > 1$, and $a \leq -1$. If $|a| < 1$, we find that the series $\sum_{i=1}^{+\infty} a^i$ is convergent:

$$\lim_{i \rightarrow +\infty} s_i = \lim_{i \rightarrow +\infty} \left(\frac{a^{i+1} - 1}{a - 1} - 1 \right) = \frac{0 - 1}{a - 1} - 1 = \frac{a}{1 - a}$$

If $a > 1$, we see that the series $\sum_{i=1}^{+\infty} a^i$ diverges to $+\infty$:

$$\lim_{i \rightarrow +\infty} s_i = \lim_{i \rightarrow +\infty} \left(\frac{a^{i+1} - 1}{a - 1} - 1 \right) = +\infty$$

Finally, if $a \leq -1$, the series $\sum_{i=1}^{+\infty} a^i$ is divergent because the sequence (s_i) alternates:

$$\frac{a^{i+1} - 1}{a - 1} - 1 = \frac{(-|a|)^{i+1} + |a|}{-|a| - 1} \Rightarrow (s_i) = -|a|, \frac{|a|^3 - |a|}{|a| + 1}, -\frac{|a|^4 + |a|}{|a| + 1}, \frac{|a|^5 - |a|}{|a| + 1}, \dots$$

As it has been shown in the above two examples, it is quite cumbersome to calculate the exact resultant sum of a series if it converges. For more complicated sequences, it may be too difficult to find a general expression for the partial sum. What we can do, however, is establish criteria to determine whether a series is convergent or divergent.

Before we discuss some of the criteria for series, let us first summarize some general properties of series:

$$\left\{ \begin{array}{ll} \text{if } \sum_{n=1}^{+\infty} u_n \text{ is convergent} & \Rightarrow \lim_{n \rightarrow +\infty} u_n = 0 \\ \text{if } \lim_{n \rightarrow +\infty} u_n \neq 0 & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is divergent} \\ \text{if } \lim_{n \rightarrow +\infty} u_n = 0 & \Rightarrow \text{no conclusion can be made} \end{array} \right.$$

For example, consider the sequence (u_n) with the n^{th} term equal to:

$$u_n = \frac{\ln\left(\frac{1}{8n^2} + 1\right)}{\ln\left(\frac{1}{5n^2} + 1\right)}$$

The limit for $n \rightarrow +\infty$ results in the undefined expression $\frac{0}{0}$. If we make the substitution $t = \frac{1}{n}$, we can subsequently Taylor expand the numerator and the denominator of the quotient around the point $a = 0$ (with a Liouville remainder term) and solve the limit:

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \frac{\ln\left(\frac{1}{8n^2} + 1\right)}{\ln\left(\frac{1}{5n^2} + 1\right)} = \lim_{t \rightarrow 0} \frac{\ln\left(\frac{t^2}{8} + 1\right)}{\ln\left(\frac{t^2}{5} + 1\right)} = \lim_{t \rightarrow 0} \frac{\frac{t^2}{8} + \frac{\lambda(t)}{2}t^2}{\frac{t^2}{5} + \frac{\lambda(t)}{2}t^2} = \lim_{t \rightarrow 0} \frac{\frac{1}{8} + \frac{\lambda(t)}{2}}{\frac{1}{5} + \frac{\lambda(t)}{2}} = \frac{5}{8}$$

As $\lim_{n \rightarrow +\infty} u_n \neq 0$, we find that the series $\sum_{n=1}^{+\infty} u_n$ is divergent.

Several criteria are available to determine whether a series is convergent or divergent. Note that the below criteria are written specifically for positive series only, i.e., $\forall n : u_n > 0$, and do not apply to alternating sequences. Similar criteria can be constructed for sequences with $u_n \geq 0$ and for negatives series, i.e., $\forall n : u_n \leq 0$.

A first set of criteria is called the **comparison test**. If we consider two sequences (u_n) and (r_n) , we can make the following statements with respect to their series (note that $k \in \mathbb{R}$):

$$u_n \leq k \cdot r_n \quad \Rightarrow \quad \begin{cases} \text{if } \sum_{n=1}^{+\infty} r_n \text{ is convergent} & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is convergent} \\ \text{if } \sum_{n=1}^{+\infty} u_n \text{ is divergent} & \Rightarrow \sum_{n=1}^{+\infty} r_n \text{ is divergent} \end{cases}$$

A second set of criteria is the **limit comparison test**. For two sequences (u_n) and (r_n) , the limit comparison test says the following:

$$\begin{aligned} \bullet \quad \lim_{n \rightarrow \infty} \frac{u_n}{r_n} = k \in \mathbb{R}_0^+ & \longrightarrow \begin{cases} \sum_{n=1}^{+\infty} u_n \text{ is convergent} & \Leftrightarrow \sum_{n=1}^{+\infty} r_n \text{ is convergent} \\ \sum_{n=1}^{+\infty} u_n \text{ is divergent} & \Leftrightarrow \sum_{n=1}^{+\infty} r_n \text{ is divergent} \end{cases} \\ \bullet \quad \lim_{n \rightarrow \infty} \frac{u_n}{r_n} = 0 & \longrightarrow \begin{cases} \text{if } \sum_{n=1}^{+\infty} r_n \text{ is convergent} & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is convergent} \\ \text{if } \sum_{n=1}^{+\infty} u_n \text{ is divergent} & \Rightarrow \sum_{n=1}^{+\infty} r_n \text{ is divergent} \end{cases} \\ \bullet \quad \lim_{n \rightarrow \infty} \frac{u_n}{r_n} = +\infty & \longrightarrow \begin{cases} \text{if } \sum_{n=1}^{+\infty} u_n \text{ is convergent} & \Rightarrow \sum_{n=1}^{+\infty} r_n \text{ is convergent} \\ \text{if } \sum_{n=1}^{+\infty} r_n \text{ is divergent} & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is divergent} \end{cases} \end{aligned}$$

Another set of criteria is known as **Cauchy's root test**:

$$\lambda = \lim_{n \rightarrow +\infty} \sqrt[n]{u_n} \in \mathbb{R} \quad \Rightarrow \quad \begin{cases} \text{if } \lambda < 1 & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is convergent} \\ \text{if } \lambda > 1 & \Rightarrow \sum_{n=1}^{+\infty} r_n \text{ is divergent} \\ \text{if } \lambda = 1 & \Rightarrow \text{no conclusion can be made} \end{cases}$$

A fourth and final set of criteria that we discuss here is called **D'Alembert's ratio test**:

$$\lambda = \lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} \in \mathbb{R} \Rightarrow \begin{cases} \text{if } \lambda < 1 & \Rightarrow \sum_{n=1}^{+\infty} u_n \text{ is convergent} \\ \text{if } \lambda > 1 & \Rightarrow \sum_{n=1}^{+\infty} r_n \text{ is divergent} \\ \text{if } \lambda = 1 & \Rightarrow \text{no conclusion can be made} \end{cases}$$

- Example 1

We wish to know whether the series $\sum_{n=1}^{+\infty} u_n$ of the sequence $(u_n) = \frac{2}{25}, \frac{1}{28}, \frac{2}{107}, \frac{1}{89}, \frac{2}{269}, \dots$, which is generally described by the below n^{th} term, converges or diverges:

$$u_n = \frac{2}{10n^2 + n + 14}$$

Given that $\lim_{n \rightarrow +\infty} u_n = 0$, the general properties of series tell us that we cannot conclude anything about the corresponding series. However, the comparison test proves to be more useful in this context. For that to work, we need to find a second sequence (r_n) . If we consider the sequence from the above Example 1, i.e., $(r_n) = \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \frac{1}{56}, \dots$, we can see that for every term $u_n \leq r_n$. Given that $\sum_{n=1}^{+\infty} r_n$ is convergent, the comparison test tells us that the series $\sum_{n=1}^{+\infty} u_n$ is also convergent.

- Example 2

We would like to examine the behaviour of the below series:

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} \frac{\cosh(2n)}{n} = \sum_{n=1}^{+\infty} \frac{e^{2n} + e^{-2n}}{2n}$$

If we now consider the n^{th} term $r_n = \frac{1}{n}$ of the harmonic sequence $(r_n) = 1, \frac{1}{2}, \frac{1}{3}, \dots$, we find that the following limit is equal to:

$$\lim_{n \rightarrow \infty} \frac{u_n}{r_n} = \lim_{n \rightarrow \infty} \frac{\cosh(2n)}{n} \cdot \frac{n}{1} = \lim_{n \rightarrow \infty} \cosh(2n) = +\infty$$

If we know that the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent, the limit comparison test tells us that the series $\sum_{n=1}^{+\infty} \frac{\cosh(2n)}{n}$ also diverges.

- Example 3

Suppose we wish to determine the behaviour of the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$, whereby $(u_n) = 1, \frac{1}{4}, \frac{1}{9}, \dots$. As a second sequence, we take $r_n = \frac{2}{10n^2 + n + 14}$ from Example 1, whose series we found to be convergent. The limit then becomes:

$$\lim_{n \rightarrow \infty} \frac{u_n}{r_n} = \lim_{n \rightarrow \infty} \frac{1}{n^2} \cdot \frac{10n^2 + n + 14}{2} = \lim_{n \rightarrow \infty} 5 + \frac{1}{2n} + \frac{7}{n^2} = 5 \in \mathbb{R}_0^+$$

According to the limit comparison test, given that the series $\sum_{n=1}^{+\infty} r_n$ converges, we find that the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ also converges.

- Example 4

We wish to understand the behaviour of the series $\sum_{n=1}^{+\infty} u_n$, whereby u_n is equal to:

$$u_n = \frac{e^{-(n^3+n)} \cdot 5^{n+1}}{e^{-3n}}$$

If we use Cauchy's root test, we obtain the following value for λ :

$$\lambda = \lim_{n \rightarrow +\infty} \sqrt[n]{u_n} = \lim_{n \rightarrow +\infty} \sqrt[n]{\frac{e^{-(n^3+n)} \cdot 5^{n+1}}{e^{-3n}}} = \lim_{n \rightarrow +\infty} \frac{e^3 \cdot 5^{1+\frac{1}{n}}}{e^{n^2+1}} = 0$$

Since $\lambda = 0 < 1$, the series $\sum_{n=1}^{+\infty} u_n$ is convergent.

- Example 5

Consider the sequence $(u_n) = \frac{1}{21}, \frac{64}{495}, \frac{128}{273}, \frac{512}{255}, \dots$ with the general term:

$$u_n = \frac{(3n+2)! \cdot 2^{3n+1}}{(3n+5)!}$$

Applying D'Alembert's ratio test, we find the following value for λ :

$$\begin{aligned} \lambda &= \lim_{n \rightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow +\infty} \frac{(3n+5)! \cdot 2^{3n+4}}{(3n+8)!} \cdot \frac{(3n+5)!}{(3n+2)! \cdot 2^{3n+1}} \\ &= \lim_{n \rightarrow +\infty} \frac{2^3(3n+3)(3n+4)(3n+5)}{(3n+6)(3n+7)(3n+8)} \\ &= \lim_{n \rightarrow +\infty} \frac{2^3(3+\frac{3}{n})(3+\frac{4}{n})(3+\frac{5}{n})}{(3+\frac{6}{n})(3+\frac{7}{n})(3+\frac{8}{n})} \\ &= 8 \end{aligned}$$

As $\lambda = 8 > 1$, we find that the series $\sum_{n=1}^{+\infty} u_n$ is divergent.

8 Complex Analysis

Within the field of mathematics, the branch of **complex analysis** studies the phenomenon of complex numbers as well as related functions.

8.1 Definitions

A **complex number** z is defined as $z = a + bi$ with $a, b \in \mathbb{R}$ and i the **imaginary unit**, which is equal to $i = \sqrt{-1}$ and has the property $i^2 = -1$. We notice hereby that $i \notin \mathbb{R}$, since the square of any real number cannot be negative.

If we extend the set of real numbers and include the imaginary unit i we find the **set of complex numbers**, which is defined as $\mathbb{C} = \{z = a + bi \mid a, b \in \mathbb{R}\}$.

We call a the **real part** of the complex number z and is denoted by $\operatorname{Re}(z) = a$, whereas b is the **imaginary part** of z and is written as $\operatorname{Im}(z) = b$. If $\operatorname{Re}(z) = 0$, we call z a **purely imaginary number**, while if $\operatorname{Im}(z) = 0$, z is referred to as a **purely real number**.

For example, 5 is a purely real number, given that $5 = 5 + 0 \cdot i$, and $2i$ is a purely imaginary number because $2i = 0 + 2i$.

8.2 Graphical Interpretation

We can graphically represent a complex number by constructing the **complex plane**, also called the **Argand plane** or **Gauss plane**, whereby the purely imaginary numbers are located on the **imaginary y-axis** and the purely real numbers on the **real x-axis**.

A complex number $z = a + bi$ has therefore the coordinates (a, b) in the complex plane, with $x = a$ and $y = b$. Writing the coordinates of z as (a, b) is called the **Cartesian form**. The left-hand side of Fig. 8.1 shows two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

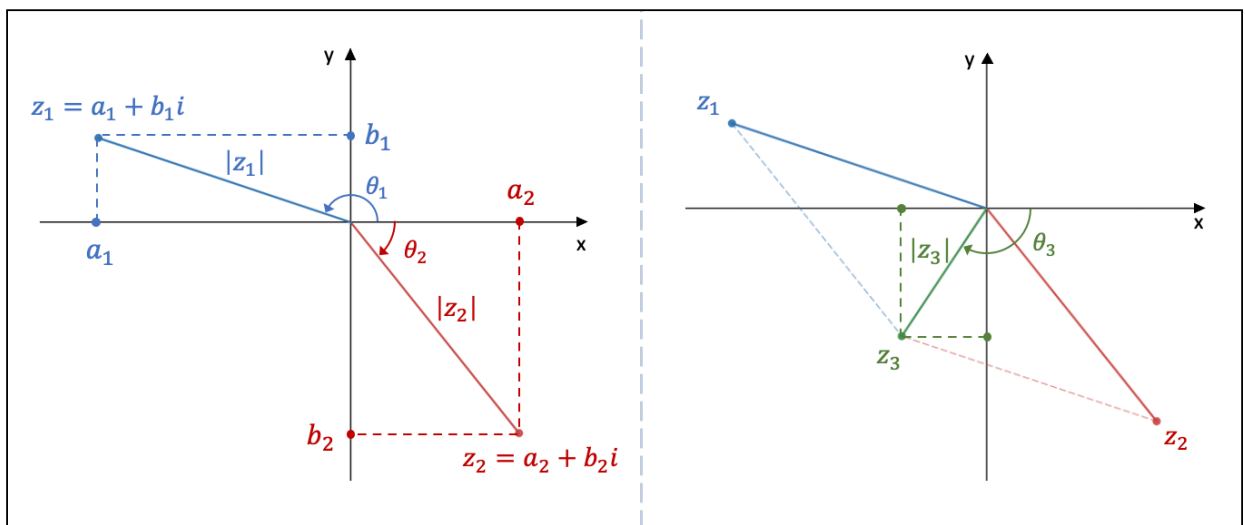


Figure 8.1: The graphical representation of complex numbers

The length of the line segment between the origin $(0, 0)$ and the point (a, b) is known as the **modulus** of the complex number, which is denoted by $|z|$ and defined with the assistance of the Pythagorean theorem as $|z| = \sqrt{a^2 + b^2}$.

The smallest angle θ between the positive real x -axis and the line segment as defined above is called the **principal argument** of the complex number and is denoted by $\arg(z)$, whereby $\arg(z) \in]-\pi, \pi]$. Above the real axis, we have that $\arg(z) \in [0, \pi]$, while below the real axis $\arg(z) \in]0, -\pi[$. Writing the coordinates of a complex number z as $(|z|, \theta)$ is called the **polar form**.

Based on the left-hand side of Fig. 8.1, trigonometry tells us that $a_1 = |z_1| \cos \theta_1$ and $b_1 = |z_1| \sin \theta_1$, so that we can write the **trigonometric form** of a complex number as follows:

$$z_1 = a_1 + ib_1 = |z_1| \cos \theta_1 + i|z_1| \sin \theta_1 = |z_1| (\cos \theta_1 + i \sin \theta_1)$$

For instance, suppose that $z_1 = -6 + 2i$. We know that the modulus is equal to $|z| = \sqrt{(-6)^2 + 2^2} = 2\sqrt{10}$. To find the argument θ_1 , consider, for example, the imaginary part $b_1 = |z_1| \sin \theta_1 \Leftrightarrow 2 = 2\sqrt{10} \sin \theta_1$. We find that $\theta_1 = \sin^{-1}\left(\frac{2}{2\sqrt{10}}\right) = 18.4^\circ$. However, since the complex number z_1 is located in the second quadrant, we know that $\theta_1 = 180^\circ - 18.4^\circ = 162^\circ$. We can thus write:

$$z_1 = -6 + 2i = 2\sqrt{10} (\cos 162^\circ + i \sin 162^\circ)$$

If we wish to graphically represent the **sum of two complex numbers**, we first draw a parallelogram with sides $|z_1|$ and $|z_2|$. The complex number that represents the sum is then identified as the end point of the parallelogram's diagonal that starts at the origin of the complex plane.

As an example, in the right-hand side of Fig. 8.1, the complex number z_3 represents the sum of the complex numbers z_1 and z_2 .

8.3 Properties and Operations

If $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, we have the following formulas for their sum, product and quotient as well as the square and the third power of a complex number (for the quotient, we demand that $z_2 \neq 0$):

- $z_1 + z_2 = (a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$
- $z_1 \cdot z_2 = (a_1 + b_1i)(a_2 + b_2i) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$
- $\frac{z_1}{z_2} = \frac{a_1 + b_1i}{a_2 + b_2i} = \left(\frac{a_1 + b_1i}{a_2 + b_2i}\right) \cdot \left(\frac{a_2 - b_2i}{a_2 - b_2i}\right) = \left(\frac{a_1a_2 + b_1b_2}{|z_2|^2}\right) + \left(\frac{a_2b_1 - a_1b_2}{|z_2|^2}\right)i$
- $z^2 = (a^2 - b^2) + 2abi$
- $z^3 = [a(a^2 - 3b^2)] + [b(3a^2 - b^2)]i$

Consider the complex numbers $z = -4 + 2i$ and $w = 4 - 4i$. We find, for instance, that:

- $z \cdot w = ((-4)4 - 2(-4)) + ((-4)(-4) + 4 \cdot 2)i = -8 + 24i$
- $\frac{w}{z} = \frac{4(-4) + (-4)2}{(2\sqrt{5})^2} + \frac{(-4)(-4) - 4 \cdot 2}{(2\sqrt{5})^2}i = -\frac{6}{5} + \frac{2}{5}i$
- $w^3 = [4(4^2 - 3(-4)^2)] + [(-4)(3 \cdot 4^2 - (-4)^2)]i = -128 - 128i$

If $n \in \mathbb{Z}$, we can calculate any power of the imaginary unit i as follows:

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

For example, $i^{39} = i^{4 \cdot 9 + 3} = -i$, $i^{101} = i^{4 \cdot 25 + 1} = i$, and $i^{-16} = i^{4 \cdot (-4)} = 1$.

The **complex conjugate** of a complex number $z = a + bi$ is defined as $\bar{z} = a - bi$. We summarize some of the main properties of complex conjugates in Fig. 8.2, whereby $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$ (and $z_2 \neq 0$ if it appears in the denominator of a quotient), and $n \in \mathbb{Z}$.

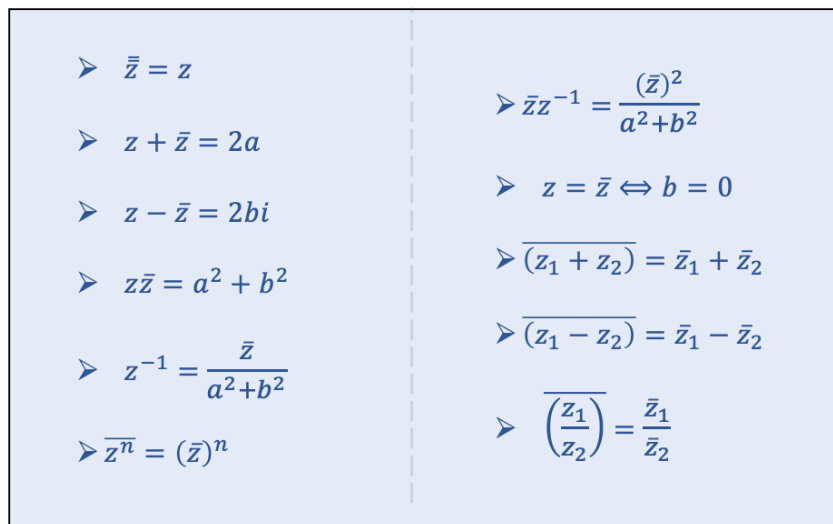


Figure 8.2: Some properties of conjugate complex numbers

For example, if we have the complex numbers $z = -3 + 2i$ and $w = 12 - i$, we find that:

- $\bar{z}z^{-1} = \frac{(-3-2i)^2}{(-3)^2+2^2} = \frac{5}{13} + \frac{12}{13}i$
- $\overline{w - z} = (12 + i) - (-3 - 2i) = 15 + 3i$
- $w\bar{w} = 12^2 + 1^2 = 145$

Let us now calculate z^3 if we consider the trigonometric form of z , i.e., $z = |z|(\cos \theta + i \sin \theta)$, whereby we use the trigonometric formulas of section 5.3:

$$\begin{aligned}
 z^3 &= |z|^3(\cos \theta + i \sin \theta)^2 = |z|^3(\cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta) \\
 &= |z|^3[\cos \theta(\cos^2 \theta - 3 \sin^2 \theta) + i \sin \theta(3 \cos^2 \theta - \sin^2 \theta)] \\
 &= |z|^3[\cos \theta(\cos 2\theta - 2 \sin^2 \theta) + i \sin \theta(2 \cos^2 \theta + \cos 2\theta)] \\
 &= |z|^3[(\cos \theta \cos 2\theta - \sin \theta \sin 2\theta) + i(\sin 2\theta \cos \theta + \sin \theta \cos 2\theta)] \\
 &= |z|^3(\cos 3\theta + i \sin 3\theta)
 \end{aligned}$$

The generalization of the above exercise is known as **De Moivre's Formula**:

$$z^n = |z|^n(\cos n\theta + i \sin n\theta)$$

If $z_1 = |z_1|(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = |z_2|(\cos \theta_2 + i \sin \theta_2)$, Fig. 8.3 summarizes some of the main properties of the modulus and the argument of a complex number, whereby $n \in \mathbb{Z}$ and $z_2 \neq 0$ if it appears in the denominator of a quotient.

➤ $ z_1 z_2 = z_1 z_2 $	➤ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
➤ $ z^n = z ^n$	➤ $\arg(z^n) = n \cdot \arg(z)$
➤ $ z = \bar{z} = -z $	➤ $\arg \left \frac{z_1}{z_2} \right = \arg(z_1) - \arg(z_2)$
➤ $z \bar{z} = z ^2 = \bar{z} ^2$	➤ $ z_1 + z_2 ^2 = z_1 ^2 + z_2 ^2 + 2\operatorname{Re}(z_1 \bar{z}_2)$
➤ $\left \frac{z_1}{z_2} \right = \frac{ z_1 }{ z_2 }$	➤ $ z_1 - z_2 ^2 = z_1 ^2 + z_2 ^2 - 2\operatorname{Re}(z_1 \bar{z}_2)$
➤ $ z = 0 \Leftrightarrow z = 0$	➤ $ az_1 - bz_2 ^2 + bz_1 + az_2 ^2 = (a^2 + b^2)(z_1 ^2 + z_2 ^2)$

Figure 8.3: Some properties of the modulus and argument of the complex numbers

Let us consider the complex numbers $z = -\sqrt{2} + i$ and $w = \sqrt{2} - i$ and calculate both the left-hand side and right-hand side of the expression $|z - w|^2 = |z|^2 + |w|^2 - 2\operatorname{Re}(z\bar{w})$.

With respect to the left-hand side, the complex number $z - w$ is equal to $z - w = (-\sqrt{2} + i) - (\sqrt{2} - i) = -2\sqrt{2} + 2i$, so that $|z - w|^2 = (-2\sqrt{2})^2 + 2^2 = 12$.

Regarding the right-hand side, we have that $|z|^2 = (-\sqrt{2})^2 + 1^2 = 3$, $|w|^2 = (\sqrt{2})^2 + (-1)^2 = 3$, and $z\bar{w} = (-\sqrt{2} + i)(\sqrt{2} + i) = -3$, so that $|z|^2 + |w|^2 - 2\operatorname{Re}(z\bar{w}) = 3 + 3 - 2(-3) = 12$.

We now wish to find the **square root of a complex number z** . If $z = a + bi$, then the roots $w = x + yi$ are found as follows (with $x \neq 0$ and $y \neq 0$):

$$\bullet \quad w^2 = z \Leftrightarrow (x + yi)^2 = a + bi \Leftrightarrow x^2 - y^2 + 2xyi = a + bi$$

$$\Leftrightarrow \begin{cases} x^2 - y^2 = a \\ 2xy = b \end{cases}$$

$$\bullet \quad x = \frac{b}{2y} \Rightarrow \left(\frac{b}{2y}\right)^2 - y^2 = a$$

$$\Leftrightarrow \frac{b^2}{4} - y^4 = ay^2$$

$$\Leftrightarrow y^4 + ay^2 - \frac{b^2}{4} = 0$$

$$\Leftrightarrow y^2 = \frac{-a \pm \sqrt{a^2 + b^2}}{2} = \frac{-a \pm |z|}{2}$$

$$\Leftrightarrow y = \pm \sqrt{\frac{-a \pm |z|}{2}}$$

- Only the following values of y produce correct solutions:

$$\Rightarrow y = \pm \sqrt{\frac{-a + |z|}{2}}$$

- Therefore, we obtain the following two sets of solutions:

$$\begin{cases} x = \frac{b}{2} \sqrt{\frac{2}{-a + |z|}} \\ y = \sqrt{\frac{-a + |z|}{2}} \end{cases} \quad \begin{cases} x = -\frac{b}{2} \sqrt{\frac{2}{-a + |z|}} \\ y = -\sqrt{\frac{-a + |z|}{2}} \end{cases}$$

Suppose we wish to find the roots $w = x + yi$ of the complex number $z = 16 - 30i$. Since $|z| = \sqrt{16^2 + (-30)^2} = 34$, the two solutions are the following:

$$\begin{cases} x = -\frac{30}{2} \sqrt{\frac{2}{-16 + 34}} = -5 \\ y = \sqrt{\frac{-16 + 34}{2}} = 3 \end{cases} \quad \begin{cases} x = \frac{30}{2} \sqrt{\frac{2}{-16 + 34}} = 5 \\ y = -\sqrt{\frac{-16 + 34}{2}} = -3 \end{cases}$$

The two solutions are therefore equal to $w_1 = -5 + 3i$ and $w_2 = 5 - 3i$.

If we express the complex number $z = a + bi$ in its trigonometric form, i.e., $z = |z|(\cos \theta + i \sin \theta)$, we can find its roots $w = |w|(\cos \phi + i \sin \phi)$ with the help of De Moivre's Formula:

$$\begin{aligned} \bullet \quad w^2 = z &\Leftrightarrow |w|^2(\cos 2\phi + i \sin 2\phi) = |z|(\cos \theta + i \sin \theta) \\ &\Leftrightarrow \begin{cases} |w|^2 = |z| \\ 2\phi = \theta + 2k\pi \quad \text{with } k \in \{0, 1\} \end{cases} \\ &\Leftrightarrow \begin{cases} |w| = \sqrt{|z|} \\ \phi = \frac{\theta}{2} + k\pi \quad \text{with } k \in \{0, 1\} \end{cases} \end{aligned}$$

- The two roots are therefore the following:

$$\begin{cases} w_1 = \sqrt{|z|} \left[\cos \left(\frac{\theta}{2} \right) + i \sin \left(\frac{\theta}{2} \right) \right] \\ w_2 = \sqrt{|z|} \left[\cos \left(\frac{\theta}{2} + \pi \right) + i \sin \left(\frac{\theta}{2} + \pi \right) \right] \end{cases}$$

If we generalize the above result for the n^{th} root of a complex number, we can succinctly write the n different roots w of the complex number $z = |z|(\cos \theta + i \sin \theta)$ as follows:

$$w = \sqrt[n]{|z|} \left[\cos \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right] \quad \text{with } k \in \{0, 1, 2, \dots, (n-1)\}$$

For example, taking the 4th root of the complex number $z = -1 + i$ will give us four different roots. First, we write z in its trigonometric form: $z = \sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$. The four roots are then the following:

$$\begin{aligned} \bullet \quad w_1 &= \sqrt[4]{\sqrt{2}} \left[\cos \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 0 \cdot \pi}{4} \right) + i \sin \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 0 \cdot \pi}{4} \right) \right] = \sqrt[8]{2} \left[\cos \left(\frac{3\pi}{16} \right) + i \sin \left(\frac{3\pi}{16} \right) \right] \\ \bullet \quad w_2 &= \sqrt[4]{\sqrt{2}} \left[\cos \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 1 \cdot \pi}{4} \right) + i \sin \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 1 \cdot \pi}{4} \right) \right] = \sqrt[8]{2} \left[\cos \left(\frac{11\pi}{16} \right) + i \sin \left(\frac{11\pi}{16} \right) \right] \\ \bullet \quad w_3 &= \sqrt[4]{\sqrt{2}} \left[\cos \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 2 \cdot \pi}{4} \right) + i \sin \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 2 \cdot \pi}{4} \right) \right] = \sqrt[8]{2} \left[\cos \left(\frac{19\pi}{16} \right) + i \sin \left(\frac{19\pi}{16} \right) \right] \\ \bullet \quad w_4 &= \sqrt[4]{\sqrt{2}} \left[\cos \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 3 \cdot \pi}{4} \right) + i \sin \left(\frac{3\pi}{4 \cdot 4} + \frac{2 \cdot 3 \cdot \pi}{4} \right) \right] = \sqrt[8]{2} \left[\cos \left(\frac{27\pi}{16} \right) + i \sin \left(\frac{27\pi}{16} \right) \right] \end{aligned}$$

Let us look at a couple more examples on complex numbers in general.

- Example 1

We wish to find the solutions to the following equation:

$$2z^2 - (3 + i)z + 1 + 2i = 0$$

In a first step, we write down the two solutions of this quadratic equation:

$$z = \frac{(3 + i) \pm \sqrt{(3 + i)^2 - 4 \cdot 2 \cdot (1 + 2i)}}{4} = \frac{(3 + i) \pm \sqrt{-10i}}{4}$$

In a second step, we determine the roots of $\sqrt{-10i}$:

$$\begin{aligned} \bullet \quad (x + yi)^2 = -10i &\Leftrightarrow x^2 - y^2 + 2xyi = -10i \\ &\Leftrightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = -10 \end{cases} \\ \bullet \quad y = -\frac{5}{x} &\Rightarrow x^2 - \left(-\frac{5}{x}\right)^2 = 0 \\ &\Leftrightarrow x^4 - 25 = 0 \\ &\Leftrightarrow x = \pm\sqrt{5} \end{aligned}$$

The two roots of $\sqrt{-10i}$ are therefore $w_1 = \sqrt{5} - i\sqrt{5}$ and $w_2 = -\sqrt{5} + i\sqrt{5}$. The two solutions of the initial equation then become:

$$\begin{cases} z_1 = \frac{(3 + i) + \sqrt{5} - i\sqrt{5}}{4} = \frac{(3 + \sqrt{5}) + (1 - \sqrt{5})i}{4} \\ z_2 = \frac{(3 + i) - \sqrt{5} + i\sqrt{5}}{4} = \frac{(3 - \sqrt{5}) + (1 + \sqrt{5})i}{4} \end{cases}$$

- Example 2

We will calculate $\left(\frac{z}{\bar{z}}\right)$ given that z is equal to:

$$z = \left(\frac{a + bi}{a - bi}\right)^2 - \left(\frac{a - bi}{a + bi}\right)^2 \quad \text{with } a \neq b$$

In a first step, we simplify the expression of z with the assistance of some calculation rules from Fig. 8.2:

$$\begin{aligned}
z &= \left(\frac{a+bi}{a-bi}\right)^2 - \left(\frac{a-bi}{a+bi}\right)^2 \\
&= \left[(a+bi) \cdot \frac{(a+bi)}{a^2+b^2}\right]^2 - \left[(a-bi) \cdot \frac{(a-bi)}{a^2+b^2}\right]^2 \\
&= \left[\frac{(a^2-b^2+2abi)^2}{(a^2+b^2)^2}\right] - \left[\frac{(a^2-b^2-2abi)^2}{(a^2+b^2)^2}\right] \\
&= \left[\frac{(a^2-b^2)^2 + 4abi(a^2-b^2) - 4a^2b^2}{(a^2+b^2)^2}\right] - \left[\frac{(a^2-b^2)^2 - 4abi(a^2-b^2) - 4a^2b^2}{(a^2+b^2)^2}\right] \\
&= \frac{8abi(a^2-b^2)}{(a^2+b^2)^2}
\end{aligned}$$

We can now calculate $\left(\frac{z}{\bar{z}}\right)$ as follows:

$$\begin{aligned}
\frac{z}{\bar{z}} &= z(\bar{z})^{-1} = z \cdot \frac{z}{[\operatorname{Re}(\bar{z})]^2 + [\operatorname{Im}(\bar{z})]^2} = z^2 \cdot \frac{1}{[\operatorname{Re}(\bar{z})]^2 + [\operatorname{Im}(\bar{z})]^2} \\
&= -\frac{64a^2b^2(a^2-b^2)^2}{(a^2+b^2)^4} \cdot \frac{1}{0^2 + \left(-\frac{8ab(a^2-b^2)}{(a^2+b^2)^2}\right)^2} \\
&= -\frac{64a^2b^2(a^2-b^2)^2}{(a^2+b^2)^4} \cdot \frac{(a^2+b^2)^4}{64a^2b^2(a^2-b^2)^2} \\
&= -1
\end{aligned}$$

- Example 3

In Fig. 8.3, we have seen that $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$. We now show that $|z_1 - z_2|^2$ is also equal to $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$.

In a first step, we calculate the left-hand side:

$$|z_1 - z_2|^2 = |(a_1 + b_1i) - (a_2 + b_2i)|^2 = |(a_1 - a_2) + (b_1 - b_2)i|^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2$$

The right-hand side is equal to:

$$\begin{aligned}
(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) &= [(a_1 + b_1i) - (a_2 + b_2i)][(a_1 - b_1i) - (a_2 - b_2i)] \\
&= [(a_1 - a_2) + (b_1 - b_2)i][(a_1 - a_2) - (b_1 - b_2)i] \\
&= (a_1 - a_2)^2 + (b_1 - b_2)^2
\end{aligned}$$

Both the left-hand side and the right-hand side are equal to each other, so we have proven the equation.

- Example 4

Let us identify the trigonometric form of z if we know that:

$$1 + \frac{z - \bar{z}}{1 + \bar{z}} = z$$

In a first step, we simplify the above equation:

$$1 + \frac{z - \bar{z}}{1 + \bar{z}} = z \Leftrightarrow \frac{(1 + \bar{z}) + (z - \bar{z})}{1 + \bar{z}} = z \Leftrightarrow \frac{1 + z}{1 + \bar{z}} = z \Leftrightarrow 1 + z = z(1 + \bar{z}) \Leftrightarrow 1 = z\bar{z}$$

Since $z\bar{z} = |z|^2$, we find that $|z| = 1$. But given that $|z| = |\bar{z}|$, the trigonometric form of z can be either $z = \cos \theta + i \sin \theta$ or $z = \cos \theta - i \sin \theta$.

- Example 5

We wish to demonstrate that the equation $z^2 + \bar{z}^2 - 2a(z + \bar{z}) + 2|w|^2 = 0$, with $z = x + iy$ and $w = a + bi$, represents a vertical hyperbola located at the center point $\vec{m} = (\operatorname{Re}(w), 0) = (a, 0)$ and with the distance between \vec{m} and a vertex equal to $\operatorname{Im}(w) = b$.

$$\begin{aligned} z^2 + \bar{z}^2 - 2a \cdot (z + \bar{z}) + 2|w|^2 &= 0 \\ \Leftrightarrow (x^2 - y^2 + 2xyi) + (x^2 - y^2 - 2xyi) - 2a[(x + iy) + (x - iy)] + 2(a^2 + b^2) &= 0 \\ \Leftrightarrow 2x^2 - 2y^2 - 4ax + 2a^2 + 2b^2 &= 0 \\ \Leftrightarrow (x - a)^2 - y^2 + b^2 &= 0 \\ \Leftrightarrow \frac{y^2}{b^2} - \frac{(x - a)^2}{b^2} &= 1 \end{aligned}$$

8.4 Complex Functions

The **complex exponential function** is defined as:

$$\exp : \mathbb{C} \rightarrow \mathbb{C} : z = c + i\theta \mapsto \exp(z) = e^c(\cos \theta + i \sin \theta)$$

Since the complex exponential of purely imaginary numbers have the form $e^{i\theta} = \cos \theta + i \sin \theta$, we can write any complex number in the following way:

$$z = a + bi = |z|(\cos \theta + i \sin \theta) = |z|e^{i\theta}$$

If $z = i\pi$, we obtain **Euler's identity**:

$$e^{i\pi} = e^0(\cos \pi + i \sin \pi) \Leftrightarrow e^{i\pi} = -1$$

If θ is replaced by $\theta + 2k\pi$ (with $k \in \mathbb{Z}$) in the definition of the complex exponential function, the function value for $\exp(z)$ does not change. Therefore, if we wish to define the inverse function of $\exp(z)$, we restrict its domain to the complex numbers $z = c + i\theta$, so that the principal argument θ lies in the interval $\theta \in]-\pi, \pi]$, and its codomain to \mathbb{C}_0 .

The inverse function of $\exp(z)$, which is called the **complex logarithmic function**, maps a complex number to just one logarithm of that complex number and is defined as follows:

$$\text{Ln} : \mathbb{C}_0 \rightarrow \{v \in \mathbb{C} \mid -\pi < \text{Im}(v) \leq \pi\} : z \mapsto v = \text{Ln}(z) = \text{Ln}(|z|e^{i\theta}) = \ln(|z|) + i\theta$$

Note that *all* the logarithms of the complex number z are represented by $\ln(|z|) + i(\theta + 2k\pi)$, with $k \in \mathbb{Z}$.

The **complex polynomial function** is defined as follows, with $n \in \mathbb{N}$:

$$P : \mathbb{C} \rightarrow \mathbb{C} : z \mapsto P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-1} z^{n-1} + \dots + a_2 z^2 + a_1 z + a_0$$

Let us now have a look at the **complex trigonometric and hyperbolic functions**. If we construct the Maclaurin series around the point $a = 0$ for the functions $\cos x$, $\sin x$, $\cosh x$, and $\sinh x$, we find the following expressions:

$$\begin{aligned} \bullet \cos x &= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} & \bullet \cosh x &= \sum_{n=1}^{+\infty} \frac{x^{2n-2}}{(2n-2)!} \\ \bullet \sin x &= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} & \bullet \sinh x &= \sum_{n=1}^{+\infty} \frac{x^{2n-1}}{(2n-1)!} \end{aligned}$$

If we use the argument ix instead of x in the above series, we find the following relations:

$$\begin{aligned} \bullet \cos(ix) &= \cosh x & \bullet \sin(ix) &= i \sinh x \\ \bullet \cosh(ix) &= \cos x & \bullet \sinh(ix) &= i \sin x \end{aligned}$$

Let us now construct the definition of the complex cosine function:

$$\begin{aligned} \cos : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \cos(z) &= \cos(x + iy) = \cos(x) \cos(iy) - \sin(x) \sin(iy) \\ &= \cos(x) \cosh(y) - i \sin(x) \sinh(y) \end{aligned}$$

Similarly, the definition of the complex sine function is found to be:

$$\begin{aligned}\sin : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \sin(z) &= \sin(x + iy) = \sin(x) \cos(iy) + \cos(x) \sin(iy) \\ &= \sin(x) \cosh(y) + i \cos(x) \sinh(y)\end{aligned}$$

We can also construct a definition of the complex cosine function based on the complex exponential function:

Given that

- $e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i \sin x)$
- $e^{-iz} = e^{-i(x+iy)} = e^{y-ix} = e^y(\cos x - i \sin x)$

We find that

$$\begin{aligned}e^{iz} + e^{-iz} &= e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x) \\ &= (e^{-y} + e^y) \cos x - i(e^y - e^{-y}) \sin x \\ &= (2 \cosh y) \cos x - i(2 \sinh y) \sin x \\ &= 2 \cos(iy) \cos x - 2 \sin(iy) \sin x \\ &= 2 \cos(x + iy) \\ &= 2 \cos z\end{aligned}$$

Similar derivations for the complex sine, complex hyperbolic cosine, and complex hyperbolic sine function gives us the following definitions:

$$\left\{ \begin{array}{l} \cos : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \\ \sin : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \\ \cosh : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \cosh(z) = \frac{e^z + e^{-z}}{2} \\ \sinh : \mathbb{C} \rightarrow \mathbb{C} : z = x + iy \mapsto \sinh(z) = \frac{e^z - e^{-z}}{2} \end{array} \right.$$

Note that the trigonometric formulas of section 5.3 are also applicable to the complex trigonometric functions.

Let us consider a couple of examples.

- Example 1

We will factor the complex polynomial function $P(z) = z^4 + z^3 + z + 1$ into both real and complex terms. We find a first root of $P(z)$ when $z = -1$, because $P(-1) = 0$. We then write, either through educated guess work or via Horner's method, that $P(z) = (z+1)(z^3+1)$.

In order to further factor the term $(z^3 + 1)$, we have to find the three different 3rd roots of -1 , since $z^3 + 1 = 0 \Leftrightarrow z = \sqrt[3]{-1}$. We can write -1 as $-1 = \cos \pi + i \sin \pi$, so that the three roots are equal to:

- $z_1 = \sqrt[3]{1} \left[\cos \left(\frac{\pi}{3} + \frac{2 \cdot 0 \cdot \pi}{3} \right) + i \sin \left(\frac{\pi}{3} + \frac{2 \cdot 0 \cdot \pi}{3} \right) \right] = \frac{1}{2} + i \frac{\sqrt{3}}{2}$
- $z_2 = \sqrt[3]{1} \left[\cos \left(\frac{\pi}{3} + \frac{2 \cdot 1 \cdot \pi}{3} \right) + i \sin \left(\frac{\pi}{3} + \frac{2 \cdot 1 \cdot \pi}{3} \right) \right] = -1$
- $z_3 = \sqrt[3]{1} \left[\cos \left(\frac{\pi}{3} + \frac{2 \cdot 2 \cdot \pi}{3} \right) + i \sin \left(\frac{\pi}{3} + \frac{2 \cdot 2 \cdot \pi}{3} \right) \right] = \frac{1}{2} - i \frac{\sqrt{3}}{2}$

Factoring the complex polynomial function then gives us the following result:

$$P(z) = (z + 1)^2 \left(z - \frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \left(z - \frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$

If we would multiply the two complex terms, the polynomial would obtain the form $P(z) = (z + 1)^2(z^2 - z + 1)$, which is what we would get when factoring the polynomial only into real terms, because the determinant of the term $(z^2 - z + 1)$ is negative and we would not be able to further factor this term.

• Example 2

We wish to find all the complex logarithms of the complex number $z = -6\sqrt{3} - 6i$.

In a first step, we write z in its trigonometric form. The modulus is equal to $|z| = \sqrt{(-6\sqrt{3})^2 + (-6)^2} = 12$. Since $-6\sqrt{3} = 12 \cos \theta$, we find that $\cos \theta = -\frac{\sqrt{3}}{2} \Leftrightarrow \theta = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. But since the principal argument must lie within the interval $]-\pi, \pi]$, we have to write θ as $\theta = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$. We thus write:

$$z = -6\sqrt{3} - 6i = 12 \left[\cos \left(\frac{5\pi}{6} \right) - i \sin \left(\frac{5\pi}{6} \right) \right]$$

In a next step, we find the complex logarithm for the principal argument, with the complex number v equal to $v = c + i\theta$:

$$\begin{aligned} \operatorname{Ln}(-6\sqrt{3} - 6i) = v &\Leftrightarrow -6\sqrt{3} - 6i = e^v = e^{c+i\theta} \\ &\Leftrightarrow 12 \left[\cos \left(\frac{5\pi}{6} \right) - i \sin \left(\frac{5\pi}{6} \right) \right] = e^c e^{i\theta} = e^c (\cos \theta + i \sin \theta) \\ &\Leftrightarrow \begin{cases} 12 = e^c &\Leftrightarrow c = \ln(12) \\ -\frac{5\pi}{6} = \theta \end{cases} \end{aligned}$$

The complex logarithm v is therefore equal to $v = \ln(12) - i\frac{5\pi}{6}$, if we only consider its principal argument. All the complex logarithms are then found to be equal to the set of solutions $O_v = \{v \in \mathbb{C} \mid v = \ln(12) + i\left(-\frac{5\pi}{6} + 2k\pi\right), k \in \mathbb{Z}\}$.

- Example 3

We wish to find the set of complex numbers O_v to which the complex number $v = c + i\theta$ belongs, if we know that $z = 9\left(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right)$ and that the following relationship exists:

$$\frac{\bar{z}}{z} = \frac{e^{v+\ln(|z|)}}{|z|}$$

We identify a solution for the principal argument of v as follows:

$$\begin{aligned} \frac{\bar{z}}{z} = \frac{e^{v+\ln(|z|)}}{|z|} &\Leftrightarrow \frac{\bar{z}^2}{|z|} = e^{v+\ln(|z|)} \Leftrightarrow \frac{81\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)}{9} = e^{c+\ln(|z|)+i\theta} \\ &\Leftrightarrow 9\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right) = e^c e^{\ln(9)}(\cos\theta + i\sin\theta) \\ &\Leftrightarrow \cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8} = e^c(\cos\theta + i\sin\theta) \\ &\Leftrightarrow \begin{cases} 1 = e^c & \Leftrightarrow c = \ln(1) = 0 \\ \frac{7\pi}{8} = \theta \end{cases} \end{aligned}$$

The complex logarithm v is equal to $v = i\frac{7\pi}{8}$ with respect to its principal argument. All the complex logarithms of v belong to the set of solutions $O_v = \{v \in \mathbb{C} \mid v = i\left(\frac{7\pi}{8} + 2k\pi\right), k \in \mathbb{Z}\}$.

- Example 4

We would like to solve the equation $\operatorname{sech} z = i$, with $z \in \mathbb{C}$. In a first step, let us rewrite the equation as follows:

$$\begin{aligned} \operatorname{sech} z = i &\Leftrightarrow \frac{1}{\cosh z} = i \Leftrightarrow \frac{2}{e^z + e^{-z}} = i \Leftrightarrow -2i = e^z + e^{-z} \\ &\Leftrightarrow e^{2z} + 2ie^z + 1 = 0 \end{aligned}$$

If we make the substitution $e^z = t$, we find the quadratic equation $t^2 + 2it + 1 = 0$, which we solve as follows:

$$t^2 + 2it + 1 = 0 \Leftrightarrow t = \frac{-2i \pm \sqrt{-8}}{2} = \frac{-2i \pm \sqrt{i^2 8}}{2} = (-1 \pm \sqrt{2})i$$

In a next step, we consider the solution $t = (-1 + \sqrt{2})i$:

- $e^z = t = (-1 + \sqrt{2})i \Leftrightarrow z = \ln [(-1 + \sqrt{2})i]$
- $\ln [(-1 + \sqrt{2})i] = v = c + i\theta \Leftrightarrow (-1 + \sqrt{2})i = e^v = e^c e^{i\theta}$

$$\Leftrightarrow (-1 + \sqrt{2}) \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = e^c (\cos \theta + i \sin \theta)$$

$$\Leftrightarrow \begin{cases} -1 + \sqrt{2} = e^c & \Leftrightarrow c = \ln(-1 + \sqrt{2}) \\ \frac{\pi}{2} = \theta \end{cases}$$

$$\Leftrightarrow v = \ln(-1 + \sqrt{2}) + i \left(\frac{\pi}{2} + 2k\pi \right)$$
- $z = \ln [(-1 + \sqrt{2})i]$

$$= \ln(-1 + \sqrt{2}) + i \left(\frac{\pi}{2} + 2k\pi \right)$$

A similar calculation can be done for $t = (-1 - \sqrt{2})i$:

- $e^z = t = (-1 - \sqrt{2})i \Leftrightarrow z = \ln [(-1 - \sqrt{2})i]$
- $\ln [(-1 - \sqrt{2})i] = v = c + i\theta \Leftrightarrow (-1 - \sqrt{2})i = e^v = e^c e^{i\theta}$

$$\Leftrightarrow (1 + \sqrt{2}) \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right) = e^c (\cos \theta + i \sin \theta)$$

$$\Leftrightarrow \begin{cases} 1 + \sqrt{2} = e^c & \Leftrightarrow c = \ln(1 + \sqrt{2}) \\ -\frac{\pi}{2} = \theta \end{cases}$$

$$\Leftrightarrow v = \ln(1 + \sqrt{2}) + i \left(-\frac{\pi}{2} + 2k\pi \right)$$
- $z = \ln [(-1 - \sqrt{2})i]$

$$= \ln(1 + \sqrt{2}) + i \left(-\frac{\pi}{2} + 2k\pi \right)$$

The solution S is equal to $S = \{z \in \mathbb{C} \mid z = \ln(\pm 1 + \sqrt{2}) + i(\pm \frac{\pi}{2} + 2k\pi)\}$.

- Example 5

We would like to calculate $\operatorname{arccoth}(i)$. In a first step, we will derive a formula for the inverse of the complex cotangent hyperbolic function:

$$\begin{aligned}
y = \operatorname{arccoth}(z) &\Leftrightarrow z = \operatorname{coth}(y) \Leftrightarrow z = \frac{e^y + e^{-y}}{e^y - e^{-y}} \Leftrightarrow z(e^y - e^{-y}) = e^y + e^{-y} \\
&\Leftrightarrow (z-1)e^y = (z+1)e^{-y} \\
&\Leftrightarrow e^{2y} = \frac{z+1}{z-1} \\
&\Leftrightarrow y = \frac{1}{2} \ln \left(\frac{z+1}{z-1} \right)
\end{aligned}$$

Note that this is the same expression as in the case of real variables (see Fig. 7.45). We can now calculate $\operatorname{arccoth}(i)$:

$$\operatorname{arccoth}(i) = \frac{1}{2} \ln \left(\frac{i+1}{i-1} \right) = \frac{1}{2} \ln \left(\frac{i+1}{i-1} \right) = \frac{1}{2} \ln(-i)$$

In a next step, we calculate the complex logarithms of $\ln(-i)$:

$$\begin{aligned}
\ln(-i) = v = c + i\theta &\Leftrightarrow -i = e^v = e^c e^{i\theta} \\
&\Leftrightarrow \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right) = e^c (\cos \theta + i \sin \theta) \\
&\Leftrightarrow \begin{cases} 1 = e^c & \Leftrightarrow c = 0 \\ -\frac{\pi}{2} = \theta \end{cases} \\
&\Leftrightarrow v = i \left(-\frac{\pi}{2} + 2k\pi \right) \quad \text{with } k \in \mathbb{Z}
\end{aligned}$$

In a final step, we determine $\operatorname{arccoth}(i)$:

$$\operatorname{arccoth}(i) = \frac{1}{2} \ln(-i) = \frac{1}{2} \left[i \left(-\frac{\pi}{2} + 2k\pi \right) \right] = i \left(-\frac{\pi}{4} + k\pi \right)$$

We can check this result if we would calculate $\operatorname{coth}(z)$ for, let's say, $z = -i\frac{\pi}{4}$; we should obtain $\operatorname{coth}(z) = i$.

- Example 6

In a final example, we wish to solve the equation $\sin(iz) = i^i$. Let us first rewrite the equation a bit:

$$\sin(iz) = i^i \Leftrightarrow \frac{e^{i(iz)} - e^{-i(iz)}}{2} = e^{\ln(i^i)} \Leftrightarrow \frac{e^{-z} - e^z}{2} = e^{i \ln(i)}$$

In a next step, we calculate the complex logarithms $\ln(i)$:

$$\begin{aligned}
\ln(i) = v = c + i\theta &\Leftrightarrow i = e^v = e^c e^{i\theta} \\
&\Leftrightarrow \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = e^c (\cos \theta + i \sin \theta) \\
&\Leftrightarrow \begin{cases} 1 = e^c &\Leftrightarrow c = 0 \\ \frac{\pi}{2} = \theta \end{cases} \\
&\Leftrightarrow v = i \left(\frac{\pi}{2} + 2k\pi\right) \quad \text{with } k \in \mathbb{Z}
\end{aligned}$$

Inserting this result into the above equation, we obtain the following solutions for z :

$$\begin{aligned}
\frac{e^{-z} - e^z}{2} = e^{i \ln(i)} &\Leftrightarrow \frac{e^{-z} - e^z}{2} = e^{i[\frac{\pi}{2} + 2k\pi]} = e^{-\left(\frac{\pi}{2} + 2k\pi\right)} \\
&\Leftrightarrow e^z + 2e^{-\left(\frac{\pi}{2} + 2k\pi\right)} - e^{-z} = 0 \\
&\Leftrightarrow e^{2z} + 2e^z e^{-\left(\frac{\pi}{2} + 2k\pi\right)} - 1 = 0 \\
&\Leftrightarrow e^z = \frac{-2e^{-\left(\frac{\pi}{2} + 2k\pi\right)} \pm \sqrt{4e^{-(\pi + 4k\pi)} + 4}}{2} \\
&\quad = -e^{-\left(\frac{\pi}{2} + 2k\pi\right)} \pm \sqrt{e^{-(\pi + 4k\pi)} + 1} \\
&\Leftrightarrow z = \ln\left(-e^{-\left(\frac{\pi}{2} + 2k\pi\right)} \pm \sqrt{e^{-(\pi + 4k\pi)} + 1}\right)
\end{aligned}$$

In the case of a plus sign in front of the square root, the expression within the natural logarithm is positive. However, in case of a minus sign, the expression becomes negative and we need to calculate the complex logarithm:

$$\begin{aligned}
&\ln\left(-e^{-\left(\frac{\pi}{2} + 2k\pi\right)} - \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) = v = c + i\theta \\
&\Leftrightarrow -e^{-\left(\frac{\pi}{2} + 2k\pi\right)} - \sqrt{e^{-(\pi + 4k\pi)} + 1} = e^v = e^c e^{i\theta} \\
&\Leftrightarrow \left(e^{-\left(\frac{\pi}{2} + 2k\pi\right)} + \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) (\cos \pi + i \sin \pi) = e^c (\cos \theta + i \sin \theta) \\
&\Leftrightarrow \begin{cases} c = \ln\left(e^{-\left(\frac{\pi}{2} + 2k\pi\right)} + \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) \\ \theta = \pi \end{cases} \\
&\Leftrightarrow v = \ln\left(e^{-\left(\frac{\pi}{2} + 2k\pi\right)} + \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) + i(\pi + 2k\pi) \quad \text{with } k \in \mathbb{Z}
\end{aligned}$$

The two solutions are therefore equal to:

$$\begin{cases} z = \ln\left(-e^{-\left(\frac{\pi}{2} + 2k\pi\right)} + \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) \\ z = \ln\left(e^{-\left(\frac{\pi}{2} + 2k\pi\right)} + \sqrt{e^{-(\pi + 4k\pi)} + 1}\right) + i(\pi + 2k\pi) \end{cases}$$

9 Integral Calculus

Integral calculus is the branch within mathematics that not only studies the concept of the integral and its properties, but also deals with the different methods to solve integrals as well as the various ways to calculate length, area, and volume.

Integral calculus is intimately connected to differential calculus (see section 7.5 and 10) and together they form the basis of mathematical analysis and calculus.

9.1 Indefinite Integrals

9.1.1 Definitions and Properties

A **primitive function** F of a certain real function $f(x)$ is defined as the function whose derivative is equal to $f(x)$:

$$F' = f(x)$$

The **indefinite integral** is then defined as the set of all primitive functions of $f(x)$ and is denoted by:

$$\int f(x)dx = F(x) + c$$

whereby $c \in \mathbb{R}$ is known as the **integration constant**. The function $f(x)$ that we wish to integrate is called the **integrand**.

For example, let us consider the function:

$$f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 9x^{12} + 5x^7 - 6x^3 + 5$$

The primitive function $F(x)$ of $f(x)$ is then equal to:

$$F : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto F(x) = \frac{9}{13}x^{13} + \frac{5}{8}x^8 - \frac{3}{2}x^4 + 5x$$

because, if we take the derivative of $F(x)$, we obtain the above function $f(x)$. We can then write the integral of $f(x)$ as follows:

$$\int (9x^{12} + 5x^7 - 6x^3 + 5) dx = \frac{9}{13}x^{13} + \frac{5}{8}x^8 - \frac{3}{2}x^4 + 5x + c$$

The process of taking the integral of a function $f(x)$ is called **integration** and is the reverse process of differentiation. Therefore, if we go from right to left in Fig. 7.18 (see section 7.5.1), we obtain a list in below Fig. 9.1 with the most common integrals, which can be directly integrated without the need for the introduction of special techniques. Assume hereby that all functions and variables are well-defined, and note furthermore that the hyperbolic functions

have been added to the list.

$\int f(x) dx = F(x) + c$	$\int f(x) dx = F(x) + c$
$\int kx^{k-1} dx = x^k + c$	$\int -\coth(x) \operatorname{csch} x dx = \operatorname{csch} x + c$
$\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + c$	$\int \tan(x) \sec x dx = \sec x + c$
$\int e^x dx = e^x + c$	$\int -\tanh(x) \operatorname{sech} x dx = \operatorname{sech} x + c$
$\int a^x \ln a dx = a^x + c$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$
$\int \frac{1}{x} dx = \ln x + c$	$\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arcsinh} x + c$
$\int \frac{1}{x \ln a} dx = \log_a x + c$	$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + c$
$\int \cos x dx = \sin x + c$	$\int \frac{1}{\sqrt{x^2-1}} dx = \operatorname{arccosh} x + c$
$\int \cosh x dx = \sinh x + c$	$\int \frac{1}{1+x^2} dx = \arctan x + c$
$\int -\sin x dx = \cos x + c$	$\int \frac{1}{1-x^2} dx = \operatorname{arctanh} x + c$
$\int \sinh x dx = \cosh x + c$	$\int \frac{-1}{1+x^2} dx = \operatorname{arccot} x + c$
$\int \sec^2 x dx = \tan x + c$	$\int \frac{1}{1-x^2} dx = \operatorname{arccoth} x + c$
$\int \operatorname{sech}^2 x dx = \tanh x + c$	$\int \frac{-1}{x\sqrt{x^2-1}} dx = \operatorname{arccsc} x + c$
$\int -\csc^2 x dx = \cot x + c$	$\int \frac{-1}{x\sqrt{x^2+1}} dx = \operatorname{arccsch} x + c$
$\int -\operatorname{csch}^2 x dx = \operatorname{coth} x + c$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + c$
$\int -\cot(x) \csc x dx = \csc x + c$	$\int \frac{-1}{x\sqrt{1-x^2}} dx = \operatorname{arcsech} x + c$

Figure 9.1: A list of common integrals

A remark must be made with respect to the integral $\int \left(\frac{1}{1-x^2}\right) dx$, which results either in the arc tangent hyperbolic function or in the arc cotangent hyperbolic function. In order to know what the right primitive function is, we have to look at the value of x (see the domain of both functions in Fig. 7.45): if $|x| < 1$, we choose the arc tangent hyperbolic function, whereas if $|x| > 1$ we have to opt for the arc cotangent hyperbolic function.

An important property of the indefinite integral is its **linearity**, as shown below, whereby $a, b \in \mathbb{R}$:

$$\int [af(x) + bg(x)] dx = a \int f(x) dx + b \int g(x) dx$$

Let us consider a couple of examples.

- Example 1

$$\begin{aligned} & \int [-4 \sin x \cos x \cos 2x \sin 5x + 2 \cos 5x \cos^2 2x - \cos 5x] dx \\ &= \int [-2 \sin 2x \cos 2x \sin 5x + \cos 5x (2 \cos^2 2x - 1)] dx \\ &= \int [-\sin 4x \sin 5x + \cos 5x \cos 4x] dx \\ &= \int \cos 9x dx \\ &= \frac{1}{9} \sin 9x + c \end{aligned}$$

- Example 2

$$\begin{aligned} & \int \left[-\sqrt{1-x^2} - \frac{2x^2-3}{2\sqrt{1-x^2}} + \frac{4\sqrt{x^2-1}}{(3x-1)^2+6x-10} \right] dx \\ &= \int \left[\frac{-2(1-x^2) - (2x^2-3)}{2\sqrt{1-x^2}} + \frac{4\sqrt{x^2-1}}{9x^2-6x+1+6x-10} \right] dx \\ &= \int \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} dx + \int \frac{4}{9} \cdot \frac{\sqrt{x^2-1}}{x^2-1} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx + \frac{4}{9} \int \frac{1}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} \arcsin x + \frac{4}{9} \operatorname{arccosh} x + c \end{aligned}$$

- Example 3

$$\begin{aligned}
 \int \left[x e^{5x} + \frac{2 \sin^3 3x + 2 \sin 3x \cos^2 3x}{1 + \cos 6x} \right] dx &= \int \left[x e^{5x} + \frac{2 \sin 3x (\sin^2 3x + \cos^2 3x)}{1 + \cos 6x} \right] dx \\
 &= \int \left[x e^{5x} + \frac{2 \sin 3x}{1 + \cos 6x} \right] dx \\
 &= \int \left[x e^{5x} + \frac{\sin 3x}{\cos^2 3x} \right] dx \\
 &= \int x e^{5x} dx + \int \tan 3x \sec 3x dx \\
 &= \frac{1}{5} \left(x - \frac{1}{5} \right) e^{5x} + \frac{1}{3} \sec 3x + c
 \end{aligned}$$

- Example 4

$$\begin{aligned}
 \int \left[\frac{(x^2 + 1) \left(\ln \frac{1}{100} \right)^{-1}}{x^3} \right] dx &= \int \left[\frac{\left(\ln \frac{1}{100} \right)^{-1}}{x} + \frac{\left(\ln \frac{1}{100} \right)^{-1}}{x^3} \right] dx \\
 &= \int \left[\frac{(-2 \ln 10)^{-1}}{x} + \frac{(-2 \ln 10)^{-1}}{x^3} \right] dx \\
 &= \int -\frac{1}{2} \cdot \frac{1}{x \ln 10} dx + \int -\frac{1}{2} \cdot \frac{1}{x^3 \ln 10} dx \\
 &= -\frac{1}{2} \int \frac{1}{x \ln 10} dx - \frac{1}{2 \ln 10} \int \frac{1}{x^3} dx \\
 &= -\frac{1}{2} \log x + \frac{1}{4x^2 \ln 10} + c \\
 &= \log \frac{1}{\sqrt{x}} + \frac{1}{4x^2 \ln 10} + c
 \end{aligned}$$

9.1.2 Integration Techniques

A. Partial Integration

Consider two differentiable functions $u, v : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$. If we take the differential of the product of these two functions and subsequently integrate the obtained expression, we find the formula for partial integration:

$$\begin{aligned} d[u(x)v(x)] &= u(x)d[v(x)] + v(x)d[u(x)] \\ \Leftrightarrow u(x)d[v(x)] &= d[u(x)v(x)] - v(x)d[u(x)] \\ \Leftrightarrow \int u(x)d[v(x)] &= \int d[u(x)v(x)] - \int v(x)d[u(x)] \\ \Leftrightarrow \int u(x)d[v(x)] &= u(x)v(x) - \int v(x)d[u(x)] \end{aligned}$$

When a certain integral is given, the first step is to identify $u(x)$ and write the integral in the form of the left-hand side of the above formula. Since more than one choice can be made with respect to the function $u(x)$ for a given integral, a useful guideline is the LIAGE rule, which stands for Logarithm, Inverse, Algebraic, Goniometric, and Exponential. This list of five types of functions is arranged in descending order in terms of priority as to the choice of the function $u(x)$.

- Example 1

Consider the following integral:

$$\int e^x(24x^2 + 6x + 1)dx$$

According to the LIAGE rule, the algebraic function has a higher priority than the exponential function, so that we choose $u(x) = 24x^2 + 6x + 1$. We then find that $d[u(x)] = (48x + 6)dx$, $d[v(x)] = e^x dx$, and, after integrating the last equation, $v(x) = e^x$. We can now write:

$$\begin{aligned} \int u(x)d[v(x)] &= u(x)v(x) - \int v(x)d[u(x)] \\ \Leftrightarrow \int (24x^2 + 6x + 1)(e^x dx) &= (24x^2 + 6x + 1)e^x - \int e^x [(48x + 6)dx] \end{aligned}$$

We apply again partial integration to the integral of the last term of the above equation. We set $u(x) = 48x + 6$, $d[u(x)] = 48dx$, $d[v(x)] = e^x dx$, and $v(x) = e^x$, so that this integral becomes:

$$\int e^x [(48x + 6)dx] = (48x + 6)e^x - \int 48e^x dx = (48x + 6)e^x - 48e^x - c$$

If we substitute this integral back into the original integral, we find the final solution:

$$\begin{aligned} \int e^x (24x^2 + 6x + 1) dx &= (24x^2 + 6x + 1) e^x - \int e^x [(48x + 6)dx] \\ &= (24x^2 + 6x + 1) e^x - [(48x + 6)e^x - 48e^x - c] \\ &= (24x^2 - 42x + 43) e^x + c \end{aligned}$$

- Example 2

Suppose we wish to calculate the following integral:

$$\int \ln^2 \left(\frac{x}{2} \right) dx$$

Following the LIAGE rule, we take $u(x) = \ln^2 \left(\frac{x}{2} \right)$, so that we have $d[u(x)] = \frac{2}{x} \ln \left(\frac{x}{2} \right) dx$, $d[v(x)] = dx$, and $v(x) = x$. The integral then becomes:

$$\int \ln^2 \left(\frac{x}{2} \right) dx = x \ln^2 \left(\frac{x}{2} \right) - 2 \int \ln \left(\frac{x}{2} \right) dx$$

Applying partial integration again to the integral of the last term within the above equation, we have that $u(x) = \ln \left(\frac{x}{2} \right)$, $d[u(x)] = \frac{1}{x} dx$, $d[v(x)] = dx$, and $v(x) = x$. This integral can then be written as:

$$\int \ln \left(\frac{x}{2} \right) dx = x \ln \left(\frac{x}{2} \right) - \int dx = x \ln \left(\frac{x}{2} \right) - (x + c)$$

Inserting the above result into the original integral, we find the final result:

$$\begin{aligned} \int \ln^2 \left(\frac{x}{2} \right) dx &= x \ln^2 \left(\frac{x}{2} \right) - 2 \int \ln \left(\frac{x}{2} \right) dx \\ &= x \ln^2 \left(\frac{x}{2} \right) - 2 \left[x \ln \left(\frac{x}{2} \right) - (x + c) \right] \\ &= x \ln \left(\frac{x}{2} \right) \left[\ln \left(\frac{x}{2} \right) - 2 \right] + 2x + 2c \end{aligned}$$

B. Integration of Rational Functions

As a first case, assume that the degree of the polynomial $P(x)$ in the numerator of a rational function $f(x) = \frac{P(x)}{Q(x)}$ is lower than the degree of the polynomial $Q(x)$ in the denominator. We can then apply the method of **partial fraction decomposition**.

In a first step, we need to factor $Q(x)$ so that the denominator obtains the form $Q(x) =$

$\prod_{i=1}^n p_i^{k_i}(x) \cdot \prod_{j=1}^r s_j^{m_j}(x)$, whereby $p_i(x)$ are first-degree polynomials and $s_j(x)$ second-degree polynomials with a non-zero negative discriminant, and $n, k_i, r, m_j \in \mathbb{N}_0$.

The method of partial fraction decomposition tells us that we can then write $f(x)$ as (with $a_{i,l}, b_{j,t}, c_{j,t} \in \mathbb{R}$):

$$f(x) = \sum_{i=1}^n \sum_{l=1}^{k_i} \frac{a_{i,l}}{p_i^l(x)} + \sum_{j=1}^r \sum_{t=1}^{m_j} \frac{b_{j,t}x + c_{j,t}}{s_j^t(x)}$$

- Example

Consider the following integral:

$$\int \frac{x^2 - 5x + 1}{x^4 - 2x^3 - 3x^2} dx$$

The denominator can be rewritten as $x^4 - 2x^3 - 3x^2 = x^2(x-3)(x+1)$. This means that we have $n = 3$ first-degree polynomials, i.e., $p_1 = x^2$ with $k_1 = 2$, $p_2 = x - 3$ with $k_2 = 1$, and $p_3 = x + 1$ with $k_3 = 1$, and $r = 0$ second-degree polynomials (with a non-zero negative discriminant). According to the method of partial fraction decomposition, we can now write the integrand $f(x)$ as:

$$f(x) = \frac{a_{1,1}}{p_1^1(x)} + \frac{a_{1,2}}{p_1^2(x)} + \frac{a_{2,1}}{p_2^1(x)} + \frac{a_{3,1}}{p_3^1(x)} = \frac{a_{1,1}}{x} + \frac{a_{1,2}}{x^2} + \frac{a_{2,1}}{x-3} + \frac{a_{3,1}}{x+1}$$

If we provide each fraction with the same denominator, we obtain the following integrand:

$$\begin{aligned} f(x) &= \frac{a_{1,1}x(x-3)(x+1) + a_{1,2}(x-3)(x+1) + a_{2,1}x^2(x+1) + a_{3,1}x^2(x-3)}{x^2(x-3)(x+1)} \\ &= \frac{(a_{1,1} + a_{2,1} + a_{3,1})x^3 + (-2a_{1,1} + a_{1,2} + a_{2,1} - 3a_{3,1})x^2 + (-3a_{1,1} - 2a_{1,2})x + (-3a_{1,2})}{x^2(x-3)(x+1)} \end{aligned}$$

As this integrand must be the same as the original integrand, we want the coefficients of the polynomial $P(x) = x^2 - 5x + 1$ to match the coefficients of the above polynomial in the numerator. In other words, we obtain a set of four equations with four unknown variables:

$$\begin{cases} a_{1,1} + a_{2,1} + a_{3,1} & = 0 \\ -2a_{1,1} + a_{1,2} + a_{2,1} - 3a_{3,1} & = 1 \\ -3a_{1,1} - 2a_{1,2} & = -5 \\ -3a_{1,2} & = 1 \end{cases}$$

The solutions to this system of equations are $a_{1,1} = \frac{17}{9}$, $a_{1,2} = -\frac{1}{3}$, $a_{2,1} = -\frac{5}{36}$, and $a_{3,1} = -\frac{7}{4}$. So, the integrand of our original integral becomes:

$$\frac{x^2 - 5x + 1}{x^4 - 2x^3 - 3x^2} = \frac{17}{9x} - \frac{1}{3x^2} - \frac{5}{36(x-3)} - \frac{7}{4(x+1)}$$

We can now solve the integral as follows:

$$\begin{aligned} \int \frac{x^2 - 5x + 1}{x^4 - 2x^3 - 3x^2} dx &= \frac{17}{9} \int \frac{1}{x} dx - \frac{1}{3} \int \frac{1}{x^2} dx - \frac{5}{36} \int \frac{1}{x-3} dx - \frac{7}{4} \int \frac{1}{x+1} dx \\ &= \frac{17}{9} \ln|x| + \frac{1}{3x} - \frac{5}{36} \ln|x-3| - \frac{7}{4} \ln|x+1| + c \end{aligned}$$

Note that the integration constant c is equal to the sum of all the integration constants produced by the four individual integrals (and multiplied by the respective factor). From now on, when we write an integration constant c , it refers to an aggregate term consisting of a number of individual integration constants.

As a second case, consider a rational function whereby the degree of the polynomial in the numerator is higher or equal than the degree of the polynomial in the denominator. Now, we first have to perform a **long division** (see section 2.3.2).

- Example

Let us have a look at the following integral:

$$\int \frac{x^3 - 6x + 2}{2x + 7} dx$$

Once we have performed long division, we can write the integrand as:

$$\frac{x^3 - 6x + 2}{2x + 7} = \left(\frac{x^2}{2} - \frac{7x}{4} + \frac{25}{8} \right) + \left(-\frac{159}{8(2x+7)} \right)$$

The integral is then solved in the following way:

$$\begin{aligned} \int \frac{x^3 - 6x + 2}{2x + 7} dx &= \frac{1}{2} \int x^2 dx - \frac{7}{4} \int x dx + \frac{25}{8} \int dx - \frac{159}{8} \int \frac{1}{2x+7} dx \\ &= \frac{x^3}{6} - \frac{7x^2}{8} + \frac{25x}{8} - \frac{159}{16} \ln|2x+7| + c \\ &= \frac{x^3}{6} - \frac{1}{8} \left(7x^2 - 25x + 159 \ln \sqrt{2x+7} \right) + c \end{aligned}$$

C. Substitution

The method of **substitution** consists of expressing the integrand $f(x)$ of a given integral in terms of another variable t , whereby $x = g(t)$ and $dx = g'(t)dt$, so that the original integral can be written as follows

$$\int f(x) dx = \int f(g(t)) \cdot g'(t) dt$$

• Example

Consider the following integral:

$$\int \frac{e^{3x+2} - 1}{e^{-3x} + e^{3x+1}} dx$$

We introduce the substitution $t = e^{3x}$, whereby $dx = \frac{dt}{3t}$, so that the integral obtains the following form:

$$\int \frac{e^{3x+2} - 1}{e^{-3x} + e^{3x+1}} dx = \int \left(\frac{e^2 t - 1}{t^{-1} + et} \right) \frac{dt}{3t} = \frac{1}{3} \int \frac{e^2 t - 1}{1 + et^2} dt$$

We now solve this integral as follows:

$$\begin{aligned} \frac{1}{3} \int \frac{e^2 t - 1}{1 + et^2} dt &= \frac{1}{3} \int \frac{e^2 t}{1 + et^2} dt - \frac{1}{3} \int \frac{1}{1 + et^2} dt \\ &= \frac{e}{6} \ln(et^2 + 1) - \frac{1}{3} \int \frac{1}{1 + (\sqrt{et})^2} dt + c \\ &= \frac{e}{6} \ln(et^2 + 1) - \frac{1}{3\sqrt{e}} \int \frac{1}{1 + s^2} ds + c \\ &= \frac{e}{6} \ln(et^2 + 1) - \frac{1}{3\sqrt{e}} \arctan s + c \\ &= \frac{e}{6} \ln \left[e (e^{3x})^2 + 1 \right] - \frac{1}{3\sqrt{e}} \arctan (\sqrt{e} \cdot e^{3x}) + c \\ &= \frac{e}{6} \ln(e^{6x+1} + 1) - \frac{1}{3\sqrt{e}} \arctan \left(e^{3x+\frac{1}{2}} \right) + c \end{aligned}$$

whereby in line 3 we introduced a second substitution $s = \sqrt{e} \cdot t$, with $dt = \frac{ds}{\sqrt{e}}$, and in line 5 we reversed the substitutions and expressed the result in terms of the variable x .

C. Integration via Reduction Formulas

If an integral cannot be integrated directly, we can rely on the **reduction formula** of an integral which is an expression of the original integral in terms of a recurring pattern of the original integral but in a form of reduced complexity. This method is typically used when the original integral contains an expression raised to a certain power.

- Example

Consider the following integral:

$$\int \frac{1}{(x^2 + 1)^2} dx$$

In a first step, we need to find its reduction formula. As a starting point, we consider the generalized version of the above integral and apply partial integration with $u(x)$ equal to the integrand:

$$\begin{aligned} I_n &= \int \frac{1}{(ax^2 + b)^n} dx = \frac{x}{(ax^2 + b)^n} + 2n \int \frac{ax^2}{(ax^2 + b)^{n+1}} dx \\ &= \frac{x}{(ax^2 + b)^n} + 2n \int \frac{ax^2 + b - b}{(ax^2 + b)^{n+1}} dx \\ &= \frac{x}{(ax^2 + b)^n} + 2n \left[\int \frac{1}{(ax^2 + b)^n} dx - b \int \frac{1}{(ax^2 + b)^{n+1}} dx \right] \\ &= \frac{x}{(ax^2 + b)^n} + 2n [I_n - bI_{n+1}] \\ &\Leftrightarrow I_{n+1} = \frac{x}{2bn(ax^2 + b)^n} + \left(\frac{2n-1}{2bn} \right) I_n \\ &\Leftrightarrow I_n = \frac{x}{2b(n-1)(ax^2 + b)^{n-1}} + \left[\frac{2n-3}{2b(n-1)} \right] I_{n-1} \end{aligned}$$

If we apply this reduction formula to our original integral with $a = 1$, $b = 1$, and $n = 2$, we find the solution as follows:

$$\begin{aligned} \int \frac{1}{(x^2 + 1)^2} dx &= \frac{x}{2 \cdot 1 \cdot (2-1)(x^2 + 1)^{2-1}} + \left[\frac{2 \cdot 2 - 3}{2 \cdot 1 \cdot (2-1)} \right] \int \frac{1}{(x^2 + 1)^{2-1}} dx \\ &= \frac{x}{2(x^2 + 1)} + \frac{1}{2} \arctan x + c \end{aligned}$$

D. Integration of Trigonometric Functions

In the general case, to solve integrals that contain trigonometric functions, we can use the **tangent half-angle substitution formulas**, whereby we introduce the substitution $t = \tan \frac{x}{2}$ and which we have discussed in section 5.3 (see Fig. 5.14).

- Example

Consider the following integral:

$$\int \frac{\sec x}{3 + \tan x} dx$$

If we implement the tangent half-angle substitution formulas, we can solve the integral as follows:

$$\begin{aligned} \int \frac{\sec x}{3 + \tan x} dx &= \int \left[\frac{1+t^2}{1-t^2} \right] \cdot \left(\frac{1}{3 + \left[\frac{2t}{1+t^2} \right]} \right) \cdot \left[\frac{2dt}{1+t^2} \right] \\ &= 2 \int \frac{1}{-3t^2 + 2t + 3} dt \\ &= -\frac{2}{3} \int \frac{1}{\left(t - \frac{1}{3}\right)^2 - \frac{10}{9}} dt \\ &= \frac{3}{5} \int \frac{1}{1 - \left[\frac{3}{\sqrt{10}} \left(t - \frac{1}{3}\right) \right]^2} dt \\ &= \frac{\sqrt{10}}{5} \int \frac{1}{1 - s^2} ds \\ &= \frac{\sqrt{10}}{5} \operatorname{arccoth} s \\ &= \frac{\sqrt{10}}{5} \cdot \frac{1}{2} \ln \left| \frac{s+1}{s-1} \right| \\ &= \frac{\sqrt{10}}{10} \ln \left| \frac{3 \tan \frac{x}{2} - 1 + \sqrt{10}}{3 \tan \frac{x}{2} - 1 - \sqrt{10}} \right| \end{aligned}$$

whereby in line 5 we have implemented the substitution $s = \frac{3}{\sqrt{10}} \left(t - \frac{1}{3}\right)$, with $dt = \frac{\sqrt{10}}{3} ds$, and in the last line we reversed the substitutions and expressed the result in terms of the variable x . Note that we would have gotten the same result if we opted for the method of partial fraction decomposition at line 2.

In the case where the integrand $f(x)$ is **not a rational function** and is written in the form of $f(x) = \cos^a(c_1x) \sin^b(c_2x)$, with $a, b \in \mathbb{Z}^+$ and $c_1, c_2 \in \mathbb{R}$, we can use the trigonometric formulas of section 5.3 to solve the integral. If $c_1 = c_2$ and at least a or b is an odd number, we can use the substitution method: if a (b) is odd, we implement the substitution $u = \sin(c_2x)$ ($u = \cos(c_1x)$).

- Example

Consider the following integral:

$$\int \sin^2(3x) \cos^3(5x) dx$$

We solve this integral as follows:

$$\begin{aligned} & \int \sin^2(3x) \cos^3(5x) dx \\ &= \int \left(\frac{1 - \cos 6x}{2} \right) \cos(5x) \left(\frac{1 + \cos 10x}{2} \right) dx \\ &= \frac{1}{4} \int [1 - \cos 6x + \cos 10x - \cos(6x) \cos(10x)] \cos 5x dx \\ &= \frac{1}{4} \int \left[1 - \cos 6x + \cos 10x - \frac{1}{2} (\cos 4x + \cos 16x) \right] \cos 5x dx \\ &= \frac{1}{8} \int [2 \cos 5x - 2 \cos(5x) \cos(6x) + 2 \cos(5x) \cos(10x) - \cos(4x) \cos(5x) - \\ & \quad \cos(5x) \cos(16x)] dx \\ &= \frac{1}{8} \int \left[2 \cos 5x - (\cos x + \cos 11x) + (\cos 5x + \cos 15x) - \frac{1}{2} (\cos x + \cos 9x) - \right. \\ & \quad \left. \frac{1}{2} (\cos 11x + \cos 21x) \right] dx \\ &= \frac{1}{8} \int \left(-\frac{3}{2} \cos x + 3 \cos 5x - \frac{1}{2} \cos 9x - \frac{3}{2} \cos 11x + \cos 15x - \frac{1}{2} \cos 21x \right) dx \\ &= \frac{1}{8} \left(-\frac{3}{2} \sin x + \frac{3}{5} \sin 5x - \frac{1}{18} \sin 9x - \frac{3}{22} \sin 11x + \frac{1}{15} \sin 15x - \frac{1}{42} \sin 21x \right) + c \end{aligned}$$

In case where the integrand $f(x)$ is an **odd function** with respect to the cosine of x , i.e., $f(\sin x, \cos x) = -f(\sin x, -\cos x)$, we perform the substitution $t = \sin x$, with $dt = \cos x dx$. In contrast, if $f(x)$ is an odd function with respect to the sine of x , i.e., $f(\sin x, \cos x) = -f(-\sin x, \cos x)$, we implement the substitution $t = \cos x$, with $dt = -\sin x dx$.

- Example

Let us have a look at the following integral:

$$\int \frac{\sec^2 x + 1}{\sin x} dx$$

Since the integrand is odd with respect to the sine of x , we introduce the substitution $t = \cos x$, with $dt = -\sin x dx$, and solve the integral as follows:

$$\begin{aligned} \int \frac{\sec^2 x + 1}{\sin x} dx &= - \int \frac{\frac{1}{t^2} + 1}{\sin^2 x} dt = - \int \frac{\frac{1}{t^2} + 1}{1 - t^2} dt \\ &= - \int \frac{\frac{1}{t^2} + 1}{(1 - t)(1 + t)} dt \\ &= - \int \frac{1 + t^2}{t^2(1 - t)(1 + t)} dt \\ &= - \int \left[\frac{1}{t^2} + \frac{1}{1 - t} + \frac{1}{1 + t} \right] dt \\ &= \frac{1}{t} + \ln |1 - t| - \ln |1 + t| + c \\ &= \frac{1}{t} + \ln \left| \frac{1 - t}{1 + t} \right| + c \\ &= \sec x + \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| + c \end{aligned}$$

whereby in line 4 we performed partial fraction decomposition.

In the case that the integrand $f(x)$ is a **rational function** and an **even function** with respect to **both the sine and the cosine of x** , i.e., $f(\sin x, \cos x) = f(-\sin x, -\cos x)$, we implement the substitution $t = \tan x$, with:

- $dx = \frac{dt}{1+t^2}$
- $\cos x = \frac{1}{\sqrt{1+t^2}}$
- $\sin x = \frac{t}{\sqrt{1+t^2}}$

- Example

Let us have a look at the following integral:

$$\int \frac{4 \sin^2(x) \cos x}{\sin x + 2 \cos x} dx$$

We solve this integral as follows:

$$\begin{aligned} & \int \frac{4 \sin^2(x) \cos x}{\sin x + 2 \cos x} dx \\ &= 4 \int \frac{\frac{t}{1+t^2} \cdot \frac{1}{\sqrt{1+t^2}}}{\frac{t}{\sqrt{1+t^2}} + \frac{2}{\sqrt{1+t^2}}} \cdot \frac{dt}{1+t^2} \\ &= 4 \int \frac{t^2}{(t+2)(1+t^2)^2} dt \\ &= 4 \int \left[\frac{4}{25} \frac{1}{(t+2)} - \frac{4}{25} \frac{(t-2)}{(1+t^2)} + \frac{1}{5} \frac{(t-2)}{(1+t^2)^2} \right] dt \\ &= \frac{4}{5} \int \left[\frac{4}{5} \frac{1}{(t+2)} - \frac{4}{5} \frac{(t-2)}{(1+t^2)} + \left(\frac{t}{(1+t^2)^2} - \frac{2}{(1+t^2)^2} \right) \right] dt \\ &= \frac{4}{5} \left[\frac{4}{5} \int \frac{1}{t+2} dt - \frac{4}{5} \int \frac{t-2}{1+t^2} dt + \int \frac{t}{(1+t^2)^2} dt - \int \frac{2}{(1+t^2)^2} dt \right] \\ &= \frac{4}{5} \left[\frac{4}{5} \int \frac{1}{t+2} dt - \frac{4}{5} \left(\int \frac{t}{1+t^2} dt - \int \frac{2}{1+t^2} dt \right) + \int \frac{t}{(1+t^2)^2} dt - \left(\frac{t}{t^2+1} + \int \frac{1}{t^2+1} dt \right) \right] \\ &= \frac{4}{5} \left[\frac{4}{5} \ln |t+2| - \frac{4}{5} \left(\frac{1}{2} \ln |1+t^2| - 2 \arctan t \right) - \frac{1}{2(1+t^2)} - \left(\frac{t}{t^2+1} + \arctan t \right) \right] + c \\ &= \frac{16}{25} \ln |t+2| - \frac{8}{25} \ln |1+t^2| + \frac{12}{25} \arctan t - \frac{2}{5(1+t^2)} - \frac{4t}{5(t^2+1)} + c \\ &= \frac{16}{25} \ln |\tan x + 2| - \frac{8}{25} \ln |1 + \tan^2 x| + \frac{12}{25} \arctan (\tan x) - \frac{2}{5(1+\tan^2 x)} - \frac{4 \tan x}{5(\tan^2 x+1)} + c \\ &= \frac{2}{25} [8 \ln |\sin x + 2 \cos x| - 5 \cos(x) (\cos x + 2 \sin x) + 6x] + c \end{aligned}$$

whereby going from line 3 to 4 we have applied partial fraction decomposition, and going from line 6 to 7, we have rewritten the last integral with the assistance of its reduction formula that we calculated under point C of this section.

E. Integration of Hyperbolic Functions

In the general case, we can use the **hyperbolic tangent half-angle substitution formulas**, which we discussed in section 7.6.9, whereby $t = \tanh \frac{x}{2}$.

• Example

Let us have a look at the following integral:

$$\int \frac{5 \operatorname{sech}(x)}{1 + 2 \sinh(x)} dx$$

We solve this integral as follows:

$$\begin{aligned} \int \frac{5 \operatorname{sech}(x)}{1 + 2 \sinh(x)} dx &= 5 \int \frac{(1-t^2)}{(1+t^2)} \cdot \frac{1}{1+2\left[\frac{2t}{1-t^2}\right]} \cdot \frac{2dt}{(1-t^2)} \\ &= 10 \int \frac{1-t^2}{(1+t^2)(1-t^2+4t)} dt \\ &= 10 \int \frac{1-t^2}{(1+t^2)(t-2+\sqrt{5})(-t+2+\sqrt{5})} dt \\ &= 10 \int \left[\frac{-\frac{2}{5}t + \frac{1}{5}}{1+t^2} + \frac{\frac{1}{5}}{t-2+\sqrt{5}} + \frac{-\frac{1}{5}}{-t+2+\sqrt{5}} \right] dt \\ &= -2 \left[\int \frac{2t-1}{1+t^2} dt - \int \frac{1}{t-2+\sqrt{5}} dt - \int \frac{1}{t-2-\sqrt{5}} dt \right] \\ &= -2 \left[\int \frac{2t}{1+t^2} dt - \int \frac{1}{1+t^2} dt - \int \frac{1}{t-2+\sqrt{5}} dt - \int \frac{1}{t-2-\sqrt{5}} dt \right] \\ &= -2 \left(\ln|1+t^2| - \arctan t - \ln|t-2+\sqrt{5}| - \ln|t-2-\sqrt{5}| \right) \\ &= 2 \arctan t + 2 \ln \left| \frac{t^2 - 4t - 1}{1+t^2} \right| \\ &= 2 \arctan \left(\tanh \frac{x}{2} \right) + 2 \ln \left| \frac{1 + 2 \sinh(x)}{\cosh(x)} \right| \end{aligned}$$

whereby we applied partial fraction decomposition to line 3 to obtain line 4.

In the case that the integrand $f(x)$ is **not a rational function** and is written in the form of $f(x) = \cosh^a(c_1x) \sinh^b(c_2x)$, with $a, b \in \mathbb{Z}^+$ and $c_1, c_2 \in \mathbb{R}$, we can use the hyperbolic formulas (see section 7.6.9) to solve the integral. If $c_1 = c_2$ and at least a or b is an odd number, we can use the substitution method: if a (b) is odd, we implement the substitution $u = \sinh(c_2x)$ ($u = \cosh(c_1x)$).

- Example

Consider the following integral:

$$\int [\cosh^2(7x) \sinh^5(3x) \cosh^2(4x) - 2 \sinh(7x) \cosh(4x) \sinh^5(3x) \cosh(7x) \sinh(4x) + \sinh^2(7x) \sinh^5(3x) \sinh^2(4x)] dx$$

We solve this integral as follows:

$$\begin{aligned} & \int [\cosh^2(7x) \sinh^5(3x) \cosh^2(4x) - 2 \sinh(7x) \cosh(4x) \sinh^5(3x) \cosh(7x) \sinh(4x) + \\ & \quad \sinh^2(7x) \sinh^5(3x) \sinh^2(4x)] dx \\ &= \int \sinh^5(3x) [\cosh^2(7x) \cosh^2(4x) - 2 \sinh(7x) \cosh(4x) \cosh(7x) \sinh(4x) + \\ & \quad \sinh^2(7x) \sinh^2(4x)] dx \\ &= \int \sinh^5(3x) [\cosh(7x) \cosh(4x) - \sinh(7x) \sinh(4x)]^2 dx \\ &= \int \sinh^5(3x) \cosh^2(3x) dx \\ &= \frac{1}{3} \int (u^2 - 1)^2 u^2 du \\ &= \frac{1}{3} \int (u^6 - 2u^4 + u^2) du \\ &= \frac{1}{3} \left(\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} \right) \\ &= \frac{\cosh^3(3x)}{3} \left(\frac{\cosh^4(3x)}{7} - \frac{2 \cosh^2(3x)}{5} + \frac{1}{3} \right) \end{aligned}$$

whereby we implemented in line 5 the substitution $u = \cosh(3x)$.

In case where the integrand $f(x)$ is an **odd function** with respect to the hyperbolic cosine of x , i.e., $f(\sinh x, \cosh x) = -f(\sinh x, -\cosh x)$, we perform the substitution $u = \sinh x$, with $du = \cosh(x)dx$. In contrast, if $f(x)$ is an odd function with respect to the hyperbolic sine of x , i.e., $f(\sinh x, \cosh x) = -f(-\sinh x, \cosh x)$, we implement the substitution $u = \cosh x$, with $du = \sinh(x)dx$.

- Example

Let us have a look at the following integral:

$$\int \frac{2 \sinh x + \tanh x}{\operatorname{csch}^2(x)(1 + \cosh x)} dx$$

We solve this integral in the following way:

$$\begin{aligned} \int \frac{2 \sinh x + \tanh x}{\operatorname{csch}^2(x)(1 + \cosh x)} dx &= \int \frac{\sinh^3(x)(2 + \operatorname{sech} x)}{(1 + \cosh x)} dx \\ &= \int \frac{(u^2 - 1)(2 + u^{-1})}{(1 + u)} du \\ &= \int \frac{(u - 1)(u + 1)(2 + u^{-1})}{(1 + u)} du \\ &= \int \frac{(u - 1)(2u + 1)}{u} du \\ &= \int \frac{2u^2 - u - 1}{u} du \\ &= \int 2u du - \int du - \int \frac{1}{u} du \\ &= u^2 - u - \ln |u| + c \\ &= \cosh^2 x - \cosh x - \ln |\cosh x| + c \end{aligned}$$

whereby in line 2 we introduced the substitution $u = \cosh x$.

In the case that the integrand $f(x)$ is a **rational function** and an **even function** with respect to **both the hyperbolic sine and hyperbolic cosine of x** , i.e., $f(\sinh x, \cosh x) = f(-\sinh x, -\cosh x)$, we implement the substitution $t = \tanh x$, with:

$$\bullet \quad dx = \frac{dt}{1-t^2} \qquad \bullet \quad \cosh x = \frac{1}{\sqrt{1-t^2}} \qquad \bullet \quad \sinh x = \frac{t}{\sqrt{1-t^2}}$$

- Example

Consider the following integral:

$$\int \frac{2 \coth(x) \operatorname{csch}^2 x}{\sinh(x) \cosh x + 1} dx$$

We solve this integral as follows:

$$\begin{aligned} \int \frac{2 \coth(x) \operatorname{csch}^2 x}{\sinh(x) \cosh x + 1} dx &= 2 \int \frac{\cosh(x)}{\sinh^3(x) (\sinh(x) \cosh x + 1)} dx \\ &= 2 \int \frac{\frac{1}{\sqrt{1-t^2}}}{\left[\frac{t}{\sqrt{1-t^2}}\right]^3 \left(\frac{t}{\sqrt{1-t^2}} \cdot \frac{1}{\sqrt{1-t^2}} + 1\right)} \cdot \frac{dt}{1-t^2} \\ &= 2 \int \frac{1-t^2}{t^3(-t^2+t+1)} dt \\ &= 2 \int \frac{1-t^2}{t^3 \left(t - \frac{1+\sqrt{5}}{2}\right) \left(-t + \frac{1-\sqrt{5}}{2}\right)} dt \\ &= 2 \int \left[\frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3} + \frac{-\frac{1}{2} + \frac{3\sqrt{5}}{10}}{t - \frac{1+\sqrt{5}}{2}} - \frac{\frac{1}{2} + \frac{3\sqrt{5}}{10}}{t - \frac{1-\sqrt{5}}{2}} \right] dt \\ &= 2 \left[\int \frac{1}{t} dt - \int \frac{1}{t^2} dt + \int \frac{1}{t^3} dt \right] + \left(-1 + \frac{3\sqrt{5}}{5}\right) \int \frac{2}{2t-1-\sqrt{5}} dt - \left(1 + \frac{3\sqrt{5}}{5}\right) \int \frac{2}{2t-1+\sqrt{5}} dt \\ &= 2 \ln |t| + \frac{2}{t} - \frac{1}{t^2} + \left(-1 + \frac{3\sqrt{5}}{5}\right) \ln |2t-1-\sqrt{5}| - \left(1 + \frac{3\sqrt{5}}{5}\right) \ln |2t-1+\sqrt{5}| + c \\ &= 2 \ln |\tanh x| + \coth(x)(2 - \coth x) - \ln |4(\tanh^2 x - \tanh x - 1)| \\ &\quad + \frac{3\sqrt{5}}{5} \ln \left| \frac{2 \tanh x - 1 - \sqrt{5}}{2 \tanh x - 1 + \sqrt{5}} \right| + c \end{aligned}$$

whereby we applied partial fraction decomposition to line 4.

F. Integration of Irrational Functions

In the case where the integrand $f(x)$ contains the expression $\sqrt[n]{(ax+b)^m}$, with $a, b, m \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $n \geq 2$, we can implement either the substitution $u = ax + b$ or the substitution $u = \sqrt[n]{(ax+b)^m}$.

- Example

Let us have a look at the following integral:

$$\int \frac{4x(x-1)+1}{3+\sqrt[3]{(2x-1)^2}} dx$$

We can solve this integral as follows:

$$\begin{aligned} \int \frac{4x(x-1)+1}{3+\sqrt[3]{(2x-1)^2}} dx &= \int \frac{(2x-1)^2}{3+\sqrt[3]{(2x-1)^2}} dx \\ &= \frac{3}{4} \int \frac{u^{\frac{7}{2}}}{3+u} du \\ &= \frac{3}{2} \int \frac{t^8}{3+t^2} dt \\ &= \frac{3}{2} \int \left[t^6 - 3t^4 + 9t^2 - 27 + \frac{81}{3+t^2} \right] dt \\ &= \frac{3}{2} \left[\int t^6 dt - 3 \int t^4 dt + 9 \int t^2 dt - 27 \int dt + 81 \int \frac{1}{3+t^2} dt \right] \\ &= \frac{3}{2} \left[\frac{t^7}{7} - \frac{3t^5}{5} + 3t^3 - 27t + \frac{81\sqrt{3}}{3} \int \frac{1}{1+s^2} ds \right] \\ &= \frac{3}{2} \left[\frac{u^{\frac{7}{2}}}{7} - \frac{3u^{\frac{5}{2}}}{5} + 3u^{\frac{3}{2}} - 27u^{\frac{1}{2}} + \frac{81\sqrt{3}}{3} \arctan s \right] + c \\ &= \frac{3(2x-1)^{\frac{7}{3}}}{14} - \frac{9(2x-1)^{\frac{5}{3}}}{10} + \frac{9(2x-1)}{2} - \frac{81(2x-1)^{\frac{1}{3}}}{2} \\ &\quad + \frac{81\sqrt{3}}{2} \arctan \left[\frac{(2x-1)^{\frac{1}{3}}}{\sqrt{3}} \right] + c \end{aligned}$$

whereby we have implemented the substitution $u = \sqrt[3]{(2x-1)^2}$ in line 2, the substitution $t = \sqrt{u}$ in line 3, and the substitution $s = \frac{t}{\sqrt{3}}$ in the last integral of line 6. In addition, we have performed long division in line 4.

In the case that the integrand $f(x)$ contains the expression $\sqrt[n]{\left(\frac{ax+b}{cx+d}\right)^m}$, with $a, b, c, d, m \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $n \geq 2$, we can implement the substitution $t^n = \left(\frac{ax+b}{cx+d}\right)$.

- Example

Consider the following integral:

$$\int \sqrt[4]{\frac{9x^2 - 6x + 1}{16x^2 - 24x + 9}} dx$$

We can solve this integral as follows:

$$\begin{aligned} & \int \sqrt[4]{\frac{9x^2 - 6x + 1}{16x^2 - 24x + 9}} dx \\ &= \int \sqrt[4]{\left(\frac{3x-1}{4x-3}\right)^2} dx \\ &= \int \sqrt{\frac{3x-1}{4x-3}} dx \\ &= -10 \int \frac{t^2}{(4t^2-3)^2} dt \\ &= -10 \left[-\frac{t}{8(4t^2-3)} + \frac{1}{8} \int \frac{1}{4t^2-3} dt \right] \\ &= -10 \left[-\frac{t}{8(4t^2-3)} + \frac{1}{8} \left(\frac{1}{4\sqrt{3}} \int \frac{1}{t-\frac{\sqrt{3}}{2}} dt - \frac{1}{\sqrt{3}} \int \frac{1}{4t+2\sqrt{3}} dt \right) \right] \\ &= \frac{5t}{4(4t^2-3)} - \frac{5}{16\sqrt{3}} \ln \left| t - \frac{\sqrt{3}}{2} \right| + \frac{5}{16\sqrt{3}} \ln |4t + 2\sqrt{3}| + c \\ &= \frac{1}{4} \sqrt{(3x-1)(4x-3)} + \frac{5}{16\sqrt{3}} \ln \left| \frac{4}{5}(24x-13) + \frac{16\sqrt{3}}{5} \sqrt{(3x-1)(4x-3)} \right| + c \end{aligned}$$

whereby we introduced the substitution $t^2 = \frac{3x-1}{4x-3}$ in line 4 and subsequently applied the below reduction formula in line 5 (with $m = 0$, $n = 1$, $a = 4$, and $b = -3$). Finally, in line 6 we performed partial fraction decomposition.

$$\int \frac{t^{m+2}}{(at^2+b)^{n+1}} dt = -\frac{t^{m+1}}{2an(at^2+b)^n} + \frac{(m+1)}{2an} \int \frac{t^m}{(at^2+b)^n} dt$$

In the case that the integrand $f(x)$ contains the expression $\sqrt{(ax^2 + bx + c)^m}$, we first rewrite the quadratic polynomial under the square root in one of the three ways as indicated in Fig. 9.2. Based on the specific form, a number of substitutions can then be implemented to solve the integral. Note that D refers to the discriminant $D = b^2 - 4ac$.

If $a < 0, D < 0$ and m is even or If $a > 0$ and $D < 0$	\Rightarrow	$\sqrt{(x^2 + s^2)^m}$	\Rightarrow	$\begin{cases} x = s \tan \theta \\ x = s \sinh \theta \end{cases}$
If $a > 0$ and $D > 0$	\Rightarrow	$\sqrt{(x^2 - s^2)^m}$	\Rightarrow	$\begin{cases} x = s \sec \theta \\ x = s \cosh \theta \end{cases}$
If $a < 0$ and $D > 0$	\Rightarrow	$\sqrt{(s^2 - x^2)^m}$	\Rightarrow	$\begin{cases} x = s \sin \theta & & x = s \tanh \theta \\ x = s \cos \theta & & x = s \operatorname{sech} \theta \end{cases}$

Figure 9.2: Method to solve integrals whereby a quadratic expression is written under a square root

- Example 1

Let us have a look at the following integral:

$$\int \sqrt{x^2 - x - 1} dx$$

We can solve this integral as follows:

$$\begin{aligned}
 \int \sqrt{x^2 - x - 1} dx &= \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}} dx \\
 &= \int \sqrt{s^2 - \frac{5}{4}} ds \\
 &= \int \sqrt{\frac{5}{4} \sec^2 \theta - \frac{5}{4}} \cdot \frac{\sqrt{5}}{2} \sec(\theta) \tan(\theta) d\theta \\
 &= \frac{5}{4} \int \sec(\theta) \tan^2(\theta) d\theta \\
 &= \frac{5}{4} \int \frac{u^2}{(1 - u^2)^2} du
 \end{aligned}$$

$$\begin{aligned}
&= \frac{5}{4} \left[\frac{u}{2(1-u^2)} - \frac{1}{2} \int \frac{1}{1-u^2} du \right] \\
&= \frac{5}{4} \left[\frac{u}{2(1-u^2)} - \frac{1}{4} \ln \left| \frac{1+u}{1-u} \right| \right] + c \\
&= \frac{1}{4} (2x-1) \sqrt{x^2-x-1} - \frac{5}{16} \ln \left| \frac{2x-1+2\sqrt{x^2-x-1}}{2x-1-2\sqrt{x^2-x-1}} \right| + c
\end{aligned}$$

whereby we inserted the substitution $s = x - \frac{1}{2}$ in line 2 and the substitution $s = \frac{\sqrt{5}}{2} \sec \theta$ in line 3, and we applied the reduction formula of the previous example (with $m = 0$, $n = 1$, $a = -1$, and $b = 1$) to the integral of line 5.

- Example 2

Consider the following integral:

$$\int \sqrt{(3x^2 - x + 1)^3} dx$$

We can solve this integral as follows:

$$\begin{aligned}
\int \sqrt{(3x^2 - x + 1)^3} dx &= \int \sqrt{\left[3 \left(x - \frac{1}{6} \right)^2 + \frac{11}{12} \right]^3} dx \\
&= \frac{11\sqrt{33}}{72} \int \sqrt{\left[\left[\frac{6}{\sqrt{11}} \left(x - \frac{1}{6} \right) \right]^2 + 1 \right]^3} dx \\
&= \frac{11\sqrt{33}}{72} \int \sqrt{(\tan^2 \theta + 1)^3} \cdot \frac{\sqrt{11}}{6} \sec^2(\theta) d\theta \\
&= \frac{121\sqrt{3}}{432} \int \sec^5(\theta) d\theta \\
&= \frac{121\sqrt{3}}{432} \left[\frac{1}{4} \tan(\theta) \sec^3(\theta) + \frac{3}{4} \int \sec^3(\theta) d\theta \right] \\
&= \frac{121\sqrt{3}}{432} \left[\frac{1}{4} \tan(\theta) \sec^3(\theta) + \frac{3}{4} \left[\frac{1}{2} \tan(\theta) \sec(\theta) + \frac{1}{2} \int \sec(\theta) d\theta \right] \right] \\
&= \frac{121\sqrt{3}}{432} \left[\frac{1}{4} \tan(\theta) \sec^3(\theta) + \frac{3}{4} \left[\frac{1}{2} \tan(\theta) \sec(\theta) + \frac{1}{2} \int \frac{1}{1-u^2} du \right] \right] \\
&= \frac{121\sqrt{3}}{432} \left[\frac{1}{4} \tan(\theta) \sec^3(\theta) + \frac{3}{8} \tan(\theta) \sec(\theta) + \frac{3}{16} \ln \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right| \right] + c
\end{aligned}$$

$$= \frac{1}{192}(6x-1)(24x^2-8x+19)\sqrt{3x^2-x+1} +$$

$$\frac{121}{768\sqrt{3}} \ln \left| \frac{1}{11} \left(4\sqrt{3}(6x-1)\sqrt{3x^2-x+1} + (72x^2-24x+13) \right) \right| + c$$

whereby we introduced the substitution $\tan \theta = \frac{6}{\sqrt{11}} \left(x - \frac{1}{6} \right)$ in line 3 and the substitution $u = \sin \theta$ in line 7. Both in line 5 and 6 we applied the below reduction formula with $n = 5$ and $n = 3$, respectively:

$$\int \sec^n(\theta) d\theta = \frac{1}{n-1} \tan(\theta) \sec^{n-2}(\theta) + \frac{n-2}{n-1} \int \sec^{n-2}(\theta) d\theta$$

- Example 3

Let us have a look at the following integral:

$$\int \frac{x^4 + 2}{\sqrt{x^2 + 2}} dx$$

We can solve this integral as follows:

$$\begin{aligned} \int \frac{x^4 + 2}{\sqrt{x^2 + 2}} dx &= \int \frac{x^4}{\sqrt{x^2 + 2}} dx + 2 \int \frac{1}{\sqrt{x^2 + 2}} dx \\ &= \int \frac{(\sqrt{2} \sinh \theta)^4}{\sqrt{(\sqrt{2} \sinh \theta)^2 + 2}} \cdot \sqrt{2} \cosh(\theta) d\theta + 2 \int \frac{1}{\sqrt{(\sqrt{2} \sinh \theta)^2 + 2}} \cdot \sqrt{2} \cosh(\theta) d\theta \\ &= 4 \int \sinh^4(\theta) d\theta + 2 \int d\theta \\ &= 4 \left[\cosh(\theta) \sinh^3(\theta) - 3 \int \cosh^2(\theta) \sinh^2(\theta) d\theta \right] + 2\theta + c \\ &= 4 \left[\cosh(\theta) \sinh^3(\theta) - \frac{3}{8} \int (\cosh(4\theta) - 1) d\theta \right] + 2\theta + c \\ &= 4 \cosh(\theta) \sinh^3(\theta) - \frac{3}{8} \sinh(4\theta) + \frac{7\theta}{2} + c \\ &= \frac{1}{4} x(x^2 - 3)\sqrt{x^2 + 2} + \frac{7}{2} \ln \left| \frac{1}{\sqrt{2}} \left(x + \sqrt{x^2 + 2} \right) \right| + c \end{aligned}$$

whereby we implemented the substitution $x = \sqrt{2} \sinh(\theta)$ in line 2 and applied the below reduction formula to the first integral of line 3.

$$\int \sinh^n(\theta) d\theta = \cosh(\theta) \sinh^{n-1}(\theta) - (n-1) \int \cosh^2(\theta) \sinh^{n-2}(\theta) d\theta$$

9.2 Definite Integrals

9.2.1 Definitions

Consider a continuous function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, whereby $f(x) \geq 0$ over the closed interval $[a, b]$, and suppose we wish to calculate the area under the graph of f restricted by $y = 0$, $x = a$, and $x = b$.

In a first instance, we identify a particular **partition** $P = (x_0 = a, x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{n-1}, x_n = b)$, whereby we have divided the interval $[a, b]$ in n **subintervals** of equal length $\Delta x = \frac{b-a}{n}$, with $i \in \{1, 2, \dots, n\}$, and whereby $[x_{i-1}, x_i]$ is defined as the i^{th} subinterval of $[a, b]$.

For each subinterval we can find the **minimum** function value $m_i = \min \{f(x) | x_{i-1} \leq x \leq x_i\}$, the **maximum** function value $M_i = \max \{f(x) | x_{i-1} \leq x \leq x_i\}$, and a **random** function value $f(x_i^*)$, with $x_{i-1} \leq x_i^* \leq x_i$.

We now define the **Riemann sum** as $R_f(P) = \sum_{i=1}^n f(x_i^*)\Delta x$, the **lower Riemann sum** as $L_f(P) = \sum_{i=1}^n m_i\Delta x$ and the **upper Riemann sum** as $U_f(P) = \sum_{i=1}^n M_i\Delta x$, whereby $L_f(P) \leq R_f(P) \leq U_f(P)$.

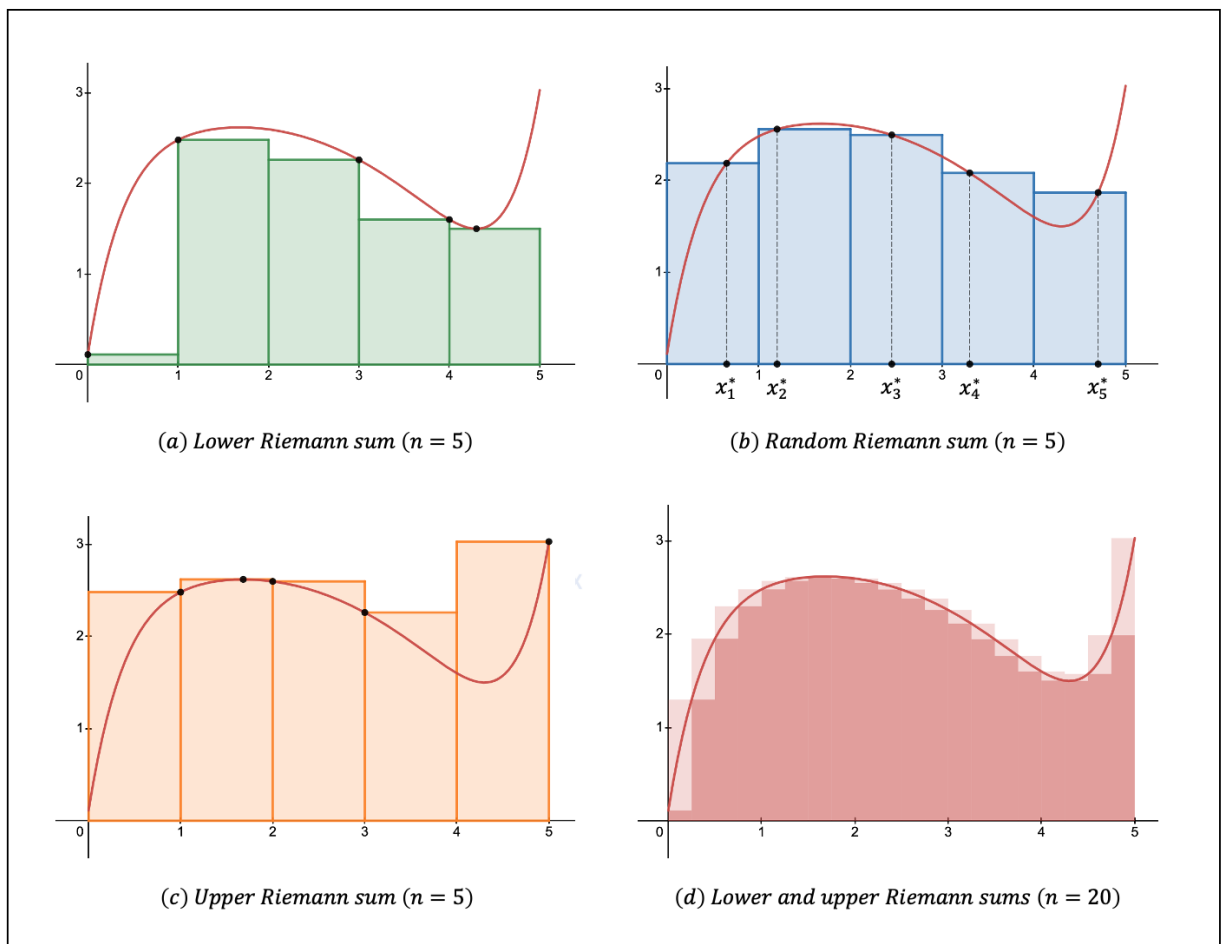


Figure 9.3: The lower, upper, and random Riemann sums of a function $f(x)$ for a given partition P

The **Riemann integral** or the **definite integral** over the interval $[a, b]$ is then defined as the area under the graph of f in the case that we consider an infinite number of subintervals and is denoted by:

$$I = \int_a^b f(x) dx = \lim_{n \rightarrow +\infty} L_f(P) = \lim_{n \rightarrow +\infty} U_f(P) = \lim_{n \rightarrow +\infty} R_f(P)$$

- Example

Suppose we wish to calculate the red-coloured area under the graph of the function $f : [0, 4] \rightarrow \mathbb{R} : x \mapsto f(x) = x^2$, as indicated in Fig. 9.4.

In a first step, we divide the interval $[0, 4]$ in n equal subintervals (with $n \neq 0$):

$$\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{8}{n}\right], \left[\frac{8}{n}, \frac{12}{n}\right], \dots, \left[\frac{4(n-1)}{n}, \frac{4n}{n}\right]$$

Given that f is a strictly increasing function over the interval $[0, 4]$, we find that for the k^{th} subinterval $\left[\frac{4(k-1)}{n}, \frac{4k}{n}\right]$ (with $k \in \{1, 2, \dots, n\}$) the smallest and largest function value, denoted by m_k and M_k , respectively, are equal to:

$$\begin{cases} m_k = \left[\frac{4(k-1)}{n}\right]^2 \\ M_k = \left[\frac{4k}{n}\right]^2 \end{cases}$$

Since the width of each subinterval is equal to $\Delta x = \frac{4}{n}$, the lower and upper Riemann sum for our particular partition are equal to:

$$\begin{cases} L_f = \sum_{k=1}^n m_k \Delta x = \sum_{k=1}^n \left[\frac{4(k-1)}{n}\right]^2 \cdot \frac{4}{n} \\ U_f = \sum_{k=1}^n M_k \Delta x = \sum_{k=1}^n \left[\frac{4k}{n}\right]^2 \cdot \frac{4}{n} \end{cases}$$

Let us, for instance, calculate the lower Riemann sum L_f :

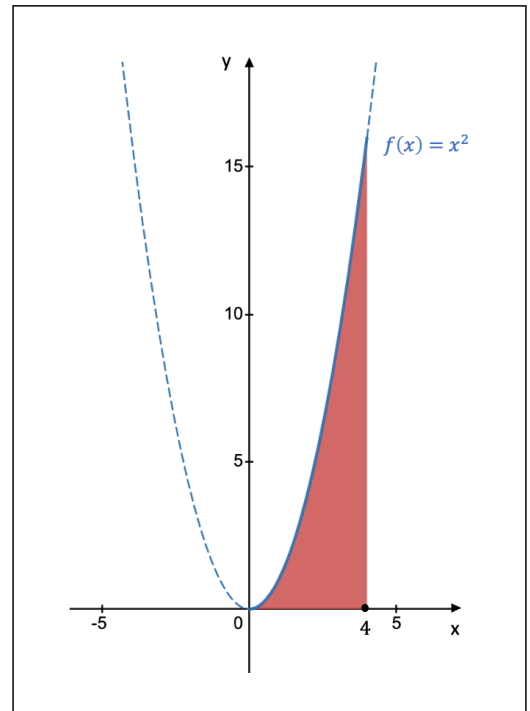


Figure 9.4: The integral of $f(x) = x^2$ over the interval $[0, 4]$

$$\begin{aligned}
L_f &= \sum_{k=1}^n \left[\frac{4(k-1)}{n} \right]^2 \cdot \frac{4}{n} = \frac{64}{n^3} \sum_{k=1}^n (k-1)^2 \\
&= \frac{64}{n^3} \left[\sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right] \\
&= \frac{64}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2 \frac{n(n+1)}{2} + n \right] \\
&= \frac{32}{3n^2} (2n^2 - 3n + 1)
\end{aligned}$$

The Riemann integral is then equal to:

$$\begin{aligned}
\int_0^4 f(x) dx &= \lim_{n \rightarrow +\infty} L_f = \lim_{n \rightarrow +\infty} \frac{32}{3n^2} (2n^2 - 3n + 1) \\
&= \lim_{n \rightarrow +\infty} \frac{64}{3} - \frac{32}{n} + \frac{32}{3n^2} \\
&= \frac{64}{3}
\end{aligned}$$

The area of the red-coloured area in Fig. 9.4 is thus equal to $\frac{64}{3}$.

From the definition of the Riemann integral it follows that $\int_a^a f(x) dx = 0$. Furthermore, in the case that $a > b$ for a given interval $[a, b]$, we can write $\int_a^b f(x) dx = -\int_b^a f(x) dx$.

If the piecewise function $f : [a, b] \rightarrow \mathbb{R}$ is continuous over the closed interval $[a, b]$ except in the points $c_1, c_2, \dots, c_{n-1}, c_n$ and if the left- and right-hand limits of f at these points exist, we can define $f_i(x)$, with $i \in \{1, 2, \dots, n\}$, as:

$$f_i : [c_{i-1}, c_i] \rightarrow \mathbb{R} : x \mapsto f_i(x) = \begin{cases} f(x) & \text{if } c_{i-1} < x < c_i \\ \lim_{x \rightarrow c_{i-1}^+} f(x) & \text{if } x = c_{i-1} \\ \lim_{x \rightarrow c_i^-} f(x) & \text{if } x = c_i \end{cases}$$

The Riemann integral of the piecewise function f is then equal to $\int_a^b f(x) dx = \sum_{i=1}^n \int_{c_{i-1}}^{c_i} f_i(x)$.

9.2.2 Properties

Similar to the indefinite integral, the Riemann integral has a property called **linearity**, which is understood to be the following, with $f, g : [a, b] \rightarrow \mathbb{R}$ continuous functions and $\alpha, \beta \in \mathbb{R}$:

$$\int_a^b [\alpha f(x) + \beta g(x)] dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

Given is a continuous function $f : [a, b] \rightarrow \mathbb{R}$ and $c_1 \leq c_2 \leq \dots \leq c_{n-2} \leq c_{n-1}$, whereby $c_i \in [a, b]$ with $i \in \{1, 2, \dots, n-1\}$. If we set $c_0 = a$ and $c_n = b$, we can **split** the definite integral of f over the interval $[a, b]$ into a sum of n definite integrals in the following way:

$$\int_a^b f(x) dx = \sum_{k=1}^n \int_{c_{k-1}}^{c_k} f(x) dx$$

Consider, for example, the continuous function $f : [-3, 3] \rightarrow \mathbb{R}$ and the points $c_1 = -2$, $c_2 = 0$, and $c_3 = 1$. Given that $c_1 \leq c_2 \leq c_3$, we can write the Riemann integral as a sum of $n = 4$ definite integrals, whereby $c_0 = -3$ and $c_4 = 3$:

$$\int_{-3}^3 f(x) dx = \int_{-3}^{-2} f(x) dx + \int_{-2}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^3 f(x) dx$$

In the example of the previous section 9.2.1, we have seen that it takes a number of steps to calculate the Riemann integral. The **fundamental theorem of calculus** provides an easier way to determine the value of a Riemann integral of a continuous function $f : [a, b] \rightarrow \mathbb{R}$:

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

whereby $F(x)$ represents the primitive function of $f(x)$. The primitive function F can be found either through direct integration of the function f (see Fig. 9.1) or via one of the methods discussed in section 9.1.2.

- Example

We wish to calculate the following Riemann integral:

$$\int_{\frac{1}{2}}^{\frac{e}{2}} \ln^2(2x) dx$$

We calculate this integral as follows:

$$\begin{aligned} \int_{\frac{1}{2}}^{\frac{e}{2}} \ln^2(2x) dx &= [(\ln |2x| - 2)(x) \ln |2x|]_{\frac{1}{2}}^{\frac{e}{2}} + 2 \int_{\frac{1}{2}}^{\frac{e}{2}} dx \\ &= \left[(\ln |e| - 2) \left(\frac{e}{2}\right) \ln |e| - (\ln |1| - 2) \left(\frac{1}{2}\right) \ln |1| \right] + 2 \left[\frac{e}{2} - \frac{1}{2} \right] \\ &= \frac{e - 2}{2} \end{aligned}$$

whereby in line 1 we applied the below reduction formula with $n = 2$, $a = 2$, and $b = 0$.

$$\int \ln^n(ax + b) dx = (\ln |ax + b| - n) \left(x + \frac{b}{a}\right) \ln^{n-1} |ax + b| + n(n-1) \int \ln^{n-2} |ax + b| dx$$

9.2.3 Applications

A first application is the concept of **improper integrals**. To calculate a Riemann integral, we considered a closed interval $[a, b]$ within the domain of the function f . As soon as one of the endpoints of the interval becomes open, we can no longer use the Riemann integral to calculate the area under the graph of f . Instead, we rely on improper integrals.

Given a continuous function $f :]a, b[\rightarrow \mathbb{R}$ and whereby $[c, d] \subset]a, b[$, we define the improper integral as follows:

$$\int_a^b f(x) dx = \lim_{\substack{c \rightarrow a, \\ d \rightarrow b}} \int_c^d f(x) dx$$

If the limit exists and is finite, we say that the integral is **convergent**, whereas if the limit produces infinity or if it does not exist, we say that the integral **diverges**.

If at least one of the endpoints of the open interval $]a, b[$ is equal to $\pm\infty$, the improper integral is known as an **improper integral of the first kind**. If the integral becomes improper in one of the two endpoints whereby $a, b \in \mathbb{R}$, we call the integral an **improper integral of the second kind**.

Note that the limit of the improper integral can be written as follows:

$$\lim_{\substack{c \rightarrow a, \\ d \rightarrow b}} \int_c^d f(x) dx = \lim_{c \rightarrow a} \left(\lim_{d \rightarrow b} \int_c^d f(x) dx \right) = \lim_{d \rightarrow b} \left(\lim_{c \rightarrow a} \int_c^d f(x) dx \right)$$

- Example 1

We wish to calculate the area below the graph of the function $f :]-1, +\infty[\setminus \{0\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{4}{x^2\sqrt{x+1}}$ for the half-open interval $[1, +\infty[$. The integral is an improper integral of the first kind and is written in the following way:

$$\int_1^{+\infty} \frac{4}{x^2\sqrt{x+1}} dx = \lim_{d \rightarrow +\infty} \int_1^d \frac{4}{x^2\sqrt{x+1}} dx$$

The area that corresponds to this improper integral is indicated in green in the below Fig. 9.5.

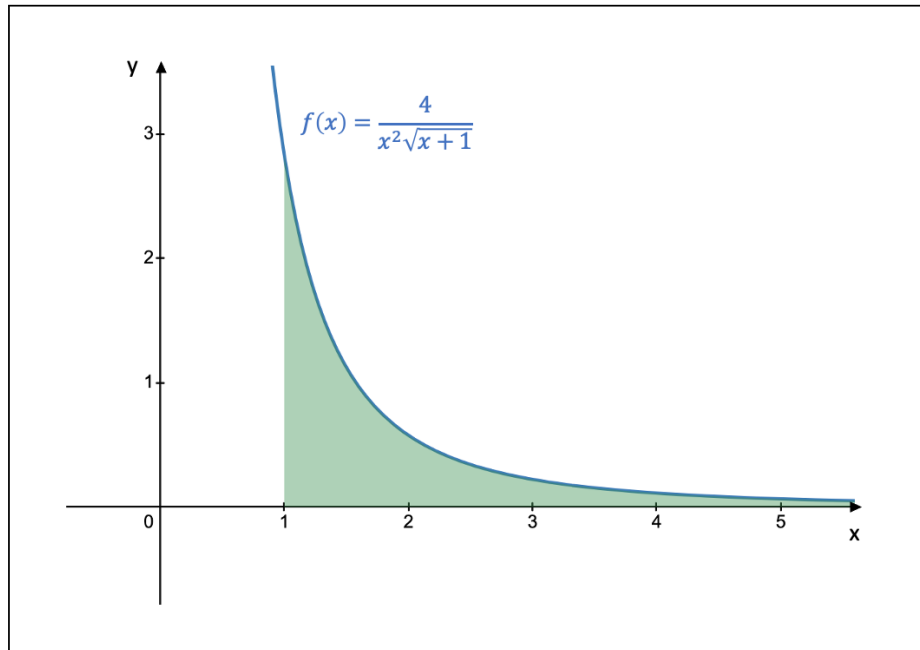


Figure 9.5: The corresponding area of the improper integral of Example 1

Let us in a first step calculate the definite integral of this improper integral:

$$\begin{aligned}
 & \int_1^d \frac{4}{x^2 \sqrt{x+1}} dx \\
 &= 8 \int_{\sqrt{2}}^{\sqrt{d+1}} \frac{1}{(t^2-1)^2} dt \\
 &= 8 \left[\left[-\frac{1}{2t(t^2-1)} \right]_{\sqrt{2}}^{\sqrt{d+1}} - \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{d+1}} \frac{1}{t^2(t^2-1)} dt \right] \\
 &= 8 \left[\left[-\frac{1}{2t(t^2-1)} \right]_{\sqrt{2}}^{\sqrt{d+1}} - \frac{1}{2} \left(- \int_{\sqrt{2}}^{\sqrt{d+1}} \frac{1}{t^2} dt - \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{d+1}} \frac{1}{t+1} dt + \frac{1}{2} \int_{\sqrt{2}}^{\sqrt{d+1}} \frac{1}{t-1} dt \right) \right] \\
 &= 8 \left[-\frac{1}{2t(t^2-1)} - \frac{1}{2t} + \frac{1}{4} \ln \left| \frac{t+1}{t-1} \right| \right]_{\sqrt{2}}^{\sqrt{d+1}} \\
 &= 8 \left[\left(-\frac{1}{2d\sqrt{d+1}} - \frac{1}{2\sqrt{d+1}} + \frac{1}{4} \ln \left| \frac{\sqrt{d+1}+1}{\sqrt{d+1}-1} \right| \right) - \left(-\frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{1}{4} \ln \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| \right) \right] \\
 &= \left[-\frac{4}{d\sqrt{d+1}} - \frac{4}{\sqrt{d+1}} + 2 \ln \left| 1 + \frac{2}{d} + 2\sqrt{\frac{1}{d} + \frac{2}{d^2}} \right| + 4\sqrt{2} - 2 \ln (3 + 2\sqrt{2}) \right]
 \end{aligned}$$

whereby we introduced the substitution $t = \sqrt{x+1}$ in line 2. In line 3, we applied the reduction formula derived in the second example of the subsection *F. Integration of Irrational Functions* of the section 9.1.2, with $a = 1$, $b = -1$, $n = 1$, and $m = -2$. Finally, in line 4, we applied partial fraction decomposition.

In a second and final step, we implement the above solution into our improper integral:

$$\begin{aligned}
 & \int_1^{+\infty} \frac{4}{x^2\sqrt{x+1}} dx \\
 &= \lim_{d \rightarrow +\infty} \int_1^d \frac{4}{x^2\sqrt{x+1}} dx \\
 &= \lim_{d \rightarrow +\infty} \left[-\frac{4}{d\sqrt{d+1}} - \frac{4}{\sqrt{d+1}} + 2 \ln \left| 1 + \frac{2}{d} + 2\sqrt{\frac{1}{d} + \frac{2}{d^2}} \right| + 4\sqrt{2} - 2 \ln(3 + 2\sqrt{2}) \right] \\
 &= 4\sqrt{2} - 2 \ln(3 + 2\sqrt{2})
 \end{aligned}$$

- Example 2

We want to find the area above the graph of the function $f : \mathbb{R} \setminus \{-1, 0\} \rightarrow \mathbb{R} : x \mapsto f(x) = \frac{8}{x^3\sqrt{(x+1)^2}}$ and below the x -axis when we restrict the function f to the interval $]-\infty, -1[$. This area corresponds to the red-coloured area in Fig. 9.6.

Given that the designated area lies below the x -axis, we will insert a minus sign into the integral in order to obtain a positive number for the area.

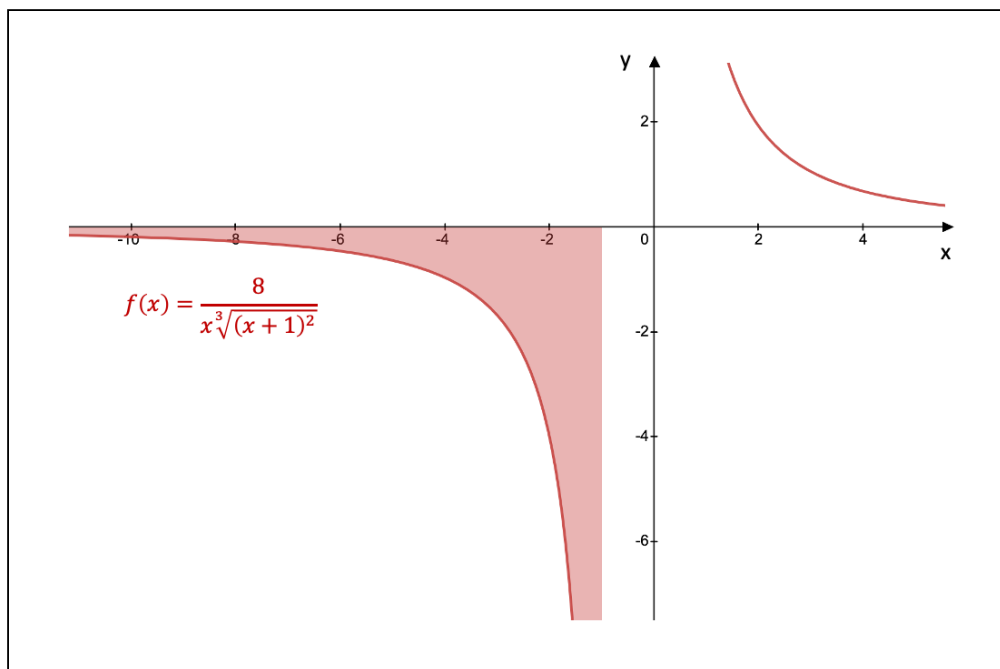


Figure 9.6: The corresponding area of the improper integral of Example 2

Taking the integral over the interval $]-\infty, -1[$, we see that the integral becomes twice improper, so we will split it as follows:

$$- \int_{-\infty}^{-1} \frac{8}{x^3\sqrt{(x+1)^2}} dx = - \int_{-\infty}^{-2} \frac{8}{x^3\sqrt{(x+1)^2}} dx - \int_{-2}^{-1} \frac{8}{x^3\sqrt{(x+1)^2}} dx$$

$$= - \lim_{c \rightarrow -\infty} \int_c^{-2} \frac{8}{x \sqrt[3]{(x+1)^2}} dx - \lim_{d \rightarrow -1} \int_{-2}^d \frac{8}{x \sqrt[3]{(x+1)^2}} dx$$

In a first step, we will calculate the following indefinite integral:

$$\begin{aligned} & \int \frac{8}{x \sqrt[3]{(x+1)^2}} dx \\ &= 24 \int \frac{1}{u^3 - 1} du \\ &= 24 \left[\frac{1}{3} \int \frac{1}{u-1} du - \frac{1}{3} \int \frac{u+2}{u^2+u+1} du \right] \\ &= 24 \left[\frac{1}{3} \int \frac{1}{u-1} du - \frac{1}{3} \int \frac{u+2}{(u+\frac{1}{2})^2 + \frac{3}{4}} du \right] \\ &= 24 \left[\frac{1}{3} \int \frac{1}{u-1} du - \frac{1}{3} \int \frac{s+\frac{3}{2}}{s^2+\frac{3}{4}} ds \right] \\ &= 24 \left[\frac{1}{3} \int \frac{1}{u-1} du - \frac{1}{3} \left[\int \frac{s}{s^2+\frac{3}{4}} ds + \frac{3}{2} \int \frac{1}{s^2+\frac{3}{4}} ds \right] \right] \\ &= 8 \ln |u-1| - 4 \ln \left| s^2 + \frac{3}{4} \right| - 8\sqrt{3} \arctan \left(\frac{2s}{\sqrt{3}} \right) + c \\ &= 8 \ln \left| \sqrt[3]{x+1} - 1 \right| - 4 \ln \left| \sqrt[3]{(x+1)^2} + \sqrt[3]{x+1} + 1 \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{x+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) + c \\ &= 4 \ln \left| \frac{1 - \frac{2}{\sqrt[3]{x+1}} + \frac{1}{\sqrt[3]{(x+1)^2}}}{1 + \frac{1}{\sqrt[3]{x+1}} + \frac{1}{\sqrt[3]{(x+1)^2}}} \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{x+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) + c \end{aligned}$$

whereby we implemented the substitution $u^3 = x+1$ in line 2 and the substitution $s = u + \frac{1}{2}$ in line 5. In addition, we applied partial fraction decomposition in line 3. In a next step, we calculate the first term of the improper integral:

$$\begin{aligned} & - \lim_{c \rightarrow -\infty} \int_c^{-2} \frac{8}{x \sqrt[3]{(x+1)^2}} dx \\ &= - \lim_{c \rightarrow -\infty} \left[4 \ln \left| \frac{1 - \frac{2}{\sqrt[3]{x+1}} + \frac{1}{\sqrt[3]{(x+1)^2}}}{1 + \frac{1}{\sqrt[3]{x+1}} + \frac{1}{\sqrt[3]{(x+1)^2}}} \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{x+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right]_c^{-2} \\ &= - \lim_{c \rightarrow -\infty} \left[4 \ln \left| \frac{1 - \frac{2}{\sqrt[3]{-2+1}} + \frac{1}{\sqrt[3]{(-2+1)^2}}}{1 + \frac{1}{\sqrt[3]{-2+1}} + \frac{1}{\sqrt[3]{(-2+1)^2}}} \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{-2+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right] \end{aligned}$$

$$\begin{aligned}
& -4 \ln \left| \frac{1 - \frac{2}{\sqrt[3]{c+1}} + \frac{1}{\sqrt[3]{(c+1)^2}}}{1 + \frac{1}{\sqrt[3]{c+1}} + \frac{1}{\sqrt[3]{(c+1)^2}}} \right| + 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{c+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \\
&= -4 \ln(4) + 8\sqrt{3} \cdot \left(-\frac{\pi}{6}\right) + 4 \ln(1) - 8\sqrt{3} \cdot \left(-\frac{\pi}{2}\right) \\
&= -8 \ln(2) + \frac{8\pi\sqrt{3}}{3}
\end{aligned}$$

Similarly, the second term of the improper integral is calculated as follows:

$$\begin{aligned}
& - \lim_{d \underset{<}{\rightarrow} -1} \int_{-2}^d \frac{8}{x \sqrt[3]{(x+1)^2}} dx \\
&= - \lim_{d \underset{<}{\rightarrow} -1} \left[8 \ln |\sqrt[3]{x+1} - 1| - 4 \ln \left| \sqrt[3]{(x+1)^2} + \sqrt[3]{x+1} + 1 \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{x+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right]_{-2}^d \\
&= - \lim_{d \underset{<}{\rightarrow} -1} \left[8 \ln |\sqrt[3]{d+1} - 1| - 4 \ln \left| \sqrt[3]{(d+1)^2} + \sqrt[3]{d+1} + 1 \right| - 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{d+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right. \\
&\quad \left. - 8 \ln |\sqrt[3]{-2+1} - 1| + 4 \ln \left| \sqrt[3]{(-2+1)^2} + \sqrt[3]{-2+1} + 1 \right| + 8\sqrt{3} \arctan \left(\frac{2\sqrt[3]{-2+1}}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right) \right] \\
&= -8 \ln |-1| + 4 \ln |1| + 8\sqrt{3} \left(\frac{\pi}{6}\right) + 8 \ln |-2| - 4 \ln |1| - 8\sqrt{3} \left(-\frac{\pi}{6}\right) \\
&= 8 \ln(2) + \frac{8\pi\sqrt{3}}{3}
\end{aligned}$$

The area that corresponds to the improper integral is then equal to:

$$\begin{aligned}
- \int_{-\infty}^{-1} \frac{8}{x \sqrt[3]{(x+1)^2}} dx &= - \lim_{c \rightarrow -\infty} \int_c^{-2} \frac{8}{x \sqrt[3]{(x+1)^2}} dx - \lim_{d \underset{<}{\rightarrow} -1} \int_{-2}^d \frac{8}{x \sqrt[3]{(x+1)^2}} dx \\
&= \left(-8 \ln(2) + \frac{8\pi\sqrt{3}}{3} \right) + \left(8 \ln(2) + \frac{8\pi\sqrt{3}}{3} \right) \\
&= \frac{16\pi\sqrt{3}}{3}
\end{aligned}$$

In Example 2, we introduced a minus sign into the integral so that we obtained a positive value for the area. We would have gotten the same result if we calculated the integral of the absolute value of the integrand. This brings us to a second application of the Riemann integral: the **area between the graphs of two functions**. Given two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$, the area between the graph of f and g over a certain interval $[a, b]$ is equal to:

$$\int_a^b |f(x) - g(x)| dx$$

- Example 1

We would like to calculate the area between the graphs of the functions $f : \mathbb{R} \rightarrow [1, +\infty[: x \mapsto f(x) = \cosh\left(\frac{x}{2}\right)$ and $g : \mathbb{R} \rightarrow [-1, +\infty[: x \mapsto g(x) = \cosh(x) - 2$, which is represented by the purple-coloured area in Fig. 9.7.

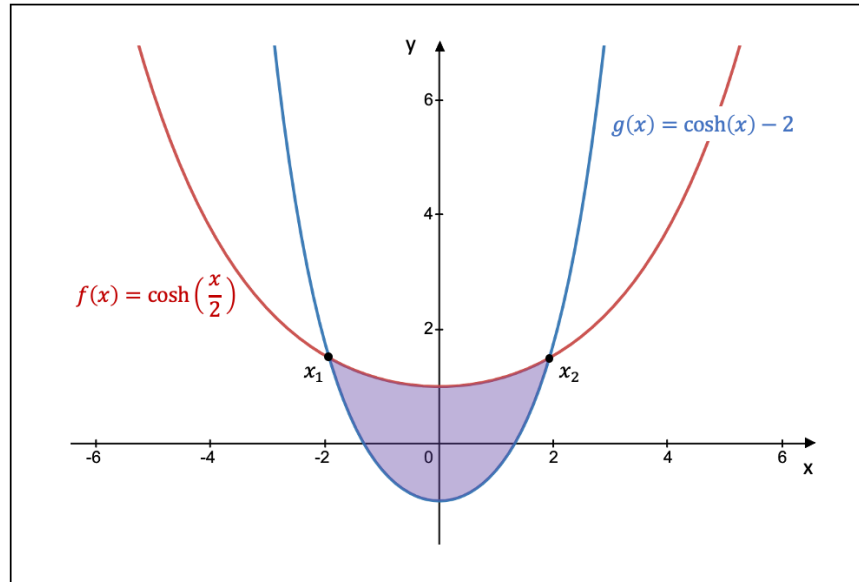


Figure 9.7: The area between the graphs of the functions f and g of Example 1

In order to determine the boundaries of our Riemann integral, we have to calculate the points of intersection x_1 and x_2 between f and g . Setting both functions equal to each other and solving for x , we find:

$$\begin{aligned}
 \cosh\left(\frac{x}{2}\right) = \cosh(x) - 2 &\Leftrightarrow \frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{2} = \frac{e^x + e^{-x}}{2} - 2 \\
 &\Leftrightarrow e^{\frac{3x}{2}} + e^{\frac{x}{2}} = e^{2x} + 1 - 4e^x \\
 &\Leftrightarrow t^3 + t = t^4 + 1 - 4t^2 \\
 &\Leftrightarrow (t+1)^2 \left(t - \frac{3+\sqrt{5}}{2}\right) \left(t - \frac{3-\sqrt{5}}{2}\right) = 0
 \end{aligned}$$

whereby we introduced the substitution $t = e^{\frac{x}{2}}$ in line 3. Converting back to the variable x , we find the two points of intersection: $x_1 = 2 \ln\left(\frac{3-\sqrt{5}}{2}\right)$ and $x_2 = 2 \ln\left(\frac{3+\sqrt{5}}{2}\right)$. Note that the solution $t = -1$ does not represent a real solution.

The area is calculated by solving the following Riemann integral:

$$\begin{aligned}
 \int_{x_1}^{x_2} |f(x) - g(x)| dx &= \int_{x_1}^{x_2} \left| \cosh\left(\frac{x}{2}\right) - \cosh(x) + 2 \right| dx \\
 &= \left[\left| 2 \sinh\left(\frac{x}{2}\right) - \sinh(x) + 2x \right| \right]_{x_1}^{x_2}
 \end{aligned}$$

$$\begin{aligned}
&= \left| 2 \sinh \left(\frac{x_2}{2} \right) - \sinh(x_2) + 2x_2 - 2 \sinh \left(\frac{x_1}{2} \right) + \sinh(x_1) - 2x_1 \right| \\
&= \left| 2 \sinh \left[\ln \left(\frac{3 + \sqrt{5}}{2} \right) \right] - \sinh \left[2 \ln \left(\frac{3 + \sqrt{5}}{2} \right) \right] + 4 \ln \left(\frac{3 + \sqrt{5}}{2} \right) \right. \\
&\quad \left. - 2 \sinh \left[\ln \left(\frac{3 - \sqrt{5}}{2} \right) \right] + \sinh \left[2 \ln \left(\frac{3 - \sqrt{5}}{2} \right) \right] - 4 \ln \left(\frac{3 - \sqrt{5}}{2} \right) \right| \\
&= \left| 2 \sinh \left[\ln \left(\frac{3 + \sqrt{5}}{2} \right) \right] - \sinh \left[\ln \left(\frac{7 + 3\sqrt{5}}{2} \right) \right] + 4 \ln \left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}} \right) \right. \\
&\quad \left. - 2 \sinh \left[\ln \left(\frac{3 - \sqrt{5}}{2} \right) \right] + \sinh \left[\ln \left(\frac{7 - 3\sqrt{5}}{2} \right) \right] \right| \\
&= \left| \left(\frac{3 + \sqrt{5}}{2} - \frac{2}{3 + \sqrt{5}} \right) - \frac{1}{2} \left(\frac{7 + 3\sqrt{5}}{2} - \frac{2}{7 + 3\sqrt{5}} \right) + 4 \ln \left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}} \right) \right. \\
&\quad \left. - \left(\frac{3 - \sqrt{5}}{2} - \frac{2}{3 - \sqrt{5}} \right) + \frac{1}{2} \left(\frac{7 - 3\sqrt{5}}{2} - \frac{2}{7 - 3\sqrt{5}} \right) \right| \\
&= -\sqrt{5} + 4 \ln \left(\frac{7 + 3\sqrt{5}}{2} \right)
\end{aligned}$$

- Example 2

We wish to calculate the area between the graphs of the functions $f : \mathbb{R} \setminus \{0, 3\} \rightarrow \mathbb{R}_0 : x \mapsto f(x) = \frac{16}{x(x-3)^2}$ and $g : \mathbb{R} \setminus \{2, 3\} \rightarrow \mathbb{R} : x \mapsto g(x) = \frac{12}{-x^2+5x-6}$, given the following two restrictions: $x < 0$ and $f(x) \geq g(x)$. This area is equal to the blue-coloured area in Fig. 9.8.

In order to identify the interval over which we have to integrate, we must first find the points of intersection between the two functions f and g , given that $x < 0$:

$$\begin{aligned}
f(x) = g(x) &\Leftrightarrow \frac{16}{x(x-3)^2} = \frac{12}{-x^2+5x-6} \Leftrightarrow \frac{4}{x(x-3)^2} = \frac{3}{(x-3)(-x+2)} \\
&\Leftrightarrow -4x+8 = 3x^2-9x \\
&\Leftrightarrow 3x^2-5x-8 = 0
\end{aligned}$$

This quadratic equation produces two solutions: $x_1 = -1$ and $x_2 = \frac{1}{3}$. Since we must have that $x < 0$, the point of intersection of our interest is $x_1 = -1$. If we insert some random

numbers to the left and right of x_1 (and making sure that $x < 0$), we can see that $f(x) \geq g(x)$ for $x_1 \leq -1$ and $f(x) \leq g(x)$ for $-1 \leq x_1 < 0$. The interval over which we have to integrate is therefore equal to $]-\infty, -1]$.

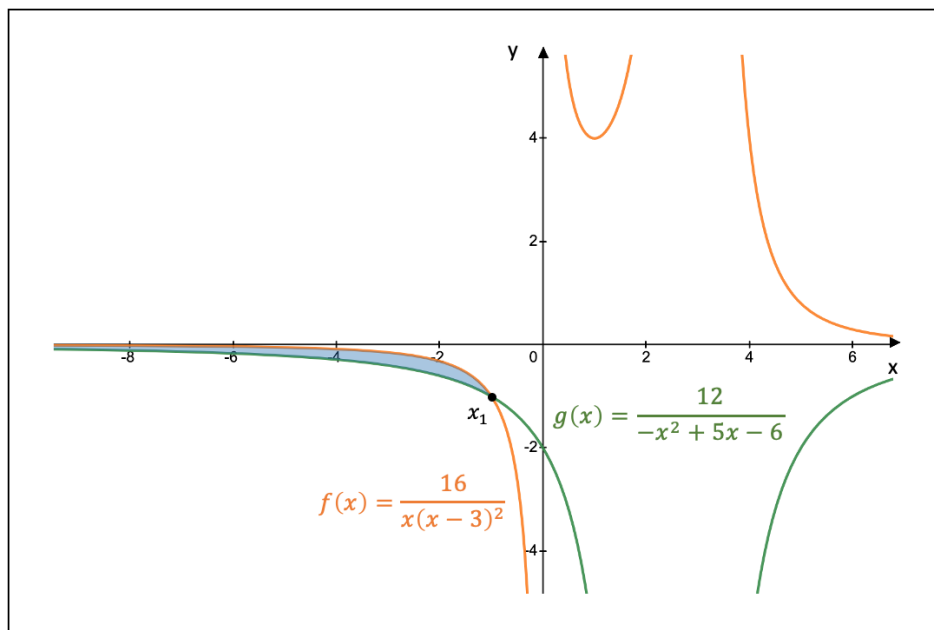


Figure 9.8: The area between the graphs of the functions f and g of Example 2

The area we wish to find is thus equal to the following improper integral of the first kind:

$$\int_{-\infty}^{-1} \left| \frac{16}{x(x-3)^2} - \frac{12}{(x-3)(-x+2)} \right| dx = \lim_{c \rightarrow -\infty} \int_c^{-1} \left| \frac{16}{x(x-3)^2} - \frac{12}{(x-3)(-x+2)} \right| dx$$

In a first step, we will calculate the indefinite integral:

$$\begin{aligned} & \int \left[\frac{16}{x(x-3)^2} - \frac{12}{(x-3)(-x+2)} \right] dx \\ &= 4 \int \frac{3x^2 - 5x - 8}{x(x-2)(x-3)^2} dx \\ &= 4 \left[\frac{4}{9} \int \frac{1}{x} dx - 3 \int \frac{1}{x-2} dx + \frac{23}{9} \int \frac{1}{x-3} dx + \frac{4}{3} \int \frac{1}{(x-3)^2} dx \right] \\ &= 4 \left[\frac{4}{9} \ln|x| - 3 \ln|x-2| + \frac{23}{9} \ln|x-3| - \frac{4}{3(x-3)} \right] + c \end{aligned}$$

whereby we applied partial fraction decomposition in line 3.

The area is calculated as follows:

$$\begin{aligned}
& \lim_{c \rightarrow -\infty} \int_c^{-1} \left| \frac{16}{x(x-3)^2} - \frac{12}{(x-3)(-x+2)} \right| dx \\
&= 4 \lim_{c \rightarrow -\infty} \left[\frac{4}{9} \ln|x| - 3 \ln|x-2| + \frac{23}{9} \ln|x-3| - \frac{4}{3(x-3)} \right]_c^{-1} \\
&= 4 \lim_{c \rightarrow -\infty} \left[\frac{4}{9} \ln|-1| - 3 \ln|-1-2| + \frac{23}{9} \ln|-1-3| - \frac{4}{3(-1-3)} - \frac{4}{9} \ln|c| + 3 \ln|c-2| \right. \\
&\quad \left. - \frac{23}{9} \ln|c-3| + \frac{4}{3(c-3)} \right] \\
&= 4 \lim_{c \rightarrow -\infty} \left[-3 \ln(3) + \frac{23}{9} \ln(4) + \frac{1}{3} + \frac{1}{9} \ln \left| \frac{(1-\frac{2}{c})^{27}}{(1-\frac{3}{c})^{23}} \right| + \frac{4}{3(c-3)} \right] \\
&= -12 \ln(3) + \frac{92}{9} \ln(4) + \frac{4}{3}
\end{aligned}$$

A third application of Riemann integrals is the calculation of the length of a curve called the **arc length**. In this section, we will limit ourselves to the study of curves in \mathbb{R}^2 .

Consider the vector function $\vec{r} : [a, b] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}(t)$. The image of $\vec{r}(t)$ is called an arc segment and if we write $\vec{r}(a) = A$ and $\vec{r}(b) = B$, the arc segment is denoted by \widehat{AB} . If $\vec{r}(t)$ is continuous, we call its image a continuous arc segment.

If the derivative of \vec{r} exists and if $\frac{d\vec{r}}{dt}$ is a continuous function over the interval $[a, b]$, we define the arc length L_{AB} of the arc segment \widehat{AB} as follows:

$$L_{AB} = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dr_x}{dt} \right)^2 + \left(\frac{dr_y}{dt} \right)^2} dt$$

If the curve is written in the form of $y = f(x)$ instead of a vector function with parameter t , we choose the variable x as the parameter and the arc length becomes:

$$L_{AB} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

If we choose polar coordinates, we can write the arc length in terms of either the parameter t or the angle θ :

$$\begin{cases} L_{AB} = \int_a^b \sqrt{\left(\frac{d\rho}{dt} \right)^2 + \rho^2 \left(\frac{d\theta}{dt} \right)^2} dt \\ L_{AB} = \int_a^b \sqrt{\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2} d\theta \end{cases}$$

- Example 1

Suppose we are given the polar coordinates $(\rho, \theta) = (\theta^2, \theta)$ and we would like to calculate the arc length of the arc segment that is created when we let the angle θ run from $A = 0$ to $B = 6\pi$.

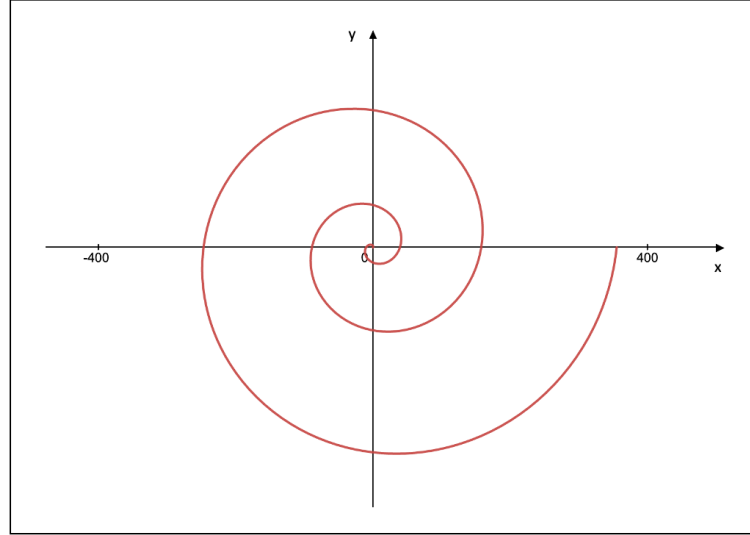


Figure 9.9: The arc segment of Example 1

We find the arc length as follows:

$$\begin{aligned}
 L_{AB} &= \int_a^b \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2} d\theta = \int_0^{6\pi} \sqrt{(2\theta)^2 + \theta^4} d\theta = \int_0^{6\pi} \theta \sqrt{4 + \theta^2} d\theta \\
 &= \int_2^{\sqrt{4+36\pi^2}} u^2 du \\
 &= \left[\frac{u^3}{3} \right]_2^{\sqrt{4+36\pi^2}} \\
 &= \frac{\sqrt{(4 + 36\pi^2)^3}}{3} - \frac{8}{3} \\
 &\approx 2268
 \end{aligned}$$

whereby we implemented the substitution $u = \sqrt{4 + \theta^2}$ in line 2.

- Example 2

Consider the vector function $\vec{r}: [-2, 3] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}(t) = \left(-\frac{t^3}{2} + \frac{3t^2}{4}, t^2 - 2t + 2\right)$, whose image is shown in Fig. 9.10, with $A = \vec{r}(a) = \vec{r}(-2) = (7, 10)$, $B = \vec{r}(b) = \vec{r}(3) = \left(-\frac{27}{4}, 5\right)$, and $\vec{r}(c) = \vec{r}(1) = \left(\frac{1}{4}, 1\right)$. Given that $\vec{r}(t)$ decreases over the interval $[-2, 1]$ and increases over the interval $[1, 3]$, we split the Riemann integral over these two intervals whereby we

introduce a minus sign in the Riemann integral that corresponds to the first interval in order to obtain a positive value for the respective arc lengths.

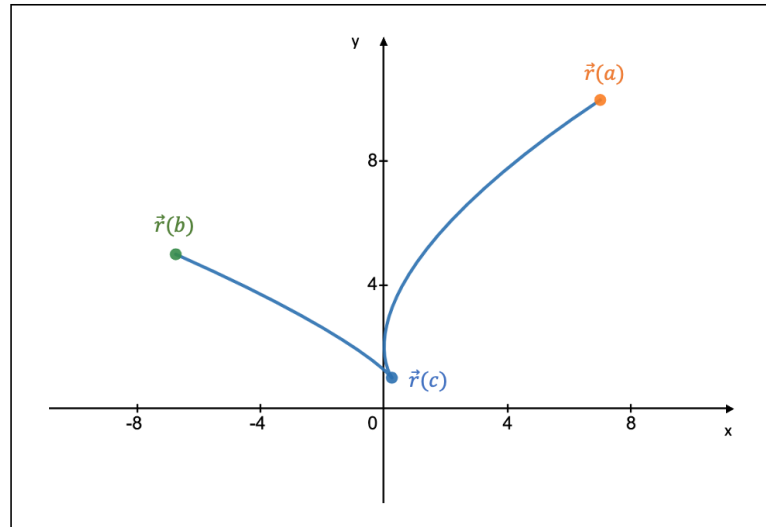


Figure 9.10: The arc segment of Example 2

We find the arc length as follows:

$$\begin{aligned}
 L_{AB} &= \int_a^c \sqrt{\left(\frac{dr_x}{dt}\right)^2 + \left(\frac{dr_y}{dt}\right)^2} dt + \int_c^b \sqrt{\left(\frac{dr_x}{dt}\right)^2 + \left(\frac{dr_y}{dt}\right)^2} dt \\
 &= - \int_{-2}^1 \sqrt{\left(-\frac{3t^2}{2} + \frac{3t}{2}\right)^2 + (2t-2)^2} dt + \int_1^3 \sqrt{\left(-\frac{3t^2}{2} + \frac{3t}{2}\right)^2 + (2t-2)^2} dt \\
 &= - \int_{-2}^1 \sqrt{\frac{9t^4}{4} - \frac{9t^3}{2} + \frac{25t^2}{4} - 8t + 4} dt + \int_1^3 \sqrt{\frac{9t^4}{4} - \frac{9t^3}{2} + \frac{25t^2}{4} - 8t + 4} dt \\
 &= - \int_{-2}^1 (t-1) \sqrt{\frac{9t^2}{4} + 4} dt + \int_1^3 (t-1) \sqrt{\frac{9t^2}{4} + 4} dt \\
 &= - \int_{-2}^1 t \sqrt{\frac{9t^2}{4} + 4} dt + \int_{-2}^1 \sqrt{\frac{9t^2}{4} + 4} dt + \int_1^3 t \sqrt{\frac{9t^2}{4} + 4} dt - \int_1^3 \sqrt{\frac{9t^2}{4} + 4} dt \\
 &= -\frac{4}{9} \int_{z_1}^{z_2} z^2 dz + \frac{8}{3} \int_{\theta_1}^{\theta_2} \sec^3(\theta) d\theta + \frac{4}{9} \int_{z_2}^{z_3} z^2 dz - \frac{8}{3} \int_{\theta_2}^{\theta_3} \sec^3(\theta) d\theta \\
 &= -\frac{4}{9} \int_{z_1}^{z_2} z^2 dz + \frac{8}{3} \left[\left[\frac{\tan(\theta) \sec(\theta)}{2} \right]_{\theta_1}^{\theta_2} + \frac{1}{2} \int_{\theta_1}^{\theta_2} \sec(\theta) d\theta \right] + \frac{4}{9} \int_{z_2}^{z_3} z^2 dz \\
 &\quad - \frac{8}{3} \left[\left[\frac{\tan(\theta) \sec(\theta)}{2} \right]_{\theta_2}^{\theta_3} + \frac{1}{2} \int_{\theta_2}^{\theta_3} \sec(\theta) d\theta \right] \\
 &= -\frac{4}{9} \left[\frac{z^3}{3} \right]_{z_1}^{z_2} + \frac{8}{3} \left[\left[\frac{\tan(\theta) \sec(\theta)}{2} \right]_{\theta_1}^{\theta_2} + \frac{1}{2} [\operatorname{arctanh}(\sin \theta)]_{\theta_1}^{\theta_2} \right] + \frac{4}{9} \left[\frac{z^3}{3} \right]_{z_2}^{z_3}
 \end{aligned}$$

$$\begin{aligned}
& -\frac{8}{3} \left[\left[\frac{\tan(\theta) \sec(\theta)}{2} \right]_{\theta_2}^{\theta_3} + \frac{1}{2} [\operatorname{arctanh}(\sin \theta)]_{\theta_2}^{\theta_3} \right] \\
&= -\frac{4}{9} \left[\frac{125}{24} - \frac{13\sqrt{13}}{3} \right] + \frac{8}{3} \left[\left[\frac{15}{32} + \frac{3\sqrt{13}}{8} \right] + \frac{1}{2} \left[\operatorname{arctanh} \left(\frac{3}{5} \right) - \operatorname{arctanh} \left(-\frac{3}{\sqrt{13}} \right) \right] \right] \\
&+ \frac{4}{9} \left[\frac{97\sqrt{97}}{24} - \frac{125}{24} \right] - \frac{8}{3} \left[\left[\frac{9\sqrt{97}}{32} - \frac{15}{32} \right] + \frac{1}{2} \left[\operatorname{arctanh} \left(\frac{9}{\sqrt{97}} \right) - \operatorname{arctanh} \left(\frac{3}{5} \right) \right] \right] \\
&= \frac{113\sqrt{97} + 316\sqrt{13} - 230}{108} + \frac{4}{3} \left[2 \operatorname{arctanh} \left(\frac{3}{5} \right) - \operatorname{arctanh} \left(\frac{9}{\sqrt{97}} \right) - \operatorname{arctanh} \left(-\frac{3}{\sqrt{13}} \right) \right] \\
&\approx 20.1
\end{aligned}$$

whereby we implemented the substitutions $z = \sqrt{\frac{9t^2}{4} + 4}$ and $t = \frac{4}{3} \tan \theta$ in line 6 with corresponding boundaries equal to $z_1 = \sqrt{13}$, $z_2 = \frac{5}{2}$, $z_3 = \frac{\sqrt{97}}{2}$, $\theta_1 = \arctan(-\frac{3}{2})$, $\theta_2 = \arctan(\frac{3}{4})$, and $\theta_3 = \arctan(\frac{9}{4})$. In line 7, we applied the reduction formula of Example 2 of the subsection *F. Integration of Irrational Functions* of section 9.1.2. Finally, note that $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$, which we used in line 11 and 12.

A fourth and final application of the Riemann integral that we discuss in this section is the calculation of the **volume of a solid of revolution**. We will discuss two methods: one is based on cylinders and the other on disks. Each method is explained by an example.

- Method of Cylinders

Consider the regions bounded by the functions $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = (x-3)^3 - 3(x-3)^2 + 6$ and $g : \mathbb{R} \rightarrow]-\infty, 6] : x \mapsto g(x) = -\frac{1}{6}(x-6)^2 + 6$. We want to calculate the volume of the solid of revolution created by rotating these bounded regions over 2π about the y -axis. The left panel of Fig. 9.11 shows the bounded regions A and B , and the right panel demonstrates the final result after rotation.

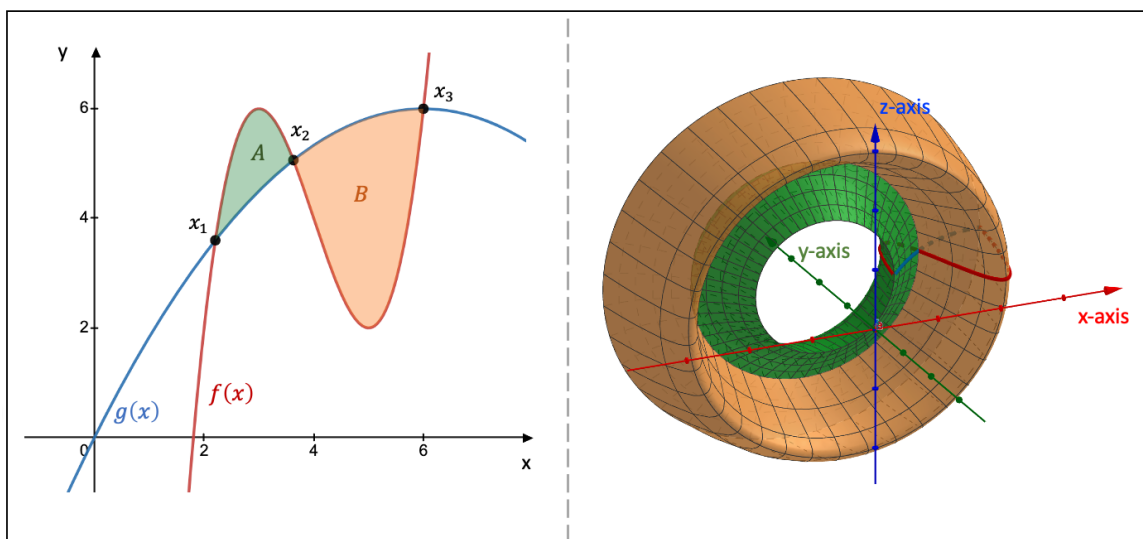


Figure 9.11: The solid of revolution (right) created by rotating the bounded regions A and B (left) about the y -axis

The method of cylinders treats the solid of revolution as a volume composed entirely of cylinders. The idea is to find an expression for the area of the curved surface of the cylinder (ignoring thus its two bases) and integrate that expression across the bounded regions. Fig. 9.12 depicts two cylinders as an example: one, in green, related to the bounded region A , and another, in orange, corresponding to the bounded region B .

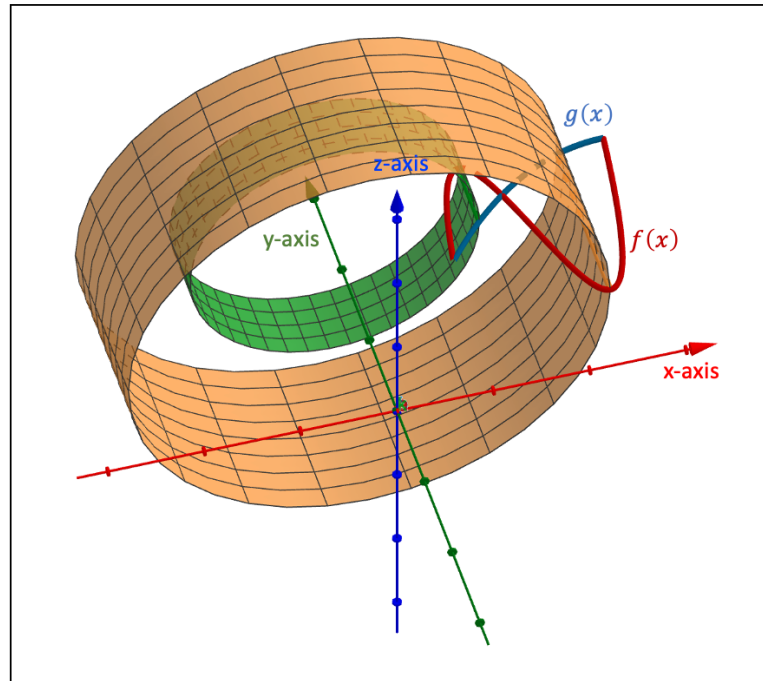


Figure 9.12: The method of cylinders

The general formula for the area of the curved surface of a cylinder is equal to $S = 2\pi Rh$, with R the radius of the cylinder and h its height. If we look at the orange cylinder within region B , its radius is equal to $R = x$ and its height to $h = g(x) - f(x)$. The area of this cylinder is therefore equal to $S_B = 2\pi x [g(x) - f(x)]$. Similarly, the area of the curved surface of the green cylinder corresponding to the bounded region A is equal to $S_A = 2\pi x [f(x) - g(x)]$.

In order to find the volume of the solid of revolution created by the region A , we integrate the expression S_A over the interval $[x_1, x_2]$. To the same extent, the volume of the solid of revolution related to the bounded region B is found by integrating the area S_B over the interval $[x_2, x_3]$.

Let us in a first instance determine the boundaries x_1 , x_2 , and x_3 , which are the points of intersection between the functions f and g :

$$\begin{aligned} f(x) = g(x) &\Leftrightarrow (x-3)^3 - 3(x-3)^2 + 6 = -\frac{1}{6}(x-6)^2 + 6 \\ &\Leftrightarrow (x-3)^2(x-6) = -\frac{1}{6}(x-6)^2 \\ &\Leftrightarrow \begin{cases} x_1 = \frac{35}{12} - \frac{\sqrt{73}}{12} \\ x_2 = \frac{35}{12} + \frac{\sqrt{73}}{12} \\ x_3 = 6 \end{cases} \end{aligned}$$

The volume of the solid of revolution created by rotating the bounded regions A and B over an angle of 2π about the y -axis is then calculated as follows:

$$\begin{aligned}
V &= \int_{x_1}^{x_2} S_A dx + \int_{x_2}^{x_3} S_B dx \\
&= \int_{x_1}^{x_2} 2\pi x [f(x) - g(x)] dx + \int_{x_2}^{x_3} 2\pi x [g(x) - f(x)] dx \\
&= 2\pi \int_{x_1}^{x_2} x \left[[(x-3)^3 - 3(x-3)^2 + 6] - \left[-\frac{1}{6}(x-6)^2 + 6 \right] \right] dx \\
&\quad + 2\pi \int_{x_2}^{x_3} x \left[\left[-\frac{1}{6}(x-6)^2 + 6 \right] - [(x-3)^3 - 3(x-3)^2 + 6] \right] dx \\
&= 2\pi \int_{x_1}^{x_2} \left[x^4 - \frac{71x^3}{6} + 43x^2 - 48x \right] dx + 2\pi \int_{x_2}^{x_3} \left[-x^4 + \frac{71x^3}{6} - 43x^2 + 48x \right] dx \\
&= 2\pi \left[\frac{x^5}{5} - \frac{71x^4}{24} + \frac{43x^3}{3} - 24x^2 \right]_{x_1}^{x_2} + 2\pi \left[-\frac{x^5}{5} + \frac{71x^4}{24} - \frac{43x^3}{3} + 24x^2 \right]_{x_2}^{x_3} \\
&= 2\pi \left[\left(\frac{(35 + \sqrt{73})^5}{5 \cdot 12^5} - \frac{(35 - \sqrt{73})^5}{5 \cdot 12^5} \right) - \left(\frac{71(35 + \sqrt{73})^4}{24 \cdot 12^4} - \frac{71(35 - \sqrt{73})^4}{24 \cdot 12^4} \right) \right. \\
&\quad \left. + \left(\frac{43(35 + \sqrt{73})^3}{3 \cdot 12^3} - \frac{43(35 - \sqrt{73})^3}{3 \cdot 12^3} \right) - \left(\frac{24(35 + \sqrt{73})^2}{12^2} - \frac{24(35 - \sqrt{73})^2}{12^2} \right) \right] \\
&\quad + 2\pi \left[- \left(\frac{6^5}{5} - \frac{(35 + \sqrt{73})^5}{5 \cdot 12^5} \right) + \left(\frac{71 \cdot 6^4}{24} - \frac{71(35 + \sqrt{73})^4}{24 \cdot 12^4} \right) \right. \\
&\quad \left. - \left(\frac{43 \cdot 6^3}{3} - \frac{43(35 + \sqrt{73})^3}{3 \cdot 12^3} \right) + \left(24 \cdot 6^2 - \frac{24(35 + \sqrt{73})^2}{12^2} \right) \right] \\
&= \left(\frac{8213737}{155520} + \frac{77891\sqrt{73}}{51840} \right) \pi \\
&\approx 206.25
\end{aligned}$$

- Method of Disks

Consider the region bounded by the two functions $f : \mathbb{R} \rightarrow]3, 5[: x \mapsto f(x) = \tanh(x) + 4$ and $g : \mathbb{R}_0 \rightarrow \mathbb{R} \setminus]1.75, 2.15[: x \mapsto g(x) = \frac{1}{4} \coth(x) + 2$ and the line $x_2 = 5$. We are interested in calculating the volume of the solid of revolution created by rotating this bounded region over an angle of 2π about the line $y = 1$. The grey-shaded region at the left-hand side of Fig. 9.13 represents the bounded region and at the right-hand side we see the final result after rotation.

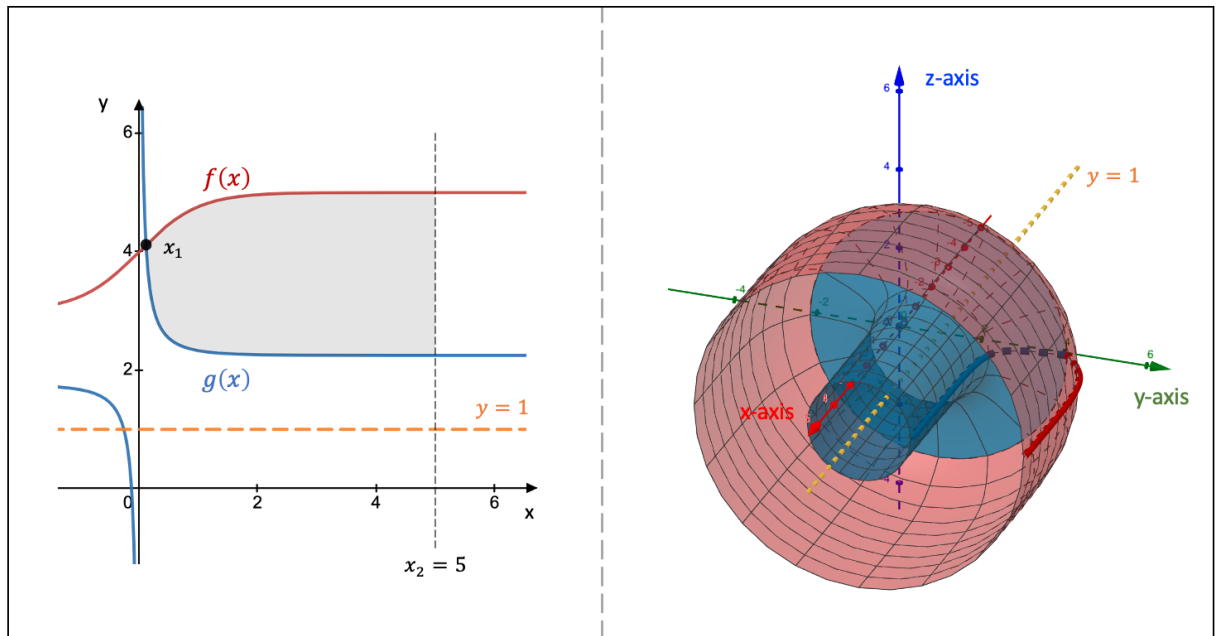


Figure 9.13: The solid of revolution (right) created by rotating the bounded region (left) about the line $y = 1$

The method of disks regards the volume of a solid of revolution as an accumulation of disks whose plane is oriented perpendicular to the axis of rotation. The method consists of first finding an expression for the surface area of a disk that corresponds to one slice of our solid of revolution and subsequently integrating that expression over the bounded region. Fig. 9.14 depicts two disks as an example.

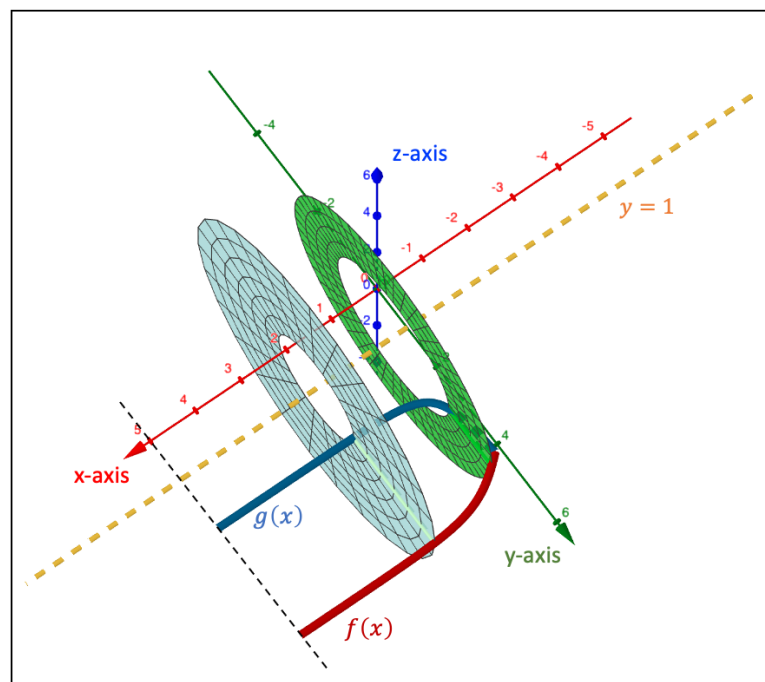


Figure 9.14: The method of disks

Since the disk has a hole in the middle in our example we have to bear in mind to subtract

the surface area of the hole when calculating the surface area of the disk. Also, remember that we are rotating the bounded region about the line $y = 1$. For a given x , the outer radius is equal to $R_{out} = f(x) - 1$, whereas the inner radius measures $R_{in} = g(x) - 1$. The surface area of the disk is therefore equal to $S = \pi [f(x) - 1]^2 - \pi [g(x) - 1]^2$.

To determine the volume of the solid of revolution, we integrate the surface area S over the interval $[x_1, x_2]$. We know that $x_2 = 5$, so we still have to find the point of intersection x_1 :

$$\begin{aligned}
 f(x) = g(x) &\Leftrightarrow \tanh(x) + 4 = \frac{1}{4} \coth(x) + 2 \\
 &\Leftrightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} + 4 = \frac{1}{4} \left(\frac{e^x + e^{-x}}{e^x - e^{-x}} \right) + 2 \\
 &\Leftrightarrow 4(e^{2x} - 2 + e^{-2x}) + 8(e^{2x} - e^{-2x}) = e^{2x} + 2 + e^{-2x} \\
 &\Leftrightarrow 11e^{4x} - 10e^{2x} - 5 = 0 \\
 &\Leftrightarrow e^{2x} = \frac{5 + 4\sqrt{5}}{11} \\
 &\Leftrightarrow x_1 = \frac{1}{2} \ln \left(\frac{5 + 4\sqrt{5}}{11} \right)
 \end{aligned}$$

We can now calculate the volume of the solid of revolution created by rotating the bounded region about the line $y = 1$ as follows:

$$\begin{aligned}
 V &= \int_{x_1}^{x_2} S \, dx \\
 &= \int_{x_1}^{x_2} [\pi [f(x) - 1]^2 - \pi [g(x) - 1]^2] \, dx \\
 &= \pi \int_{x_1}^{x_2} \left[\tanh^2(x) + 6 \tanh(x) + 9 - \frac{\coth^2(x)}{16} - \frac{\coth(x)}{2} - 1 \right] \, dx \\
 &= \pi \int_{x_1}^{x_2} \left[\frac{\cosh(2x) - 1}{\cosh(2x) + 1} + 6 \tanh(x) + 9 - \frac{\cosh^2(x)}{16 \sinh^2(x)} - \frac{\coth(x)}{2} - 1 \right] \, dx \\
 &= \pi \left[\int_{t_1}^{t_2} \left(\frac{1}{t^2} + \frac{1}{4(1-t)} + \frac{1}{4(1+t)} - \frac{1}{t^2 \sqrt{1-t^2}} \right) dt + 6 \int_{u_1}^{u_2} \frac{1}{u} \, du + 8 \int_{x_1}^{x_2} dx \right. \\
 &\quad \left. - \frac{1}{16} \int_{s_1}^{s_2} \left(\frac{1}{s^2} + \frac{1}{2(1-s)} + \frac{1}{2(1+s)} \right) ds - \frac{1}{2} \int_{r_1}^{r_2} \frac{1}{r} \, dr \right] \\
 &= \pi \left[- \left[\frac{1}{t} \right]_{t_1}^{t_2} + \frac{1}{4} \left[\ln \left| \frac{1+t}{1-t} \right| \right]_{t_1}^{t_2} + [\tan \theta]_{\theta_1}^{\theta_2} + 6 [\ln |u|]_{u_1}^{u_2} + 8 [x]_{x_1}^{x_2} + \frac{1}{16} \left[\frac{1}{s} \right]_{s_1}^{s_2} \right. \\
 &\quad \left. - \frac{1}{32} \left[\ln \left| \frac{1+s}{1-s} \right| \right]_{s_1}^{s_2} - \frac{1}{2} [\ln |r|]_{r_1}^{r_2} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \pi \left[\left(-\frac{e^{20}+1}{e^{20}-1} + \frac{113+20\sqrt{5}}{-8+20\sqrt{5}} \right) + \left(5 - \frac{1}{4} \ln \left| \frac{105+40\sqrt{5}}{121} \right| \right) + \left(\frac{2e^{10}}{e^{20}-1} - \frac{11(5+4\sqrt{5})}{-8+20\sqrt{5}} \right) \right. \\
&\quad + \left(6 \ln \left| \frac{(e^5 + e^{-5})(-3 + 2\sqrt{5})\sqrt{11(5 + 4\sqrt{5})}}{-8 + 20\sqrt{5}} \right| \right) + \left(40 - 4 \ln \left| \frac{5 + 4\sqrt{5}}{11} \right| \right) \\
&\quad + \left(\frac{e^{10} + 1}{16(e^{10} - 1)} - \frac{-4 + 10\sqrt{5}}{16(29 - 12\sqrt{5})} \right) + \left(-\frac{5}{16} + \frac{1}{32} \ln \left| \frac{25 - 2\sqrt{5}}{11(-3 + 2\sqrt{5})} \right| \right) \\
&\quad \left. + \left(-\frac{1}{2} \ln \left| \frac{(e^5 + e^{-5})(4 + \sqrt{5})\sqrt{11(5 + 4\sqrt{5})}}{-4 + 10\sqrt{5}} \right| \right) \right]
\end{aligned}$$

$$\approx 203.76$$

whereby we introduced in line 5 the substitutions $t = \tanh(2x)$, $u = \cosh(x)$, $s = \tanh(x)$, and $r = \sinh(x)$. In the same line, we also applied twice partial fraction decomposition. Note furthermore that in line 7 the angle θ is equal to $\theta = (\arcsin \sqrt{1 - t^2})$ and that $\tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{\sqrt{1 - t^2}}{t} = \operatorname{csch}(2x)$.

10 Multivariable Calculus

Multivariable calculus or vector calculus is the extension of the study of real functions of one variable to the analysis of real functions of any number of variables. Specifically, it deals with differential and integral calculus of multivariable functions.

10.1 Definitions

In section 7.1.1, we defined the general scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n)$. Recall that the scalar function $f : \mathbb{R} \rightarrow \mathbb{R}$ represents a curve in two-dimensional space and that the scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ represents a two-dimensional surface in three-dimensional space.

For a two-dimensional surface generated by the scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we define the **level curve(s)** as the collection of points for which $f(x, y) = c$ (with $c \in \mathbb{R}$). For a three-dimensional surface—which cannot be drawn—generated by the scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, we can define the **level surface(s)** as the collection of points for which $f(x, y, z) = c$, which we can draw.

The left-hand side of Fig. 10.1 shows the level curves (in yellow) of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \cos(x) - \sin(y)$ for which $f(x, y) = c = \frac{\sqrt{3}-1}{2}$. The graph of f is represented by the blue surface and note that the graph has been lowered by an amount of c for visual purposes. At the right-hand side, we see the level surface (in green) of the function $f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z) = 2x^2 + 0.2y^2 + 10z^2$ for which $f(x, y, z) = 16$.

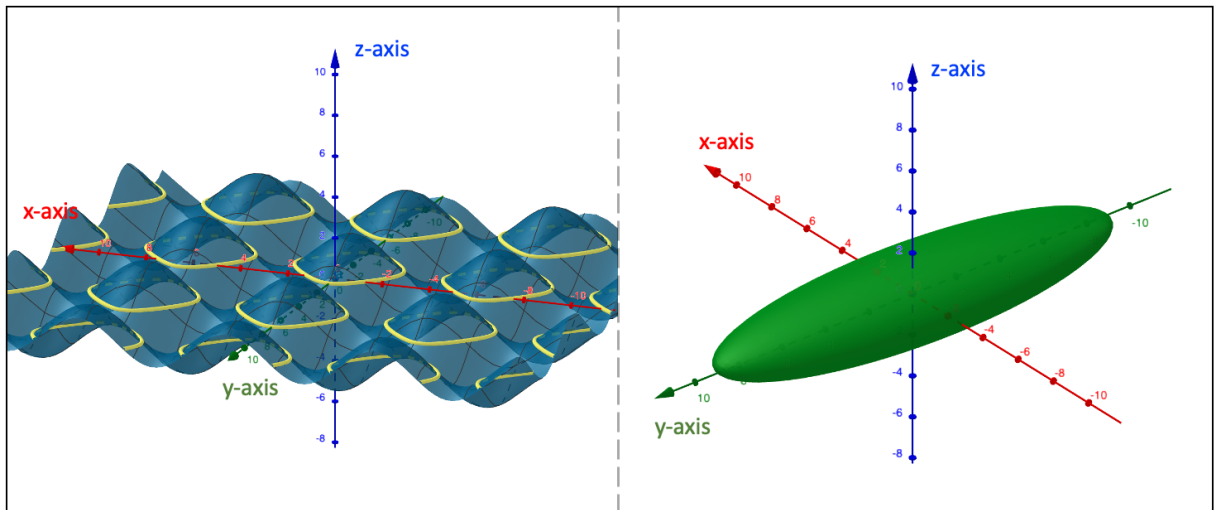


Figure 10.1: Level curves in yellow (left) and the level surface in green (right)

Also in the same section 7.1.1, we discussed the general vector function $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m : (t_1, t_2, \dots, t_n) \mapsto \vec{F}(t_1, t_2, \dots, t_n) = (f_1(t_1, t_2, \dots, t_n), f_2(t_1, t_2, \dots, t_n), \dots, f_m(t_1, t_2, \dots, t_n))$, whereby the vector function $\vec{F} : \mathbb{R} \rightarrow \mathbb{R}^2$ represents a curve in two-dimensional space, the vector function $\vec{F} : \mathbb{R} \rightarrow \mathbb{R}^3$ a curve in three-dimensional space, and the vector function $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ a two-dimensional surface in three-dimensional space.

If $n > 1$ and $n = m$, the vector function is called a **vector field**. Graphically, this means

that at every point in m -dimensional space we assign a vector of a certain magnitude and with a certain direction. In other words, a vector field is a vector whose properties depend of its position in space. If the vector field is constant, we obtain a regular vector. An example of a vector field is $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F} = (0.5x + y, y + z, 2z - x)$, which is depicted in Fig. 10.2, and an example of a constant vector field is $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto \vec{F} = (5, 2)$.

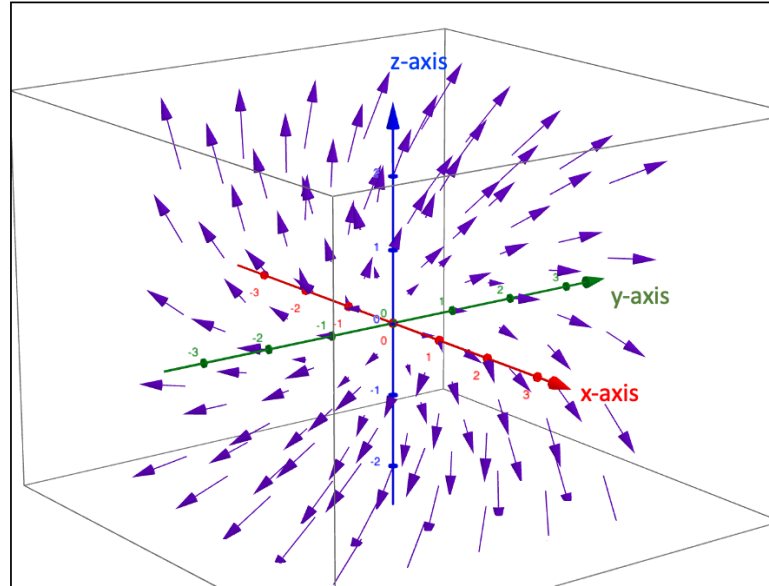


Figure 10.2: An example of a vector field

10.2 Differential Calculus

10.2.1 Partial Derivatives and Directional Derivatives

- A scalar function of n variables

Let us in a first instance consider the general scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, x_2, \dots, x_n) \mapsto f(x_1, x_2, \dots, x_n)$. The **partial derivative** of the function f with respect to the variable x_i , with $i \in \{1, 2, \dots, n\}$, at a given point $\vec{a} = (a_1, a_2, \dots, a_n)$ is defined as follows, whereby $\vec{u}_i = (0, 0, \dots, 1, \dots, 0)$ represents the i^{th} unit vector:

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\vec{a}) &= \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}_i) - f(\vec{a})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a_1, a_2, \dots, a_i + h, \dots, a_n) - f(a_1, a_2, \dots, a_i, \dots, a_n)}{h} \end{aligned}$$

As an example, consider the function $f : \mathbb{R}^4 \rightarrow \mathbb{R} : (x, y, z, u) \mapsto f(x, y, z, u) = x^2u - 2^{z+2u} + ue^{xyu}$. The partial derivative with respect to the variable u at the point $\vec{a} = (-1, 1, 0, 2)$ is equal to:

$$\frac{\partial f}{\partial u}(\vec{a}) = \lim_{h \rightarrow 0} \frac{f((-1, 1, 0, 2) + h(0, 0, 0, 1)) - f(-1, 1, 0, 2)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{(2+h) - 2^{4+2h} + (2+h)e^{-(2+h)} - 2 + 2^4 - 2e^{-2}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 - 2 \ln(2)2^{4+2h} + e^{-(2+h)} - (2+h)e^{-(2+h)}}{1} \\
&= 1 - 32 \ln 2 - e^{-2}
\end{aligned}$$

whereby we applied the L'Hôpital Rule in line 3.

Calculating the partial derivative based on the limit definition can be quite cumbersome. A more practical way is to take the derivative with respect to the variable in question and treat the other variables as constants. If, for instance, we wish to calculate the partial derivative of f with respect to the variable x , we treat f as if it is a scalar function of one variable (x) and apply the rules of differentiation as discussed in section 7.5.

For example, we would like to calculate the partial derivative of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = yx^{y^2} \sqrt{x+y^2}$ with respect to the variable y (note that the function is only well-defined if $x \geq -y^2$):

$$\begin{aligned}
\frac{\partial f}{\partial y} &= (yx^{y^2})' \sqrt{x+y^2} + yx^{y^2} (\sqrt{x+y^2})' \\
&= (x^{y^2} + 2y^2 \ln(x)x^{y^2}) \sqrt{x+y^2} + yx^{y^2} \left(\frac{y}{\sqrt{x+y^2}} \right) \\
&= \frac{x^{y^2+1} + 2x^{y^2+1}y^2 \ln x + x^{y^2}y^2 + 2x^{y^2}y^4 \ln x + x^{y^2}y^2}{\sqrt{x+y^2}} \\
&= \frac{x^{y^2} (x + 2y^2 [1 + (x+y^2) \ln x])}{\sqrt{x+y^2}}
\end{aligned}$$

The generalization of the derivative of a function f of one variable at the point $x = a$ to the multivariable situation is the **directional derivative** denoted by $D_{\vec{u}}f(\vec{a})$. That is, the directional derivative tells us to what extent f varies if we move \vec{x} , starting from the point \vec{a} , along the direction of the vector \vec{u} . The directional derivative is defined as:

$$D_{\vec{u}}f(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$$

The direction in which the directional derivative $D_{\vec{u}}f(\vec{a})$ for a given unit vector \vec{u} reaches its maximum value (in absolute value terms) is called the **gradient**, which is written as $\vec{\nabla}f(\vec{a})$. The mathematical symbol ∇ is known as the nabla symbol.

The i^{th} component of the gradient is equal to the partial derivative with respect to the variable x_i :

$$\vec{\nabla}f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \frac{\partial f}{\partial x_2}(\vec{a}), \dots, \frac{\partial f}{\partial x_i}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right)$$

Based on the definition of the gradient, the directional derivative $D_{\vec{u}}f(\vec{a})$ can also be calculated in the following way:

$$D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{u}$$

From a geometric perspective, the gradient allows us to linearly approximate the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ in the vicinity of \vec{a} through the following expression:

$$z = f(\vec{a}) + \vec{\nabla}f(\vec{a})(\vec{x} - \vec{a})$$

- A scalar function of two variables

For a function of two variables $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y)$, the linear approximation of f is interpreted as the plane tangent to f at the point $\vec{a} = (a_1, a_2, f(a_1, a_2))$ with the following equation:

$$z = f(a_1, a_2) + \left[\frac{\partial f}{\partial x}(a_1, a_2) \right] (x - a_1) + \left[\frac{\partial f}{\partial y}(a_1, a_2) \right] (y - a_2)$$

The gradient $\vec{\nabla}f(\vec{a})$ lies within the xy -plane and is oriented perpendicular to the level curve $f(x, y) = f(a_1, a_2)$ at the point \vec{a} .

The partial derivative $\frac{\partial f}{\partial x}(\vec{a})$ represents the slope of the line that is tangent at the point \vec{a} to the curve that is the result of the intersection between the surface $f(x, y)$ and the plane $y = a_2$. Similarly, the partial derivative $\frac{\partial f}{\partial y}(\vec{a})$ is equal to the slope of the line tangent to the curve that is produced when intersecting the graph of f and the plane $x = a_1$. Both these curves lie within the tangent plane and go through the point \vec{a} .

The directional derivative $D_{\vec{u}}f(\vec{a})$ represents the slope of the line tangent to the function f at the point \vec{a} along the direction of \vec{u} . This line also lies within the tangent plane. If the projection of \vec{u} onto the xy -plane is parallel to the gradient vector $\vec{\nabla}f(\vec{a})$, the directional derivative $D_{\vec{u}}f(\vec{a})$ reaches its maximum value (in absolute value terms).

- Example

Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \cosh(2x) - \sinh(2y)$, the point $\vec{a} = (1, -1)$ and the directional vector $\vec{u} = (e^2, e^2)$. The gradient of f at the point \vec{a} is equal to:

$$\begin{aligned} \vec{\nabla}f(\vec{a}) &= \left(\frac{\partial f}{\partial x}(\vec{a}), \frac{\partial f}{\partial y}(\vec{a}) \right) = (2 \sinh(2x), -2 \cosh(2y)) \\ &= (2 \sinh(2), -2 \cosh(-2)) \\ &= \left(2 \left(\frac{e^2 - e^{-2}}{2} \right), -2 \left(\frac{e^{-2} + e^2}{2} \right) \right) \\ &= (e^2 - e^{-2}, -e^2 - e^{-2}) \end{aligned}$$

At the point \vec{a} , the gradient $\vec{\nabla}f(\vec{a})$ stands perpendicular to the level curve, which has the following equation:

$$\begin{aligned} f(x, y) = f(a_1, a_2) &\Leftrightarrow \cosh(2x) - \sinh(2y) = \cosh(2) - \sinh(-2) \\ &\Leftrightarrow \cosh(2x) - \sinh(2y) = e^2 \end{aligned}$$

The directional derivative at point \vec{a} along the directional vector \vec{u} is equal to (using the limit definition):

$$\begin{aligned} D_{\vec{u}}f(\vec{a}) &= \lim_{h \rightarrow 0} \frac{f((1, -1) + h(e^2, e^2)) - f(1, -1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cosh[2(1 + he^2)] - \sinh[2(-1 + he^2)] - \cosh(2) + \sinh(-2)}{h} \\ &= \lim_{h \rightarrow 0} (2e^2 \sinh[2(1 + he^2)] - 2e^2 \cosh[2(-1 + he^2)]) \\ &= 2e^2 [\sinh(2) - \cosh(-2)] \\ &= 2e^2 \left[\left(\frac{e^2 - e^{-2}}{2} \right) - \left(\frac{e^{-2} + e^2}{2} \right) \right] \\ &= -2 \end{aligned}$$

whereby we applied the L'Hôpital Rule in line 3. If instead we use the definition based on the gradient we find:

$$D_{\vec{u}}f(\vec{a}) = \vec{\nabla}f(\vec{a}) \cdot \vec{u} = (e^2 - e^{-2}, -e^2 - e^{-2}) \cdot (e^2, e^2) = -2$$

Let us now turn the gradient vector into a normalized directional vector \vec{u}_n . This means that the norm of \vec{u}_n is equal to 1, i.e., $\|\vec{u}_n\| = 1$, so that we are dealing with a unit vector. With $u_{n,x} = \frac{\partial f}{\partial x}(\vec{a}) = e^2 - e^{-2}$ and $u_{n,y} = \frac{\partial f}{\partial y}(\vec{a}) = -e^2 - e^{-2}$, we find that:

$$\vec{u}_n = \left(\frac{u_{n,x}}{\sqrt{u_{n,x}^2 + u_{n,y}^2}}, \frac{u_{n,y}}{\sqrt{u_{n,x}^2 + u_{n,y}^2}} \right) = \left(\frac{e^2 - e^{-2}}{\sqrt{2(e^4 + e^{-4})}}, \frac{-e^2 - e^{-2}}{\sqrt{2(e^4 + e^{-4})}} \right)$$

If we consider \vec{u}_n instead of the directional vector \vec{u} to calculate the directional derivative $D_{\vec{u}_n}f(\vec{a})$, we obtain the maximum value for the directional derivative of f at the point \vec{a} since \vec{u}_n runs parallel to $\vec{\nabla}f(\vec{a})$:

$$\begin{aligned} D_{\vec{u}_n}f(\vec{a}) &= \vec{\nabla}f(\vec{a}) \cdot \vec{u}_n \\ &= (e^2 - e^{-2}, -e^2 - e^{-2}) \cdot \left(\frac{e^2 - e^{-2}}{\sqrt{2(e^4 + e^{-4})}}, \frac{-e^2 - e^{-2}}{\sqrt{2(e^4 + e^{-4})}} \right) \\ &= \sqrt{2(e^4 + e^4)} \\ &\approx 10.45 \end{aligned}$$

The equation of the plane tangent to f at \vec{a} is equal to:

$$\begin{aligned} z &= f(a_1, a_2) + \left[\frac{\partial f}{\partial x}(a_1, a_2) \right] (x - a_1) + \left[\frac{\partial f}{\partial y}(a_1, a_2) \right] (y - a_2) \\ &= \cosh(2) - \sinh(-2) + (e^2 - e^{-2})(x - 1) + (-e^2 - e^{-2})(y + 1) \\ &= (e^2 - e^{-2})x - (e^2 + e^{-2})y - e^2 \end{aligned}$$

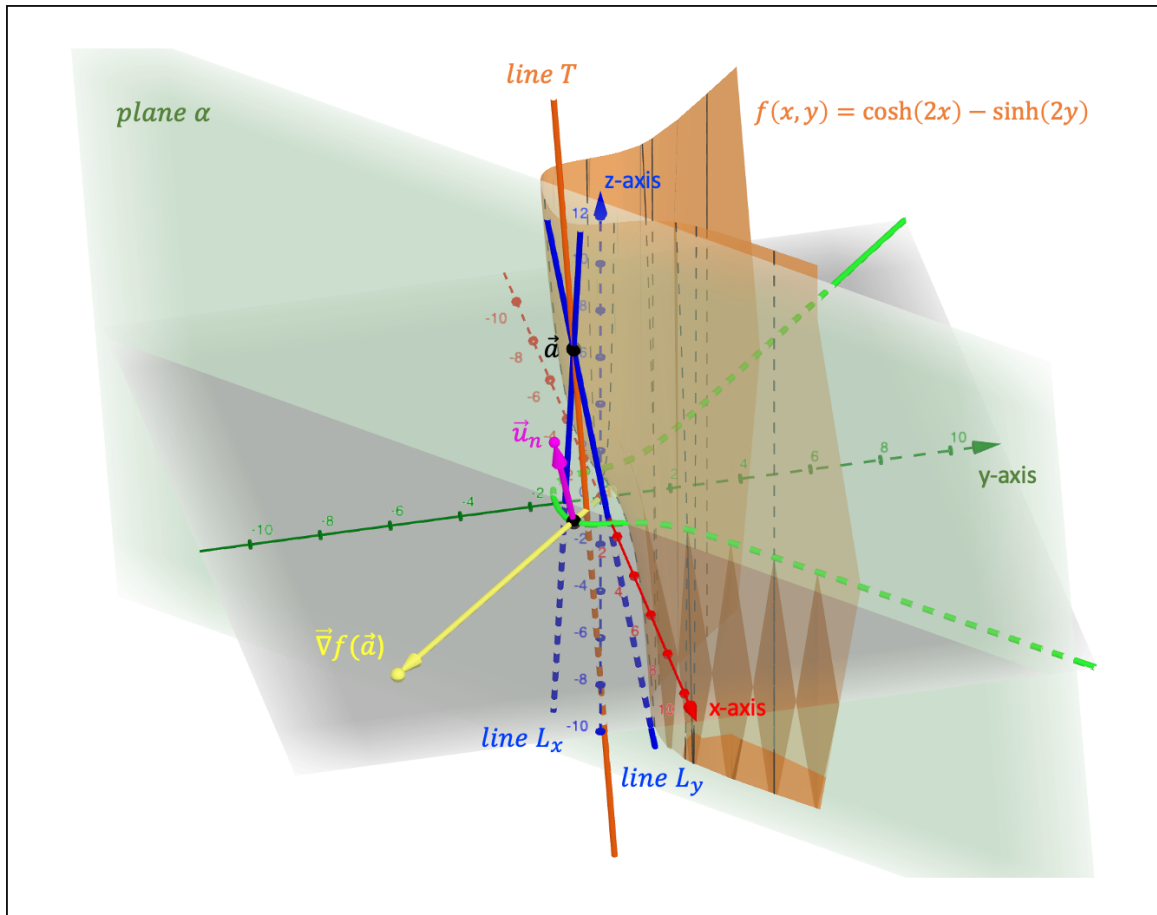


Figure 10.3: The graph of $f(x, y) = \cosh(2x) - \sinh(2y)$

In Fig. 10.3, the dark orange surface represents the graph of $f(x, y)$. The gradient $\vec{\nabla}f(\vec{a})$ is indicated in yellow and stands perpendicular to the level curve (in green) at the black dot, which is the projection of the point $\vec{a} = (1, -1, e^2)$ onto the xy -plane. The plane α shown in green represents the tangent plane to f at the point \vec{a} .

The partial derivative $\frac{\partial f}{\partial x}(\vec{a}) = e^2 - e^{-2}$ is equal to the slope of the blue line L_x , whereas the partial derivative $\frac{\partial f}{\partial y}(\vec{a}) = -e^2 - e^{-2}$ represents the slope of the blue line L_y . The two sets of parametric equations of these lines are equal to:

$$L_x : \begin{cases} x = t \\ y = -1 \\ z = e^{-2} + (e^2 - e^{-2})t \end{cases} \quad L_y : \begin{cases} x = 1 \\ y = t \\ z = -e^{-2} - (e^2 + e^{-2})t \end{cases}$$

In the case of L_x , the set of equations has been found by considering the intersection of the tangent plane α and the vertical plane $y = a_2 = -1$. Similarly, the equations of the line L_y are found by intersecting the tangent plane α and the vertical plane $x = a_1 = 1$.

The dark orange line T is the line tangent to f at \vec{a} whereby its slope is equal to the maximum value of the directional derivative: $D_{\vec{u}_n} f(\vec{a}) \approx 10.45$. The purple vector represents the unit vector \vec{u}_n and note that in Fig. 10.3 the function value $f(u_{n,x}, u_{n,y})$ has been added to the coordinates of \vec{u}_n for visual purposes.

The set of parametric equations of the tangent line T is found by intersecting the tangent plane α with the vertical plane β (not drawn in Fig. 10.3) that has the gradient $\vec{\nabla} f(\vec{a})$ as a directional vector and goes through the point \vec{a} (the equation of this vertical plane is $\beta : (e^2 + e^{-2})x + (e^2 - e^{-2})y = 2e^{-2}$):

$$T : \begin{cases} x = t \\ y = \left[\frac{2e^{-2}}{e^2 - e^{-2}} \right] - \left[\frac{e^2 + e^{-2}}{e^2 - e^{-2}} \right] t \\ z = - \left[\frac{2e^{-2}(e^2 + e^{-2})}{e^2 - e^{-2}} + e^2 \right] + \left[\frac{(e^2 - e^{-2})^2 + (e^2 + e^{-2})^2}{e^2 - e^{-2}} \right] t \end{cases}$$

- A general vector function

In the case of a general vector function $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m : \vec{x} \mapsto \vec{F}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}))$ of n variables and m component functions, whereby $\vec{x} = (x_1, x_2, \dots, x_n)$, we can arrange the partial derivatives in a $m \times n$ matrix called the **Jacobian matrix** \mathcal{L} whereby row i , with $i \in \{1, 2, \dots, m\}$, contains the n partial derivatives of the component function $f_i(\vec{x})$. The Jacobian matrix \mathcal{L} is the matrix associated with the linear transformation $\vec{F}'_{\vec{a}}$ (see section 3) and is defined as:

$$\mathcal{L} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If $m = n = 1$ the vector function becomes a scalar function of one variable, which we discussed in section 7.5. If $m = 1$ the vector function becomes a general scalar function of n variables and we discussed this case higher above in this section. If $m = n$ the matrix \mathcal{L} becomes square and its determinant is known as the **Jacobian determinant**, which is often noted as:

$$\det(\mathcal{L}) = \left| \frac{\partial(f_1, f_2, \dots, f_m)}{\partial(x_1, x_2, \dots, x_n)} \right|$$

As an example, consider the three-dimensional vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto f(x, y, z) = (x^2yz^3 - xy^2, xy^2z + yz^4, x^2y^2z + xz^3)$. The Jacobian matrix is then equal to:

$$\mathcal{L} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2xyz^3 - y^2 & x^2z^3 - 2xy & 3x^2yz^2 \\ y^2z & 2xyz + z^4 & xy^2 + 4yz^3 \\ 2xy^2z + z^3 & 2x^2yz & x^2y^2 + 3xz^2 \end{bmatrix}$$

The Jacobian determinant of this matrix then becomes:

$$\det(\mathcal{L}) = xyz [xyz^2 [xyz(-5xy - 12z^3) + [4xz^3 - y(9yz + 2)]] - 2x^2y^4 + z^5(7xz^3 - 11y)]$$

If we evaluate the Jacobian determinant in the point $\vec{a} = (2, 1, -1)$, we find that $\det(\mathcal{L}) = -14$. Since $\det(\mathcal{L}) \neq 0$ we know that the matrix is non-singular and thus invertible, so that the vector field \vec{F} is invertible in the vicinity of \vec{a} . If we image a small three-dimensional object in the near vicinity of point \vec{a} , the minus sign tells us that the linear transformation $\vec{F}'_{\vec{a}}$ has reversed the orientation of the object and, moreover, we know that its volume has been increased by a factor of 14.

In a final case, if $n = 1$, we obtain a vector function of one variable $\vec{F} : \mathbb{R} \rightarrow \mathbb{R}^m : t \mapsto \vec{F}(t) = (f_1(t), f_2(t), \dots, f_m(t))$, which describes a curve in \mathbb{R}^m . The Jacobian matrix of the linear transformation now becomes a column vector:

$$\mathcal{L} = \begin{bmatrix} \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial t} \\ \vdots \\ \frac{\partial f_m}{\partial t} \end{bmatrix}$$

Given a certain point $\vec{F}(t_*)$, the matrix \mathcal{L} evaluated in t_* represents a directional vector of the line $\vec{r}(t)$ tangent to \vec{F} at the point $\vec{F}(t_*)$ which is described by the equation $\vec{r}(t) = \vec{F}(t_*) + \mathcal{L}(t_*)t$.

As an example, let us look at the vector function $\vec{F} :]0, \frac{\pi}{2}[\rightarrow \mathbb{R}^3 : t \mapsto \vec{F}(t) = (\sec(t), \csc(t), 2t)$, whose graph is shown in Fig. 10.4. We would like to find an equation for the line $\vec{r}(t)$ tangent to the graph of \vec{F} at the point corresponding to $t_* = \frac{\pi}{4}$. The Jacobian matrix is equal to:

$$\mathcal{L} = \begin{bmatrix} \sec(t) \tan(t) \\ -\csc(t) \cot(t) \\ 2 \end{bmatrix} \Rightarrow \mathcal{L}(t_*) = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 2 \end{bmatrix}$$

Given that $\vec{F}(t_*) = (\sqrt{2}, \sqrt{2}, \frac{\pi}{2})$, the set of parametric equations of the line $\vec{r}(t)$ tangent to \vec{F} at $\vec{F}(t_*)$ is equal to:

$$\vec{r}(t) : \begin{cases} r_x = \sqrt{2} + \sqrt{2}t \\ r_y = \sqrt{2} - \sqrt{2}t \\ r_z = \frac{\pi}{2} + 2t \end{cases}$$

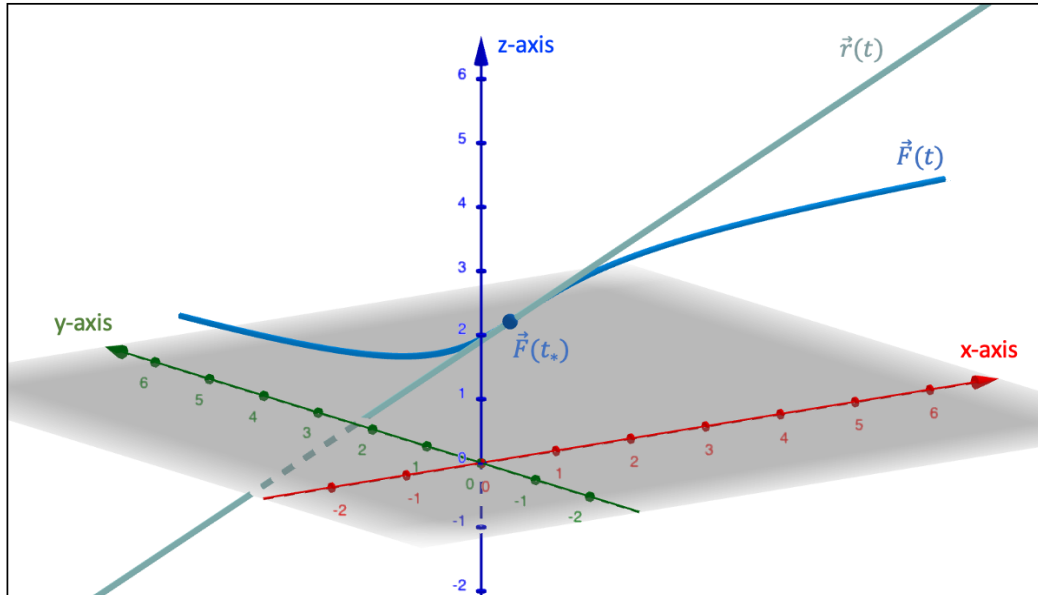


Figure 10.4: The graph of the vector function $\vec{F}(t) = (\sec(t), \csc(t), 2t)$

10.2.2 Higher Order Partial Derivatives and Total Differentials

Let us consider the scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and its partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$. We can now define the functions $\frac{\partial f}{\partial x_i} : A \subset \mathbb{R}^n \rightarrow \mathbb{R} : \vec{x} \mapsto \frac{\partial f}{\partial x_i}$, with $i \in \{1, 2, \dots, n\}$.

If the functions $\frac{\partial f}{\partial x_i}$ themselves have partial derivatives, we call them **partial derivatives of the second order** and are written as (with $j \in \{1, 2, \dots, n\}$):

$$\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{and} \quad \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_i} \right) = \frac{\partial^2 f}{\partial x_i^2} \quad (\text{if } i = j)$$

Note furthermore that:

$$\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) \Leftrightarrow \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

In the same manner as above, we define the partial derivatives of any higher order, as long as they exist.

For example, let us calculate the second derivatives of the function $f : \mathbb{R}^4 \rightarrow \mathbb{R} : (x, y, z, u) \mapsto f(x, y, z, u) = 2x^2y^4u - yz^3u^5 + x^4y^2$.

$$\begin{aligned}
 \bullet \frac{\partial^2}{\partial x^2} &= 4y^4u + 12x^2y^2 & \bullet \frac{\partial^2}{\partial x \partial z} &= 0 \\
 \bullet \frac{\partial^2}{\partial y^2} &= 24x^2y^2u + 2x^4 & \bullet \frac{\partial^2}{\partial x \partial u} &= 4xy^4 \\
 \bullet \frac{\partial^2}{\partial z^2} &= -6yzu^5 & \bullet \frac{\partial^2}{\partial y \partial z} &= -3z^2u^5 \\
 \bullet \frac{\partial^2}{\partial u^2} &= -20yz^3u^3 & \bullet \frac{\partial^2}{\partial y \partial u} &= 8x^2y^3 - 5z^3u^4 \\
 \bullet \frac{\partial^2}{\partial x \partial y} &= 16xy^3u + 8x^3y & \bullet \frac{\partial^2}{\partial z \partial u} &= -15yz^2u^4
 \end{aligned}$$

To write the **total differential** of order n we consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto z = f(x, y)$. Assume furthermore that the rate of change dx and dy is the same for every point and that these rates remain constant at every differentiation.

The first differential is equal to:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

The second differential is calculated as follows:

$$\begin{aligned}
 d^2z &= d(dz) = d\left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right) \\
 &= \left[\frac{\partial^2 z}{\partial x^2} dx^2 + \frac{\partial z}{\partial x} d(dx) + \frac{\partial^2 z}{\partial y \partial x} dx dy\right] + \left[\frac{\partial^2 z}{\partial x \partial y} dy dx + \frac{\partial^2 z}{\partial y^2} dy^2 + \frac{\partial z}{\partial y} d(dy)\right] \\
 &= \left[\frac{\partial^2 z}{\partial x^2} dx^2 + \frac{\partial z}{\partial x} \cdot 0 + \frac{\partial^2 z}{\partial y \partial x} dx dy\right] + \left[\frac{\partial^2 z}{\partial x \partial y} dy dx + \frac{\partial^2 z}{\partial y^2} dy^2 + \frac{\partial z}{\partial y} \cdot 0\right] \\
 &= \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2
 \end{aligned}$$

To find an expression for the n^{th} differential, we rely on Newton's binomial theorem, which states the following:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \quad \text{with} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The general formula for the n^{th} differential is equal to:

$$d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^{(n)} z$$

As an example, let us consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto z = f(x, y) = \tan(xy)$, whereby $xy \neq \frac{\pi}{2} + k\pi$ (with $k \in \mathbb{Z}$). Based on the above formula, we wish to write down the third differential:

$$\begin{aligned} d^3z &= \frac{\partial^3 z}{\partial x^3} dx^3 + 3 \frac{\partial^3 z}{\partial x^2 \partial y} dx^2 dy + 3 \frac{\partial^3 z}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 z}{\partial y^3} dy^3 \\ &= [2y^3 \sec^2(xy) (2 \sec^2 xy - 1)] dx^3 + 3 [2y \sec^2(xy) (2 \tan xy + 3xy \sec^2 xy - 2xy)] dx^2 dy \\ &\quad + 3 [2x \sec^2(xy) (2 \tan xy + 3xy \sec^2 xy - 2xy)] dx dy^2 + [2x^3 \sec^2(xy) (2 \sec^2 xy - 1)] dy^3 \\ &= 2 \sec^2(xy) (2 \sec^2 xy - 1) (y^3 dx^3 + x^3 dy^3) \\ &\quad + 6 \sec^2(xy) (2 \tan xy + 3xy \sec^2 xy - 2xy) (x dx dy^2 + y dx^2 dy) \end{aligned}$$

10.2.3 Taylor's Theorem

The extrapolation of Taylor's formula for a scalar function of one variable (see section 7.5.3) to the multivariable situation gives us the following expression, if we wish to Taylor expand the function $f : \mathbb{R}^n \rightarrow \mathbb{R} : \vec{x} \mapsto f(\vec{x})$ around the point $\vec{a} = (a_1, a_2, \dots, a_n)$:

$$f(\vec{x}) = \sum_{i=0}^m \frac{1}{i!} \left[(x_1 - a_1) \frac{\partial}{\partial x_1} + (x_2 - a_2) \frac{\partial}{\partial x_2} + \dots + (x_n - a_n) \frac{\partial}{\partial x_n} \right]^{(i)} f(\vec{a}) + r_{m+1}(\vec{x})$$

whereby $r_{m+1}(\vec{x})$ represents a remainder term and whereby we calculate the square brackets to the power of (i) according to Newton's binomial theorem.

Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \sqrt[3]{2x^2 - y^2}$. We would like to Taylor expand f to the second order around the point $\vec{a} = (-4, 4, 2\sqrt[3]{2})$. If we write $f(x, y)$ as $f(x, y) = g(x, y) + r_n(x, y)$, we find $g(x, y)$ with the help of Taylor's theorem as follows:

$$\begin{aligned} g(x, y) &= f(a_1, a_2) + (x - a_1) \frac{\partial f}{\partial x}(a_1, a_2) + (y - a_2) \frac{\partial f}{\partial y}(a_1, a_2) \\ &\quad + \frac{1}{2} \left[(x - a_1)^2 \frac{\partial^2 f}{\partial x^2}(a_1, a_2) + 2(x - a_1)(y - a_2) \frac{\partial^2 f}{\partial x \partial y}(a_1, a_2) + (y - a_2)^2 \frac{\partial^2 f}{\partial y^2}(a_1, a_2) \right] \\ &= 2\sqrt[3]{2} + (x + 4) \left(-\frac{2\sqrt[3]{2}}{3} \right) + (y - 4) \left(-\frac{\sqrt[3]{2}}{3} \right) \\ &\quad + \frac{1}{2} \left[(x + 4)^2 \left(-\frac{5}{9\sqrt[3]{4}} \right) + 2(x + 4)(y - 4) \left(-\frac{2\sqrt[3]{2}}{9} \right) + (y - 4)^2 \left(-\frac{7}{9\sqrt[3]{4}} \right) \right] \\ &= -\frac{2\sqrt[3]{2}}{9} \left(5 + 4x - \frac{3y}{2} + xy + \frac{5x^2}{8} + \frac{7y^2}{8} \right) \end{aligned}$$

The blue two-dimensional surface in Fig. 10.5 represents the graph of $f(x, y)$ in three-dimensional space. The yellow two-dimensional surface represents the graph of $g(x, y)$,

which is the Taylor expansion of f about the point \vec{a} . As we can see, in the vicinity of \vec{a} , the function g approximates the function f , which is what Taylor's theorem entails.

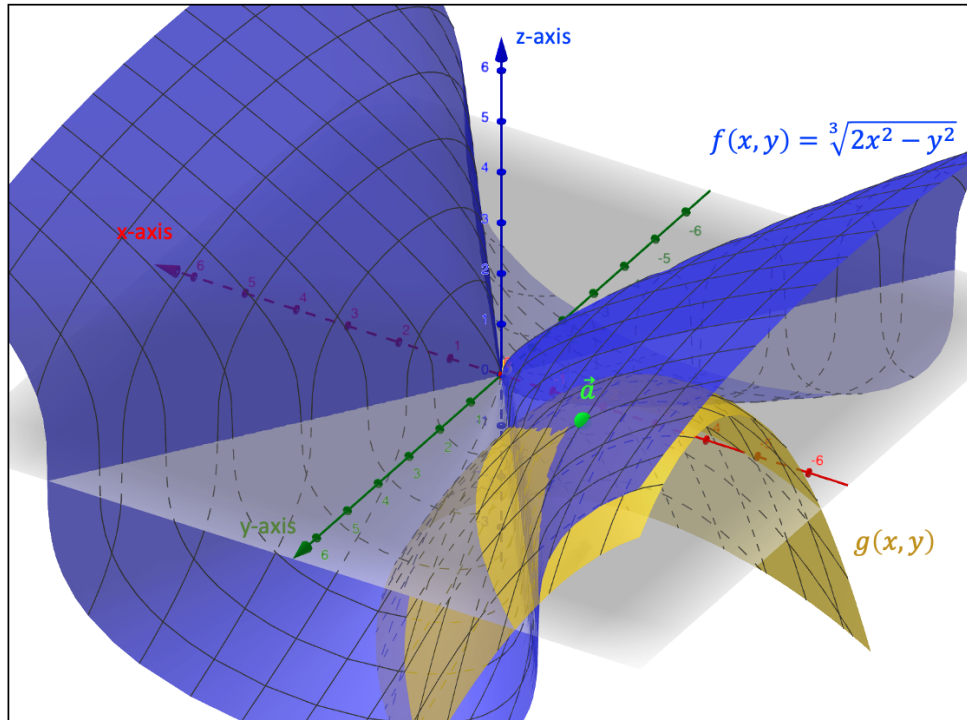


Figure 10.5: The Taylor expansion of $f(x, y)$ around the point \vec{a}

10.2.4 Extremum Points

A **stationary point** \vec{a} of the scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a point for which $\vec{\nabla} f(\vec{a}) = \vec{0}$. Similar to the situation of a scalar function of one variable (see section 7.5.4), the existence of a stationary point does not necessarily imply the existence of an extremum point.

We consider the case of \mathbb{R}^2 and define the following variables:

$$\bullet r = \frac{\partial^2 f}{\partial x^2}(\vec{a}) \quad \bullet s = \frac{\partial^2 f}{\partial x \partial y}(\vec{a}) \quad \bullet t = \frac{\partial^2 f}{\partial y^2}(\vec{a})$$

In order to investigate whether we are dealing with an extremum point, we apply the following rules:

$$\left\{ \begin{array}{ll} \text{If } s^2 - rt > 0 & \Rightarrow \text{f does not reach an extremum in } \vec{a} \\ \text{If } s^2 - rt < 0 \text{ and } r > 0 (t > 0) & \Rightarrow \text{f reaches a local minimum in } \vec{a} \\ \text{If } s^2 - rt < 0 \text{ and } r < 0 (t < 0) & \Rightarrow \text{f reaches a local maximum in } \vec{a} \\ \text{If } s^2 - rt = 0 & \Rightarrow \text{no conclusion can be reached} \end{array} \right.$$

Note that in the case where $s^2 - rt > 0$, the point \vec{a} is called a saddle point and the graph of f near \vec{a} represents a saddle surface, such as a hyperbolic paraboloid.

Let us examine the function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = 8x^3 - 10y^3 + 12x^2 + 6y^2 - 2x + 2y$. In a first step, we calculate the stationary points by demanding that $\vec{\nabla} f(\vec{x}) = \vec{0}$:

$$\begin{cases} \frac{\partial f}{\partial x} = 0 & \Leftrightarrow & 24x^2 + 24x - 2 = 0 \\ \frac{\partial f}{\partial y} = 0 & \Leftrightarrow & -30y^2 + 12y + 2 = 0 \end{cases}$$

Solving these quadratic equations gives us four stationary points:

$$\begin{aligned} \bullet \vec{a}_1 &= \left(\frac{1}{12}, \frac{3-2\sqrt{6}}{15} \right) & \bullet \vec{a}_3 &= \left(-\frac{13}{12}, \frac{3-2\sqrt{6}}{15} \right) \\ \bullet \vec{a}_2 &= \left(\frac{1}{12}, \frac{3+2\sqrt{6}}{15} \right) & \bullet \vec{a}_4 &= \left(-\frac{13}{12}, \frac{3+2\sqrt{6}}{15} \right) \end{aligned}$$

Given that $\frac{\partial^2}{\partial x^2} = 48x + 24$, $\frac{\partial^2}{\partial y^2} = -60y + 12$, and $\frac{\partial^2}{\partial x \partial y} = 0$, we find the following values for r , s , and t for the above four stationary points:

$$\vec{a}_1 = \left(\frac{1}{12}, \frac{3-2\sqrt{6}}{15} \right) \Rightarrow \begin{cases} r = 28 \\ s = 0 \\ t = 8\sqrt{6} \end{cases} \Rightarrow s^2 - rt = -224\sqrt{6} < 0$$

$$\vec{a}_2 = \left(\frac{1}{12}, \frac{3+2\sqrt{6}}{15} \right) \Rightarrow \begin{cases} r = 28 \\ s = 0 \\ t = -8\sqrt{6} \end{cases} \Rightarrow s^2 - rt = 224\sqrt{6} > 0$$

$$\vec{a}_3 = \left(-\frac{13}{12}, \frac{3-2\sqrt{6}}{15} \right) \Rightarrow \begin{cases} r = -28 \\ s = 0 \\ t = 8\sqrt{6} \end{cases} \Rightarrow s^2 - rt = 224\sqrt{6} > 0$$

$$\vec{a}_4 = \left(-\frac{13}{12}, \frac{3+2\sqrt{6}}{15} \right) \Rightarrow \begin{cases} r = -28 \\ s = 0 \\ t = -8\sqrt{6} \end{cases} \Rightarrow s^2 - rt = -224\sqrt{6} < 0$$

These results tell us that the function f does not reach an extremum in the points \vec{a}_2 and \vec{a}_3 and f represents a saddle surface in their vicinity. A local minimum is reached in the point \vec{a}_1 since $s^2 - rt < 0$ and $r > 0$ ($t > 0$), whereas f reaches a local maximum in the point \vec{a}_4 given that $s^2 - rt < 0$ and $r < 0$ ($t < 0$).

The graph of f together with the four stationary points are presented in below Fig. 10.6.

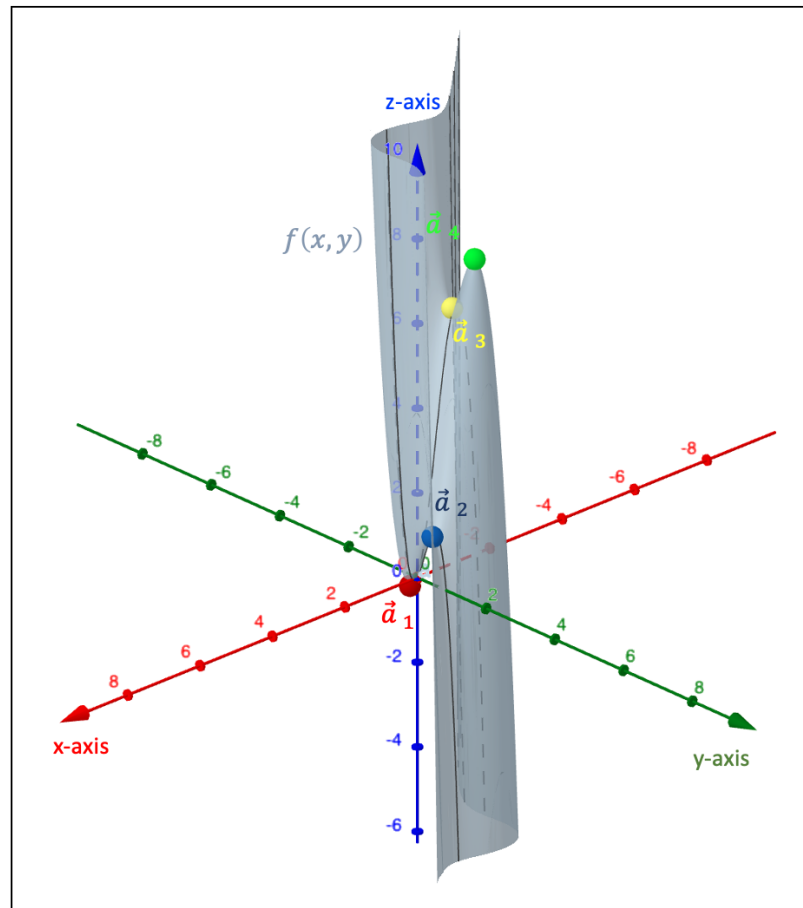


Figure 10.6: Extremum points of $f(x, y)$

In case we wish to determine the extremum points of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ under certain equations of constraints, we can turn to the **method of the Lagrange multipliers**.

In a first step, we construct the following function f_s :

$$f_s : \mathbb{R}^n \rightarrow \mathbb{R} : \vec{x} \mapsto f_s(\vec{x}) = f(\vec{x}) - \sum_{i=1}^p \alpha_i g_i(\vec{x})$$

whereby $p \in \mathbb{N}_0$ represents the number of constraints, α_i are variables called the Lagrange multipliers, and the functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ represent the constraints. Note that for a given constraint $u(\vec{x}) = d$, the function $g(\vec{x})$ obtains the form $g(\vec{x}) = u(\vec{x}) - d$.

Next, we identify the stationary points \vec{a} through the requirement that $\vec{\nabla} f_s(\vec{x}) = \vec{0}$:

$$\vec{\nabla} f_s(\vec{x}) = \vec{\nabla} f(\vec{x}) - \sum_{i=1}^p \alpha_i \vec{\nabla} g_i(\vec{x}) = \vec{0}$$

In a final step, to determine whether the points \vec{a} correspond to a minimum or maximum, we calculate the second-order differential of the function f_s : if $d^2 f_s > 0$ ($d^2 f_s < 0$), the point \vec{a} is a minimum (maximum).

Suppose we are interested in finding the shortest distance between the saddle point \vec{s} of the function $h : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto h(x, y) = -\frac{x^2}{8} + \frac{y^2}{2} - xy + \frac{7x}{4} - 3y + \frac{79}{8}$ and the point \vec{c} , which lies within the plane $\gamma : -2x + 2y + 3z = -6$ and whose sum of the coordinates is equal to 10. Fig. 10.7 shows the configuration of this problem.

The function f for which we want to find an extremum is the function that expresses the distance between the points $\vec{s} = (s_1, s_2, s_3)$ and $\vec{c} = (x, y, z)$:

$$f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z) = \sqrt{(x - s_1)^2 + (y - s_2)^2 + (z - s_3)^2}$$

In fact, since we have to take the partial derivatives of f in a later step, instead of considering the distance we consider the distance squared which reduces the amount of calculation and does not change our problem:

$$f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z) = (x - s_1)^2 + (y - s_2)^2 + (z - s_3)^2$$

Let us now first find the saddle point \vec{s} of the function h through the method discussed earlier in this section. Given that $\frac{\partial h}{\partial x} = -\frac{x}{4} - y + \frac{7}{4}$ and $\frac{\partial h}{\partial y} = y - x - 3$, we find a stationary point $\vec{b} = (-1, 2, 6)$ for the function h . The value of the expression $s^2 - rt$ is equal to:

$$\vec{b}_1 = (-1, 2, 6) \quad \Rightarrow \quad \begin{cases} r = -\frac{1}{4} \\ s = -1 \\ t = 1 \end{cases} \quad \Rightarrow \quad s^2 - rt = \frac{5}{4} > 0$$

In other words, the point $\vec{s} = (-1, 2, 6)$ indeed corresponds to a saddle point of the function h .

In our problem, we have two constraints for the point \vec{c} : it lies within the plane $\gamma : -2x + 2y + 3z = -6$, and $x + y + z = 10$, which means that the sum of its coordinates equals 10. The two functions related to these constraints are: $g_1 : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto g_1(x, y, z) = -2x + 2y + 3z + 6$ and $g_2 : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto g_2(x, y, z) = x + y + z - 10$.

The function f_s then obtains the following form:

$$f_s(\vec{x}) = (x + 1)^2 + (y - 2)^2 + (z - 6)^2 - \alpha_1(-2x + 2y + 3z + 6) - \alpha_2(x + y + z - 10)$$

In a next step, we identify the stationary points \vec{a} of f_s by demanding that $\vec{\nabla} f_s(\vec{x}) = \vec{0}$:

$$\begin{cases} \frac{\partial f_s}{\partial x} = 2x + 2 + 2\alpha_1 - \alpha_2 = 0 \\ \frac{\partial f_s}{\partial y} = 2y - 4 - 2\alpha_1 - \alpha_2 = 0 \\ \frac{\partial f_s}{\partial z} = 2z - 12 - 3\alpha_1 - \alpha_2 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \frac{\alpha_2}{2} - \alpha_1 - 1 \\ y = \frac{\alpha_2}{2} + \alpha_1 + 2 \\ z = \frac{\alpha_2}{2} + \frac{3\alpha_1}{2} + 6 \end{cases}$$

If we now insert these values for x , y , and z into the two constraints, we find the following two equations:

$$\begin{cases} -2x + 2y + 3z = -6 \\ x + y + z = 10 \end{cases} \Leftrightarrow \begin{cases} \frac{3\alpha_2}{2} + \frac{17\alpha_1}{2} + 24 = -6 \\ \frac{3\alpha_2}{2} + \frac{3\alpha_1}{2} + 7 = 10 \end{cases}$$

Solving these equations gives us the values of the two Lagrange multipliers and the coordinates of the point \vec{c} :

$$\begin{cases} \alpha_1 = -\frac{33}{7} \\ \alpha_2 = \frac{47}{7} \end{cases} \Rightarrow \vec{c} = \left(\frac{99}{14}, \frac{9}{14}, \frac{16}{7} \right)$$

To confirm that this point indeed produces a minimal value for the distance, we must have that the second-order differential of f_s is greater than zero:

$$d^2 f_s = 2dx^2 + 2dy^2 + 2dz^2 = 2(dx^2 + dy^2 + dz^2) > 0$$

In a final step, we calculate the distance d between the saddle point \vec{s} and the point \vec{c} :

$$d = \sqrt{(x - s_1)^2 + (y - s_2)^2 + (z - s_3)^2} = \sqrt{\left(\frac{99}{14} + 1\right)^2 + \left(\frac{9}{14} - 2\right)^2 + \left(\frac{16}{7} - 6\right)^2} \approx 8.99$$

The length of the blue line in Fig. 10.7 represents the distance d between \vec{s} and \vec{c} .

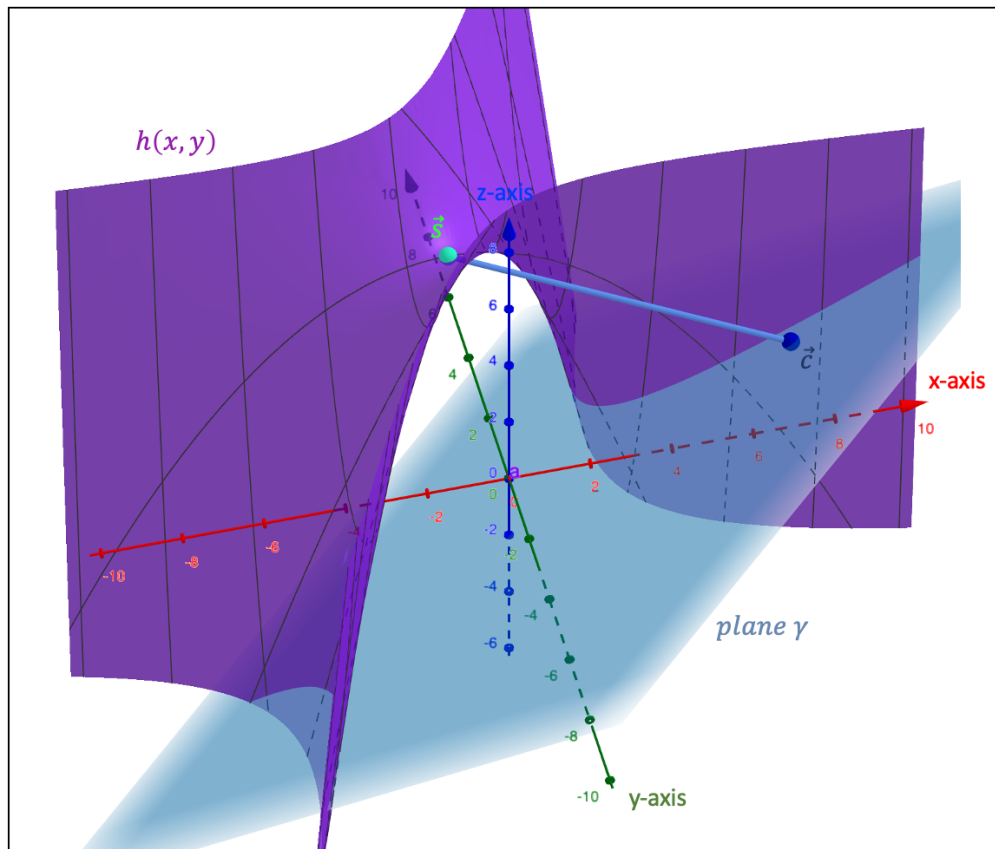


Figure 10.7: The shortest distance between the saddle point \vec{s} and the point \vec{c}

10.3 Multivariable Operators

10.3.1 The Gradient

We have defined the **gradient** in section 10.2.1 and we have seen that the gradient of a scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point \vec{a} represents the direction in which f varies maximally in the vicinity of \vec{a} .

Remember that for a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a point $\vec{a} = (a_1, a_2)$ the gradient $\vec{\nabla}f(\vec{a})$ is oriented perpendicular to the level curve $f(x, y) = f(a_1, a_2)$ at the point \vec{a} . Similarly, for a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and a point $\vec{a} = (a_1, a_2, a_3)$ the gradient $\vec{\nabla}f(\vec{a})$ is oriented perpendicular to the level surface $f(x, y, z) = f(a_1, a_2, a_3)$ at the point \vec{a} .

Given that the definition of the gradient of a scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is equal to:

$$\vec{\nabla}f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right) f$$

we can write down the following definition for the **nabla operator** $\vec{\nabla}$:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

10.3.2 The Divergence

The **divergence** of a vector field $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n : \vec{x} \mapsto \vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_n(\vec{x}))$ describes the net flow at each point of the field and is defined by the scalar product of the nabla operator and the vector field:

$$\operatorname{div}(\vec{F}(\vec{x})) = \vec{\nabla} \cdot \vec{F}(\vec{x}) = \sum_{i=1}^n \frac{\partial F_i(\vec{x})}{\partial x_i}$$

In other words, the divergence operator takes a vector field as input variable and returns a scalar value. If at a certain point \vec{a} of the field $\vec{F}(\vec{x})$ we find that $\operatorname{div}(\vec{F}(\vec{x})) > 0$, there is a net outflow at \vec{a} and we describe that point as a *source*. In contrast, if $\operatorname{div}(\vec{F}(\vec{x})) < 0$ at point \vec{a} , there is a net inflow and \vec{a} is defined as a *sink*.

Consider for example the vector field $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \vec{x} \mapsto \vec{F}(x, y) = (x^3y, xy^3)$, which is depicted in Fig. 10.8. The divergence is equal to $\operatorname{div}(\vec{F}(x, y)) = 3x^2y + 3xy^2 = 3xy(x + y)$.

On the coordinate axes, either x or y is zero, so that $\operatorname{div}(\vec{F}(x, y)) = 0$. The divergence also becomes zero if we consider a point in the second or fourth quadrant whereby $x = -y$; this is the case for all the points that lie on the bisector of these two quadrants, i.e., the red line in Fig. 10.8.

In the region A within the second quadrant, we have that $x < 0$, $y > 0$ and $y > |x|$, so that

$\operatorname{div}(\vec{F}(x, y)) < 0$. That is, all the points within region A are sinks. In the region B of the second quadrant, we can see that $x < 0$, $y > 0$ and $y < |x|$, so that $\operatorname{div}(\vec{F}(x, y)) > 0$. In other words, all the points of region B are sources. A similar reasoning can be applied to the fourth quadrant and we find that the points of region C are sources, while those of region D are sinks.

Finally, all the points of the first quadrant are sources given that $\operatorname{div}(\vec{F}(x, y)) > 0$, whereas all the points of the third quadrant are sinks because $\operatorname{div}(\vec{F}(x, y)) < 0$.

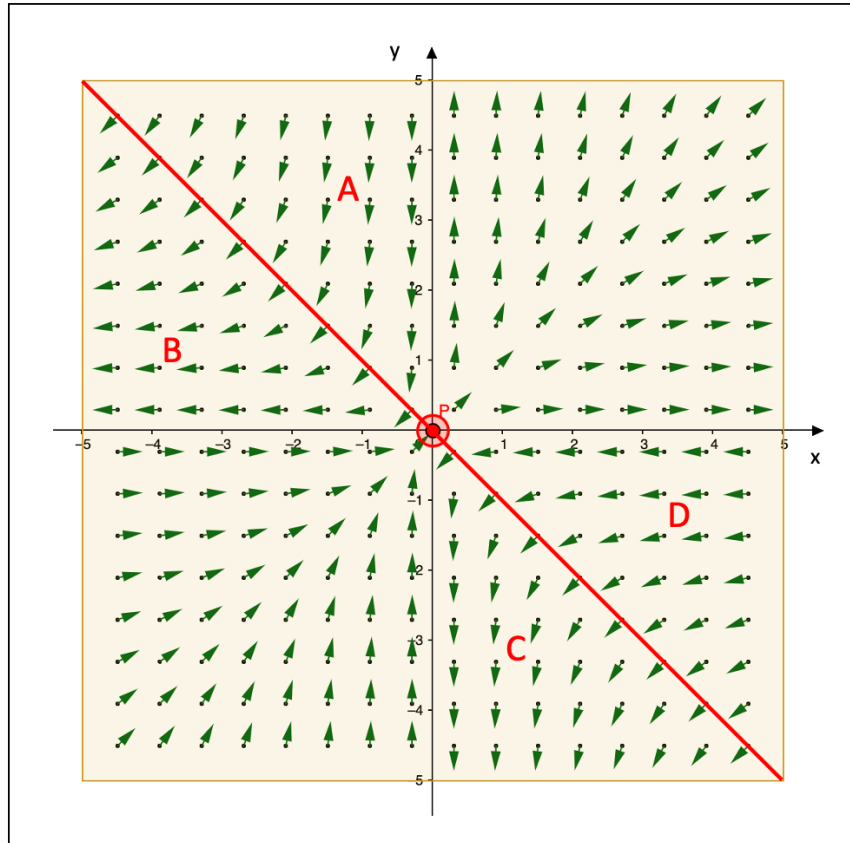


Figure 10.8: The vector field $\vec{F}(x, y) = (x^3y, yx^3)$ (Credit for the GeoGebra applet: Antonio Di Muro)

There exists a relationship between the gradient and the divergence operator that resembles the chain rule of differentiation (with f a scalar function and \vec{v} a vector field):

$$\vec{\nabla} \cdot (f\vec{v}) = \vec{\nabla} f \cdot \vec{v} + f\vec{\nabla} \cdot \vec{v}$$

Let us demonstrate this property for the case of three dimensions based on the definitions of the gradient and the divergence:

$$\begin{aligned} \vec{\nabla} \cdot (f\vec{v}) &= \frac{\partial(fv_x)}{\partial x} + \frac{\partial(fv_y)}{\partial y} + \frac{\partial(fv_z)}{\partial z} \\ &= \left[v_x \frac{\partial f}{\partial x} + f \frac{\partial v_x}{\partial x} \right] + \left[v_y \frac{\partial f}{\partial y} + f \frac{\partial v_y}{\partial y} \right] + \left[v_z \frac{\partial f}{\partial z} + f \frac{\partial v_z}{\partial z} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} \right] + \left[f \frac{\partial v_x}{\partial x} + f \frac{\partial v_y}{\partial y} + f \frac{\partial v_z}{\partial z} \right] \\
&= \left[(v_x, v_y, v_z) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \right] + f \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (v_x, v_y, v_z) \right] \\
&= \vec{v} \cdot \vec{\nabla} f + f \vec{\nabla} \cdot \vec{v}
\end{aligned}$$

10.3.3 The Curl

The **curl** of a three-dimensional vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \vec{x} \mapsto \vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), F_3(\vec{x}))$ describes the pattern of rotation of the field at a certain point \vec{a} . The curl is defined as the cross product between the nabla operator and the vector field (with \vec{i}_x , \vec{i}_y , and \vec{i}_z the respective unit vectors):

$$\text{rot}(\vec{F}(\vec{x})) = \vec{\nabla} \times (\vec{F}(\vec{x})) = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(\vec{x}) & F_2(\vec{x}) & F_3(\vec{x}) \end{vmatrix}$$

The norm of the curl $\|\vec{\nabla} \times (\vec{F}(\vec{x}))\|$ tells us the amount of rotation, while the direction of the rotation axis is given by the curl vector whereby the orientation of the rotation axis is determined through the right-hand rule (see section 6.3).

Consider the vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \vec{x} \mapsto \vec{F}(\vec{x}) = (-y^3x^2 + 2y^2 - 2x + 20, -x^3y^2 - 6y^2 + 4x^2 + 24, 0)$, which is shown in below Fig. 10.9. Given that $F_3(\vec{x}) = 0$ and that both $F_1(\vec{x})$ and $F_2(\vec{x})$ do not contain the z variable, the curl becomes the following:

$$\begin{aligned}
\vec{\nabla} \times (\vec{F}(\vec{x})) &= \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(\vec{x}) & F_2(\vec{x}) & 0 \end{vmatrix} = \left[\frac{\partial F_2(\vec{x})}{\partial x} - \frac{\partial F_1(\vec{x})}{\partial y} \right] \vec{i}_z \\
&= (-3x^2y^2 + 8x + 3y^2x^2 - 4y) \vec{i}_z \\
&= (8x - 4y) \vec{i}_z
\end{aligned}$$

Take for instance the point $\vec{a} = (\frac{37}{20}, -\frac{4}{5})$, which corresponds to the red dot in Fig. 10.9. The curl is equal to $\vec{\nabla} \times (\vec{F}(\vec{a})) = (8x - 4y) \vec{i}_z = (8 \cdot \frac{37}{20} + 4 \cdot \frac{4}{5}) \vec{i}_z = 18 \vec{i}_z$. This means that at point \vec{a} , the strength of rotation is equal to 18 and the rotation axis is oriented in the positive z -direction.

Bearing in mind the right-hand rule of the curl, we can see in Fig. 10.9 that the curl of the points to the left (right) of the red line is oriented toward the negative (positive) z -direction. The equation of the red line is found by setting the magnitude of the curl equal to zero: $8x - 4y = 0 \Leftrightarrow y = 2x$.

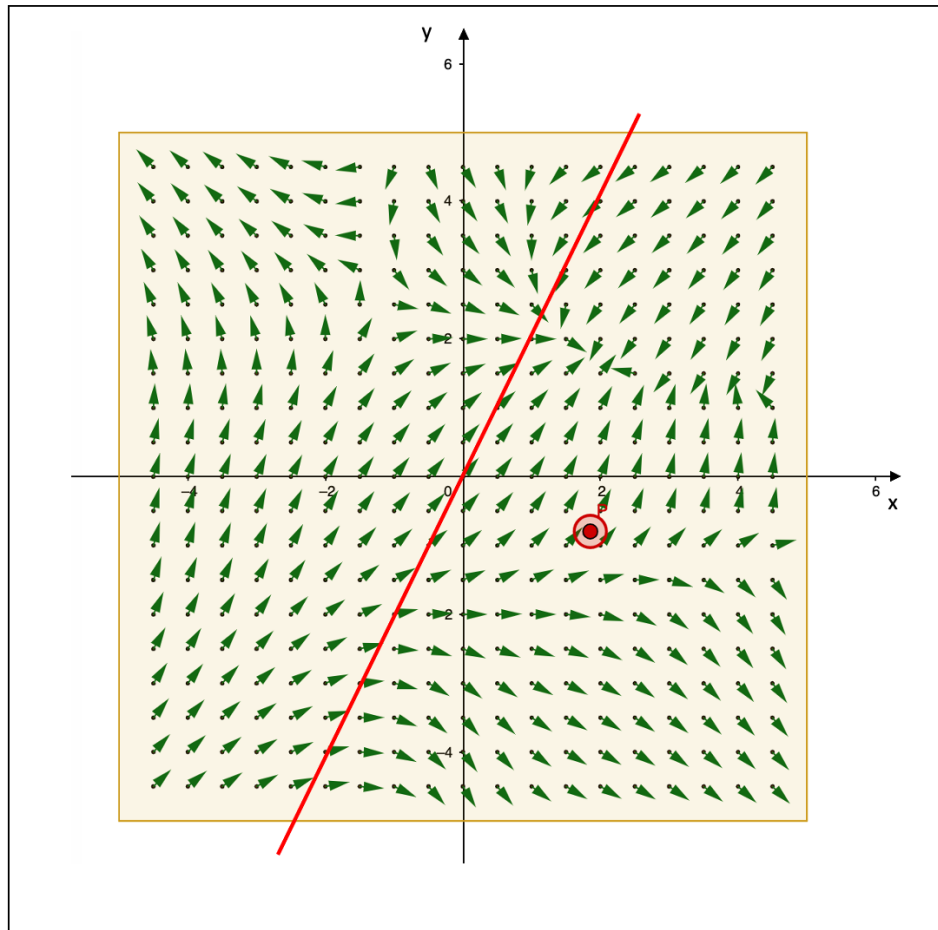


Figure 10.9: The vector field $\vec{F}(x, y) = (-y^3x^2 + 2y^2 - 2x + 20, -x^3y^2 - 6y^2 + 4x^2 + 24, 0)$ (Credit for the GeoGebra applet: Antonio Di Muro)

Fig. 10.10 lists four relationships that exist between the gradient, the divergence, and the curl, whereby f is the scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and \vec{F}_1 and \vec{F}_2 the two vector fields $\vec{F}_1, \vec{F}_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

<ul style="list-style-type: none"> ➤ $\vec{\nabla} \cdot (\vec{F}_1 \times \vec{F}_2) = (\vec{\nabla} \times \vec{F}_1) \cdot \vec{F}_2 - (\vec{\nabla} \times \vec{F}_2) \cdot \vec{F}_1$ ➤ $\vec{\nabla} \times (f\vec{F}_1) = \vec{\nabla}f \times \vec{F}_1 + f(\vec{\nabla} \times \vec{F}_1)$ 	<ul style="list-style-type: none"> ➤ $\vec{\nabla} \times (\vec{\nabla}f) = \vec{0}$ ➤ $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}_1) = 0$
---	---

Figure 10.10: Some relationships between the gradient, the divergence, and the curl

Let us demonstrate the relationship $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}_1) = 0$ using the definitions of the divergence and the curl:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}_1) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\begin{aligned}
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\
&= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y} \\
&= 0
\end{aligned}$$

10.3.4 The Laplacian

The **Laplacian** of a scalar function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ provides information about the local curvature of f or, alternatively, it compares the function value of a certain point to that of the points in its close vicinity.

The Laplacian is denoted by $\vec{\nabla}^2 f$ or $\Delta(f)$ and is defined as the divergence of the gradient of the function f :

$$\Delta(f) = \vec{\nabla} \cdot \vec{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Take as an example the scalar function $f : \mathbb{R}^5 \rightarrow \mathbb{R} : (x, y, z, u, v) \mapsto f(x, y, z, u, v) = x^4 v^2 + 3y^2 z^3 - 2u^3$. The Laplacian is calculated as:

$$\Delta(f) = 12x^2 v^2 + 6z^3 + 18y^2 z - 12u + 2x^4$$

Since the Laplacian is a scalar, we can multiply it with a vector field \vec{F} , whereby the outcome is referred to as the **vector Laplacian**. For a three-dimensional vector field \vec{F} , there exists the following relationship between the gradient, the divergence, the curl, and the vector Laplacian:

$$\text{rot}(\text{rot}(\vec{F})) = \text{grad}(\text{div}(\vec{F})) - \Delta(\vec{F}) \Leftrightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \Delta(\vec{F})$$

Let us verify the above relationship for the vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \vec{x} \mapsto \vec{F}(\vec{x}) = (2x^3 y^2 z, -x^2 y z^3, x^4 y^3 z^2)$. The divergence, the gradient of the divergence, and the curl of \vec{F} are equal to:

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{F} = 6x^2 y^2 z - x^2 z^3 + 2x^4 y^3 z \\ \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) = (12xy^2 z - 2xz^3 + 8x^3 y^3 z, 12x^2 y z + 6x^4 y^2 z, 6x^2 y^2 - 3x^2 z^2 + 2x^4 y^3) \\ \vec{\nabla} \times \vec{F} = (3x^4 y^2 z^2 + 3x^2 y z^2, 2x^3 y^2 - 4x^3 y^3 z^2, -2xyz^3 - 4x^3 y z) \end{array} \right.$$

The curl of the curl of the vector field \vec{F} is equal to:

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) &= (-2xz^3 - 4x^3 z + 8x^3 y^3 z, 6x^4 y^2 z + 6x^2 y z + 2yz^3 + 12x^2 y z, \\
&\quad 6x^2 y^2 - 12x^2 y^3 z^2 - 6x^4 y z^2 - 3x^2 z^2)
\end{aligned}$$

The vector Laplacian of \vec{F} is calculated as:

$$\Delta(\vec{F}) = (12xy^2z + 4x^3z, -2yz^3 - 6x^2yz, 12x^2y^3z^2 + 6x^4yz^2 + 2x^4y^3)$$

The right-hand side of the relationship is equal to:

$$\begin{aligned} \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \Delta(\vec{F}) &= (12xy^2z - 2xz^3 + 8x^3y^3z, 12x^2yz + 6x^4y^2z, 6x^2y^2 - 3x^2z^2 + 2x^4y^3) - \\ &\quad (12xy^2z + 4x^3z, -2yz^3 - 6x^2yz, 12x^2y^3z^2 + 6x^4yz^2 + 2x^4y^3) \\ &= (-2xz^3 - 4x^3z + 8x^3y^3z, 6x^4y^2z + 6x^2yz + 2yz^3 + 12x^2yz, \\ &\quad 6x^2y^2 - 12x^2y^3z^2 - 6x^4yz^2 - 3x^2z^2) \end{aligned}$$

and this expression is exactly equal to the left-hand side of the relationship, i.e., the curl of the curl of the vector field \vec{F} .

10.4 Integral Calculus

10.4.1 The Line Integral

Consider a two-dimensional surface in three-dimensional space that is described by the scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We now take the continuous curve \widehat{AB} that runs along the surface of f between the points A and B and is described by the vector function $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}(t)$.

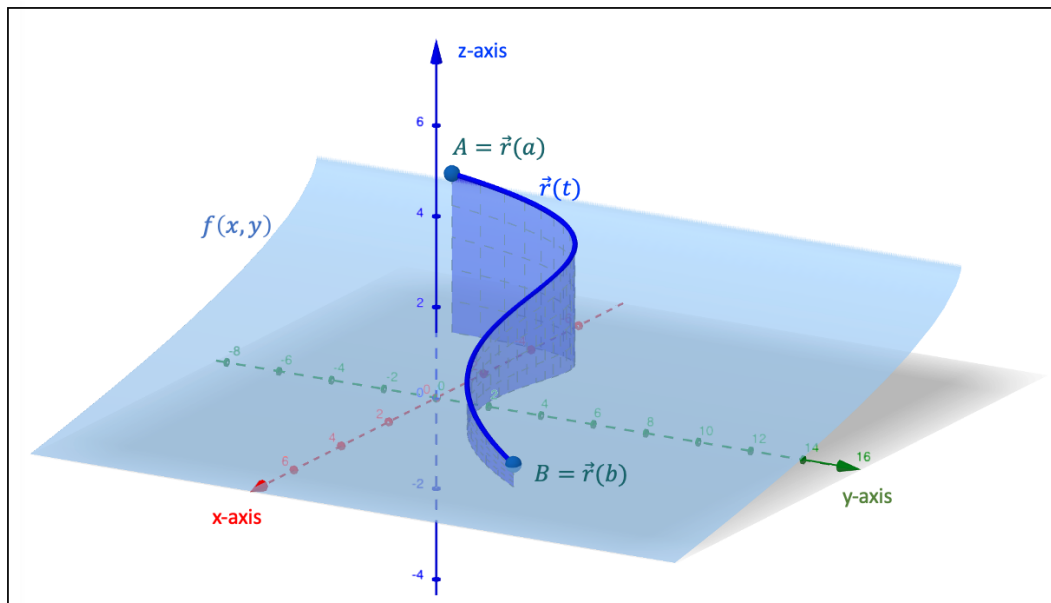


Figure 10.11: The line integral of the scalar function f

If we expand the definition of the Riemann sum (see section 9.2) to three dimensions, we can write $R_f(P) = \sum_{i=1}^n f(\vec{r}(t_i^*)) \|\vec{r}(t_i) - \vec{r}(t_{i-1})\|$ for a given partition P of the interval $[\vec{r}(t_0 = a) = A, \dots, t_{i-1}, t_i, \dots, \vec{r}(t_n = b) = B]$ and whereby $t_i^* \in [t_i, t_{i-1}]$.

The **line integral or path integral of the scalar function f along the curve \widehat{AB}** calculates the surface area of the cross-sectional region restricted by the curve $\vec{r}(t)$ and the surface generated by f . Put another way, it calculates the area right below (above) the curve and bounded by the xy -plane if the curve is positioned above (below) the xy -plane. The line integral of the scalar function f is defined as:

$$\int_{\widehat{AB}} f(\vec{r}(t)) ds = \lim_{n \rightarrow +\infty} R_f(P) = \int_a^b f(\vec{r}(t)) \|d\vec{r}(t)\|$$

In above Fig. 10.11, the graph of the scalar function $f : [-\frac{17}{4}, +\infty[\times \mathbb{R} \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \frac{1}{2}(7 - \sqrt{17 + 4x})$ is represented by the light-blue surface. The dark-blue curve is described by the vector function $\vec{r} : [-\frac{3}{2}, \frac{3}{2}] \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}(t) = (t^2 + 3t - 2, 2t^3 - t + 2, -t + 2)$. The line integral gives the surface area of the blue-shaded region straight under the curve.

Note that if we would set $f(\vec{r}(t)) = 1$, the line integral would calculate the arc length, which we discussed in section 9.2.3.

Let us now consider a curve \widehat{AB} described by the vector function $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}(t)$ and a vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \vec{r}(t) \mapsto \vec{F}(\vec{r}(t))$.

If we define the Riemann sum in a similar fashion as we have done for the scalar function, i.e., $R_F(P) = \sum_{i=1}^n \vec{F}(\vec{r}(t_i^*)) \cdot (\vec{r}(t_i) - \vec{r}(t_{i-1}))$, we can define the **line integral or path integral of the vector field \vec{F} along the curve \widehat{AB}** as:

$$\int_{\widehat{AB}} \vec{F}(\vec{r}(t)) \cdot d\vec{r} = \lim_{n \rightarrow +\infty} R_F(P) = \int_a^b \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \right] dt$$

Note that if \vec{F} would represent a force field, the line integral would calculate the work done by the force \vec{F} on an object that is moving from $\vec{r}(a) = A$ to $\vec{r}(b) = B$.

For example, given the vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z) = (2x, -y^2, -z)$, we would like to calculate the line integral of this field over the curve \widehat{AB} , which is described by the vector function $\vec{r} : [0, 1] \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}(t) = (2t^2 + 1, -t, t^2 - 3t)$. Given that $\vec{r}'(t) = (4t, -1, 2t - 3)$ and $\vec{F}(\vec{r}(t)) = (4t^2 + 2, -t^2, -t^2 + 3t)$, we find the line integral as follows:

$$\begin{aligned} \int_a^b \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \right] dt &= \int_0^1 \left[(4t^2 + 2, -t^2, -t^2 + 3t) \cdot (4t, -1, 2t - 3) \right] dt \\ &= \int_0^1 (14t^3 + 10t^2 - t) dt \\ &= \left[\frac{14t^4}{4} + \frac{10t^3}{3} - \frac{t^2}{2} \right]_0^1 \\ &= \frac{19}{3} \end{aligned}$$

If we replace the vector field $\vec{F}(\vec{r}(t))$ in the above definition for the line integral by the gradient of a scalar function $V : \mathbb{R}^3 \rightarrow \mathbb{R} : \vec{r}(t) \mapsto V(\vec{r}(t))$, we obtain the **fundamental theorem of line integrals**:

$$\int_{\widehat{AB}} \vec{\nabla} V(\vec{r}(t)) \cdot d\vec{r}(t) = V(\vec{r}(B)) - V(\vec{r}(A))$$

This theorem states that the line integral of a gradient does only depend on the initial and final position of a certain interval and not on the path taken between these two points.

Within the domain of physics, if we can write a force $\vec{F}(\vec{r}(t))$ as the gradient of a scalar function $V(\vec{r}(t))$, it means that the work done by this force does not depend on the path taken by the object on which the force acts. The force is then called a *conservative force* and the scalar function is referred to as the *potential*.

10.4.2 The Double Integral

Consider the region $S \subset \mathbb{R}^2$ and the continuous scalar function $f : S \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y)$. The **double integral** is defined as:

$$\iint_S f(x, y) dS$$

whereby dS represents a segment of the surface area of the region S . If we look at Fig. 10.12, the double integral as defined above represents the volume of the shape bounded by the xy -plane, by the surface of $f(x, y)$ and by the surface that is made up of the lines parallel to the z -axis and going through the edge of the region S .

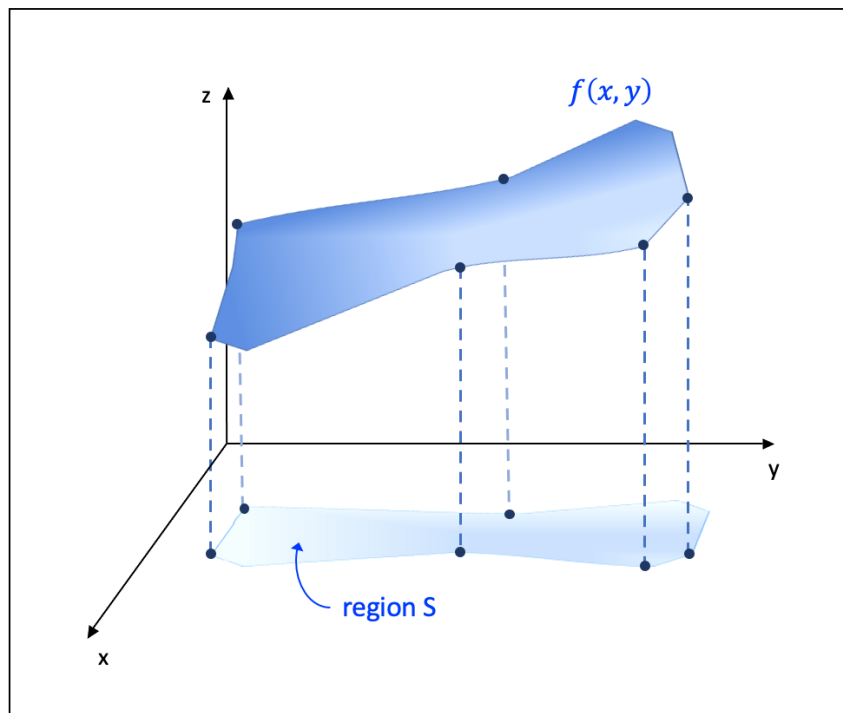


Figure 10.12: Integrating over the two-dimensional region S

To calculate the double integral, we can go about it in two ways, depending on the particular shape of the region S , whereby the boundaries of the integral corresponds to the x - and y -coordinates in Fig. 10.13:

$$\left\{ \begin{array}{l} \text{Option 1 : } \iint_S f(x, y) dS = \int_{x_1}^{x_2} dx \int_{b_1(x)}^{b_2(x)} f(x, y) dy \\ \text{Option 2 : } \iint_S f(x, y) dS = \int_{y_1}^{y_2} dy \int_{d_1(y)}^{d_2(y)} f(x, y) dx \end{array} \right.$$

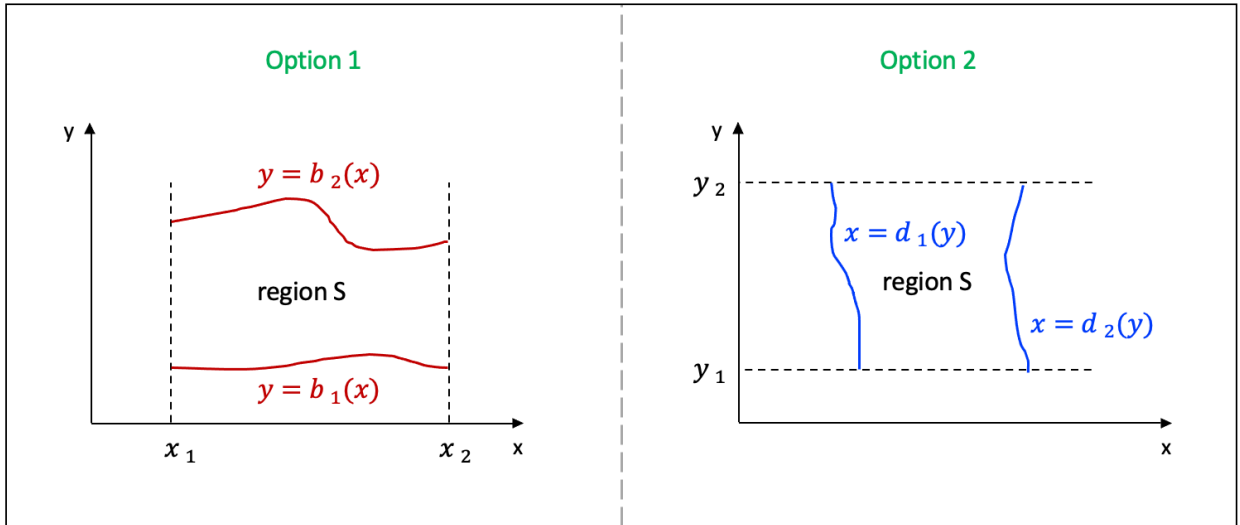


Figure 10.13: Calculating the double integral over the region S

- Example

We would like to calculate the volume of the region bounded by the xy -plane, by the right circular cylinder with radius equal to 2 and whose axis is equal to the z -axis, and by the scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = x^2 + y^2$. In below Fig. 10.14 we can see that this volume corresponds to the volume of the cylinder with height 4 minus the volume of the paraboloid of the same height.

Due to the symmetry of the region S , we can either choose option 1 or 2 to calculate this volume. Let us go for option 1. With respect to the x -dimension, we wish to integrate over the interval $[-2, 2]$. Regarding the y -direction, we can see that the edge of region S is formed by the level curve $x^2 + y^2 = 4$. At the negative side of the y -axis, we find that $y = b_1(x) = -\sqrt{4 - x^2}$, whereas at the positive side, we have $y = b_2(x) = \sqrt{4 - x^2}$. With respect to the y -dimension, we will thus integrate over the interval $[b_1(x), b_2(x)]$.

$$\begin{aligned} V &= \iint_S f(x, y) dS = \int_{x_1}^{x_2} dx \int_{b_1(x)}^{b_2(x)} f(x, y) dy \\ &= \int_{-2}^2 dx \int_{b_1(x)}^{b_2(x)} (x^2 + y^2) dy \end{aligned}$$

$$\begin{aligned}
&= \int_{-2}^2 \left(\left[\left(x^2 y + \frac{y^3}{3} \right) \right]_{b_1(x)=-\sqrt{4-x^2}}^{b_2(x)=\sqrt{4-x^2}} \right) dx \\
&= \frac{4}{3} \int_{-2}^2 x^2 \sqrt{4-x^2} dx + \frac{8}{3} \int_{-2}^2 \sqrt{4-x^2} dx \\
&= \frac{64}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\theta) \cos^2(\theta) d\theta + \frac{32}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) d\theta \\
&= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos(4\theta)) d\theta + \frac{16}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(2\theta)) d\theta \\
&= \frac{8}{3} \left[\theta - \frac{\sin(4\theta)}{4} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{16}{3} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 8\pi
\end{aligned}$$

whereby we implemented the substitution $x = 2 \sin \theta$ in line 5. We can check this result as follows: given that the volume of the cylinder is equal to $V = \pi r^2 h = 16\pi$ and that the volume of the paraboloid is equal to $V = \frac{\pi h r^2}{2} = 8\pi$, we find that the volume by the double integral is equal to $16\pi - 8\pi = 8\pi$.

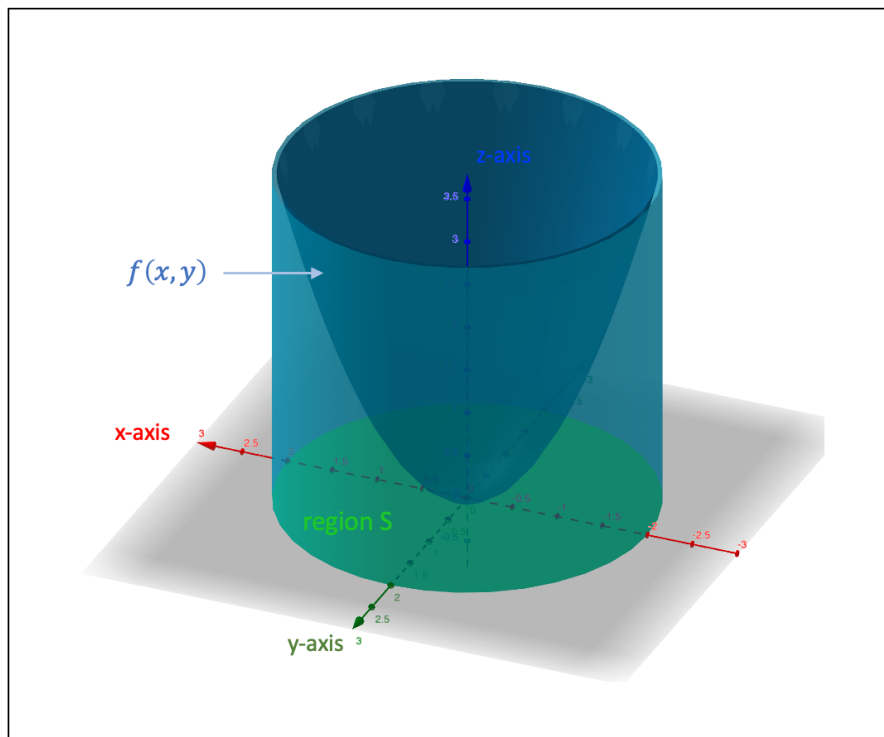


Figure 10.14: Calculating the double integral over the region S

We can link line integrals to double integrals through **Green's theorem a.k.a. the Green-Riemann formula**, which states that the line integral of the vector field $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x, y) \mapsto \vec{F}(x, y) = (M(x, y), N(x, y))$ that is enclosed by the parametrized curve C is equal to the double integral over the region S , which is enclosed by the curve C :

$$\oint_{C^+} M(x, y) dx + N(x, y) dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dS$$

whereby \oint_{C^+} stands for the line integral taken along the closed curve C^+ and the integration occurs in the counterclockwise direction, which is by convention the positive direction.

- Example

We would like to evaluate the integral $\iint_S (x + y) dS$ whereby S is the region bounded by the curves $6y = (x - 1)^2 - 4$ and $4y^2 = x^2 - 9$, as shown in Fig. 10.15. One way is to calculate the double integral, but we will use the line integral via Green's theorem.

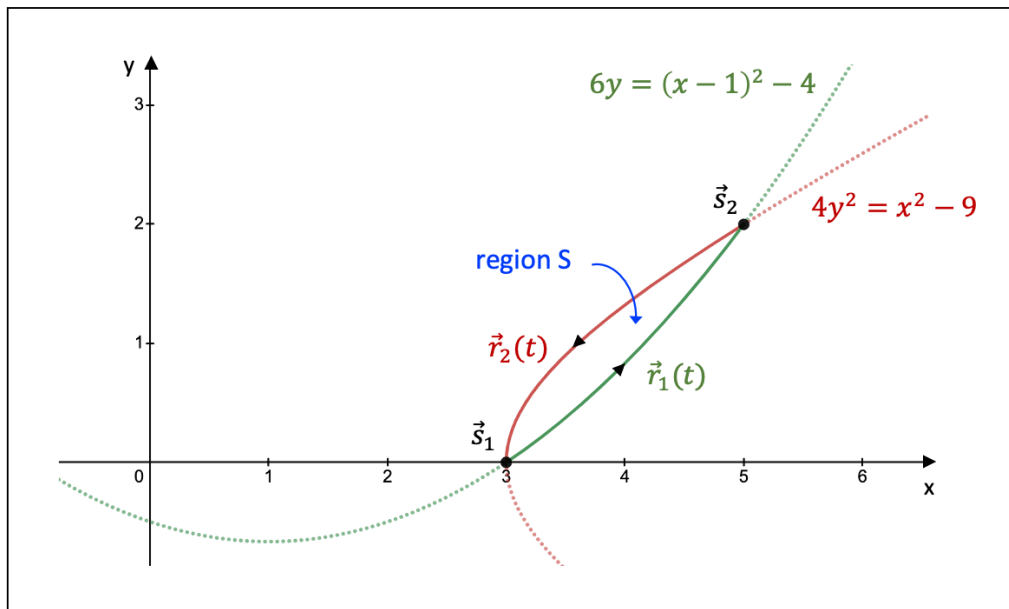


Figure 10.15: An example of Green's theorem

Let us in a first step determine the points of intersection \vec{s}_1 and \vec{s}_2 , which we need to find the boundaries of our line integral. If we rewrite the first curve $6y = (x - 1)^2 - 4$ as $2y = \frac{(x-1)^2}{3} - \frac{4}{3}$ and plug it into the second equation, we find:

$$\begin{aligned} \left[\frac{(x-1)^2}{3} - \frac{4}{3} \right]^2 &= x^2 - 9 \Leftrightarrow (x-1)^4 - 8(x-1)^2 + 16 = 9x^2 - 81 \\ &\Leftrightarrow x^4 - 4x^3 - 11x^2 + 12x + 90 = 0 \\ &\Leftrightarrow (x-3)(x-5)(x^2 + 4x + 6) = 0 \end{aligned}$$

We have the points of intersection $\vec{s}_1 = (3, 0)$ and $\vec{s}_2 = (5, 2)$. In a next step, we determine the functions $M, N : S \subset \mathbb{R}^2 \rightarrow \mathbb{R}$:

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dS = \iint_S (x + y) dS \Leftrightarrow \begin{cases} \frac{\partial N}{\partial x} = y \\ \frac{\partial M}{\partial y} = -x \end{cases} \Leftrightarrow \begin{cases} N(x, y) = xy \\ M(x, y) = -xy \end{cases}$$

Note that we also could have chosen the functions $N(x, y) = \frac{x^2}{2}$ and $M(x, y) = -\frac{y^2}{2}$, but this does not affect the outcome. The line integral obtains the following form:

$$\oint_{C^+} M(x, y) dx + N(x, y) dy = \oint_{C^+} -xy dx + xy dy$$

We will have to split up the line integral in two parts, whereby the two segments are described by \vec{r}_1 and \vec{r}_2 , respectively. We now evaluate the line integral of the first segment of the curve C^+ , which is represented by $\vec{r}_1 : [3, 5] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}_1(t) = (t, \frac{(t-1)^2}{6} - \frac{2}{3})$. We have the following values for the vectors $d\vec{r}_1$ and $\vec{F}_1(\vec{r}_1(t))$:

$$\begin{cases} d\vec{r}_1 = (1, \frac{1}{3}(t-1)) dt \\ \vec{F}_1(x, y) = (M(x, y), N(x, y)) = (-xy, xy) \Leftrightarrow \vec{F}_1(\vec{r}_1(t)) = (-\frac{t}{6}(t-1)^2 + \frac{2t}{3}, \frac{t}{6}(t-1)^2 - \frac{2t}{3}) \end{cases}$$

The line integral related to the first segment is calculated as follows:

$$\begin{aligned} \int_3^5 \vec{F}_1(\vec{r}_1(t)) \cdot d\vec{r}_1 &= \int_3^5 \left(-\frac{t}{6}(t-1)^2 + \frac{2t}{3}, \frac{t}{6}(t-1)^2 - \frac{2t}{3} \right) \cdot \left(1, \frac{1}{3}(t-1) \right) dt \\ &= \int_3^5 \left[-\frac{t}{6}(t-1)^2 + \frac{2t}{3} + \frac{t}{18}(t-1)^3 - \frac{2t}{9}(t-1) \right] dt \\ &= \int_3^5 \left[\frac{t^4}{18} - \frac{t^3}{3} + \frac{5t^2}{18} + \frac{2t}{3} \right] dt \\ &= \left[\frac{t^5}{90} - \frac{t^4}{12} + \frac{5t^3}{54} + \frac{t^2}{3} \right]_3^5 \\ &= \frac{148}{135} \end{aligned}$$

Next, we evaluate the line integral of the second segment of the curve C^+ , which is described by $\vec{r}_2 : [3, 5] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}_2(t) = (t, \frac{1}{2}\sqrt{t^2-9})$. For $d\vec{r}_2$ and $\vec{F}_2(\vec{r}_2(t))$ we have:

$$\begin{cases} d\vec{r}_2 = \left(1, \frac{t}{2\sqrt{t^2-9}} \right) dt \\ \vec{F}_2(\vec{r}_2(t)) = \left(-\frac{t}{2}\sqrt{t^2-9}, \frac{t}{2}\sqrt{t^2-9} \right) \end{cases}$$

Remember that the orientation of C^+ is counterclockwise so the parameter t of \vec{r}_2 runs from 5 to 3. The line integral of the second segment then becomes:

$$\begin{aligned}
\int_5^3 \vec{F}_2(\vec{r}_2(t)) \cdot d\vec{r}_2 &= \int_5^3 \left(-\frac{t}{2}\sqrt{t^2-9}, \frac{t}{2}\sqrt{t^2-9}\right) \cdot \left(1, \frac{t}{2\sqrt{t^2-9}}\right) dt \\
&= \int_5^3 \left[-\frac{t}{2}\sqrt{t^2-9} + \frac{t^2}{4}\right] dt \\
&= -\frac{27}{2} \int_{\theta_1}^{\theta_2} \sec^2(\theta) \tan^2(\theta) d\theta + \int_5^3 \frac{t^2}{4} dt \\
&= -\frac{27}{2} \int_{s_1}^{s_2} s^2 ds + \int_5^3 \frac{t^2}{4} dt \\
&= -\frac{27}{2} \left[\frac{s^3}{3}\right]_{s_1=\frac{4}{3}}^{s_2=0} + \left[\frac{t^3}{12}\right]_5^3 \\
&= \frac{5}{2}
\end{aligned}$$

whereby we applied the substitution $t = 3 \sec \theta$ in line 3 and $s = \tan \theta$ in line 4. Remember that $\tan(\arccos(t)) = \sqrt{\frac{1}{\cos^2(\arccos(t))} - 1} = \sqrt{\frac{1}{t^2} - 1}$.

The full line integral is then equal to:

$$\oint_{C^+} M(x, y) dx + N(x, y) dy = \int_3^5 \vec{F}_1(\vec{r}_1(t)) \cdot d\vec{r}_1 + \int_5^3 \vec{F}_2(\vec{r}_2(t)) \cdot d\vec{r}_2 = \frac{148}{135} + \frac{5}{2} = \frac{971}{270}$$

Green's theorem can also be used to calculate the **surface area** of a certain region S . If we consider the respective vector fields $\vec{F}(x, y) = (-y, x)$, $\vec{F}(x, y) = (0, x)$, and $\vec{F}(x, y) = (-y, 0)$ in Green's theorem, we obtain the following three equivalent expressions for the surface area of S :

$$\iint_S dS = \frac{1}{2} \oint_{C^+} xdy - ydx = \oint_{C^+} xdy = - \oint_{C^+} ydx$$

- Example

Consider the scalar function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = 3x - x^3$. We wish to calculate the surface area of the region S that is bounded by $f(x)$ within the first quadrant (see below Fig. 10.16). Given that f intersects the x -axis at $x = 0$ and $x = \sqrt{3}$, that $\vec{r}_1 : [0, \sqrt{3}] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}_1(t) = (t, 0)$, and that $d\vec{r}_1 = (1, 0) dt$, we can see that the line integral of $\vec{r}_1(t)$ is equal to zero:

$$- \int_0^{\sqrt{3}} y dx = - \int_0^{\sqrt{3}} 0 \cdot dt = 0$$

The surface area of the region S is thus found by evaluating the line integral of $\vec{r}_2 : [0, \sqrt{3}] \rightarrow \mathbb{R}^2 : t \mapsto \vec{r}_2(t) = (t, 3t - t^3)$, with $d\vec{r}_2 = (1, 3 - 3t^2) dt$:

$$\iint_S dS = - \oint_{C^+} y dx = - \int_{\sqrt{3}}^0 (3t - t^3) \cdot 1 dt = - \left[\frac{3t^2}{2} - \frac{t^4}{4}\right]_{\sqrt{3}}^0 = \frac{9}{4}$$

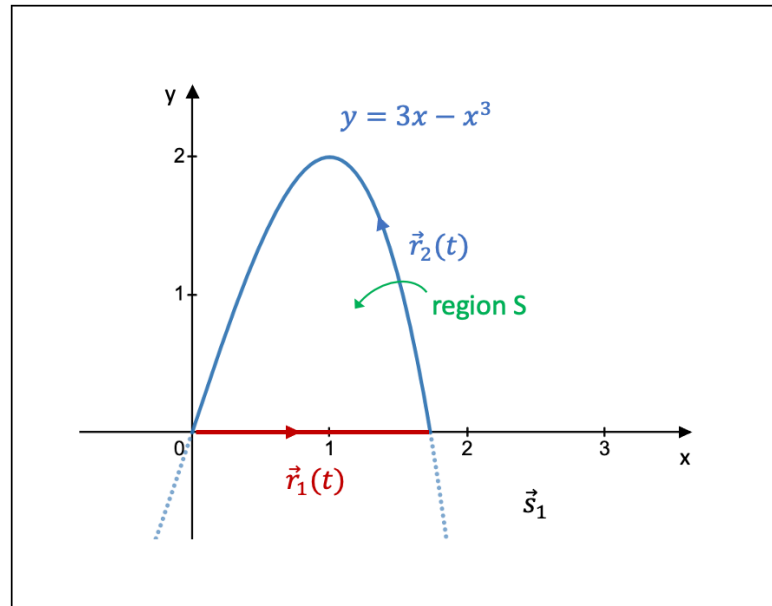


Figure 10.16: An example of Green's theorem

If we would like to introduce a **change of coordinates** whereby we exchange the variables x and y for the variables u and v , with $x = \phi_1(u, v)$ and $y = \phi_2(u, v)$, the double integral is transformed in the following way:

$$\iint_S f(x, y) dS = \iint_A f(\phi_1(u, v), \phi_2(u, v)) \left| \frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} \right| dA$$

Remember that $\left| \frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} \right|$ represents the Jacobian determinant (see section 10.2.1).

- Example

Consider the scalar function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = \frac{5+4x}{y+\sqrt{x^2+y^2}}$. We want to calculate the volume bounded by the graph of f when integrating over the region S , which is the circle sector with $\rho = 3$ and restricted by $0 \leq \theta \leq \frac{\pi}{2}$. The volume is shown in below Fig. 10.17.

Before writing down the double integral, let us first calculate the Jacobian determinant. Given the circular symmetry of the problem at hand, we choose polar coordinates to solve the integral, i.e., $x = \rho \cos \theta$ and $y = \rho \sin \theta$. The Jacobian determinant is then equal to:

$$\det(\mathcal{L}) = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -\rho \sin \theta \\ \sin \theta & \rho \cos \theta \end{vmatrix} = \rho \cos^2 \theta + \rho \sin^2 \theta = \rho$$

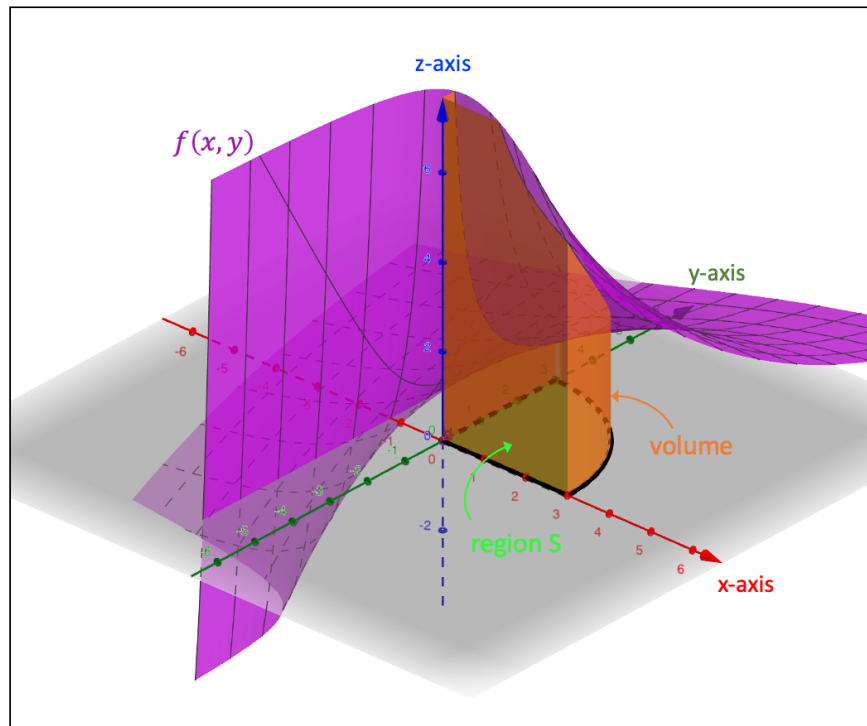


Figure 10.17: Calculating the volume through a change of coordinates

The volume is then calculated as follows:

$$\begin{aligned}
 \iint_S f(x, y) dS &= \iint_A f(\phi_1(u, v), \phi_2(u, v)) \left| \frac{\partial(\phi_1, \phi_2)}{\partial(u, v)} \right| dA \\
 &= \int_0^{\frac{\pi}{2}} \int_0^3 \frac{5 + 4(\rho \cos \theta)}{\rho \sin \theta + \sqrt{(\rho \cos \theta)^2 + (\rho \sin \theta)^2}} \cdot \rho d\rho d\theta \\
 &= \int_0^{\frac{\pi}{2}} \int_0^3 \left[\frac{5}{\sin \theta + 1} + \frac{4\rho \cos \theta}{\sin \theta + 1} \right] d\rho d\theta \\
 &= \int_0^{\frac{\pi}{2}} \left[\frac{5}{\sin \theta + 1} \cdot [\rho]_0^3 + \frac{4 \cos \theta}{\sin \theta + 1} \cdot \left[\frac{\rho^2}{2} \right]_0^3 \right] d\theta \\
 &= 15 \int_0^{\frac{\pi}{2}} \frac{1}{\sin \theta + 1} d\theta + 18 \int_0^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta + 1} d\theta \\
 &= 30 \int_0^1 \frac{1}{(1+t)^2} dt + 36 \int_0^1 \left[\frac{-t}{1+t^2} + \frac{1}{1+t} \right] dt \\
 &= 30 \left[-\frac{1}{1+t} \right]_0^1 + 36 \left[-\frac{1}{2} \ln |1+t^2| + \ln |1+t| \right]_0^1 \\
 &= 15 + 18 \ln 2
 \end{aligned}$$

whereby we introduced the substitution $t = \tan \frac{\theta}{2}$ in line 6 and subsequently performed partial fraction decomposition on the second integral of that same line.

10.4.3 The Surface Integral

While the double integral $\iint_S dS$ calculates the area of a two-dimensional surface S in two-dimensional space, the surface integral $\iint_{S^*} dS$ calculates the area of a two-dimensional surface S^* in three-dimensional space.

If the surface S^* is described by the vector function $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (u, v) \mapsto \vec{r}(u, v)$, the **surface integral** can be defined as:

$$\iint_{S^*} dS = \iint_G \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dG$$

whereby G is the projection of S^* onto the uv -plane. This integral calculates the surface area of the surface S^* .

- Density function f

If we define a scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z)$ that represents the density of a massive surface S^* , the surface integral of f over S^* calculates the mass of S^* and is given by:

$$\iint_{S^*} f(x, y, z) dS = \iint_G f(\vec{r}(u, v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dG$$

- Change of coordinates in 2D

Note that if the surface S^* would lie within the xy -plane, i.e., $z = 0$ and thus $S^* = S$, the above formula would transform into the expression of the double integral in the case of changing coordinate systems (see previous section 10.4.2). Since $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (u, v) \mapsto \vec{r}(u, v) = (r_1(u, v), r_2(u, v), 0)$, we find that:

$$\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| = \left\| \begin{pmatrix} \frac{\partial r_1}{\partial u} \\ \frac{\partial r_2}{\partial u} \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial r_1}{\partial v} \\ \frac{\partial r_2}{\partial v} \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ 0 \\ \frac{\partial(r_1, r_2)}{\partial(u, v)} \end{pmatrix} \right\| = \left| \frac{\partial(r_1, r_2)}{\partial(u, v)} \right|$$

In other words:

$$\iint_S f(x, y, 0) dS = \iint_G f(\vec{r}(u, v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dG = \iint_G f(\vec{r}(u, v)) \left| \frac{\partial(r_1, r_2)}{\partial(u, v)} \right| dG$$

- Parametrization in terms of x and y

If we use the parameters x and y , the vector function \vec{r} has the form $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (x, y) \mapsto \vec{r}(x, y) = (x, y, z(x, y))$, so that we get:

$$\left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| = \left\| \begin{pmatrix} 1 \\ 0 \\ \frac{\partial z}{\partial x} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial z}{\partial y} \end{pmatrix} \right\| = \left\| \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix} \right\| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

The surface integral then obtains the following form:

$$\begin{aligned} \iint_{S^*} f(x, y, z) dS &= \iint_G f(\vec{r}(x, y)) \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| dG \\ &= \iint_G f(\vec{r}(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dG \end{aligned}$$

whereby G is the projection of S^* onto the xy -plane.

Let us for instance consider the surface S^* , which is described by the vector function $\vec{r} : [-2, 4] \times [-1, 3] \rightarrow \mathbb{R}^3 : (x, y) \mapsto \vec{r}(x, y) = (x, y, x^2 - 2(x + y))$ and shown in below Fig. 10.18.

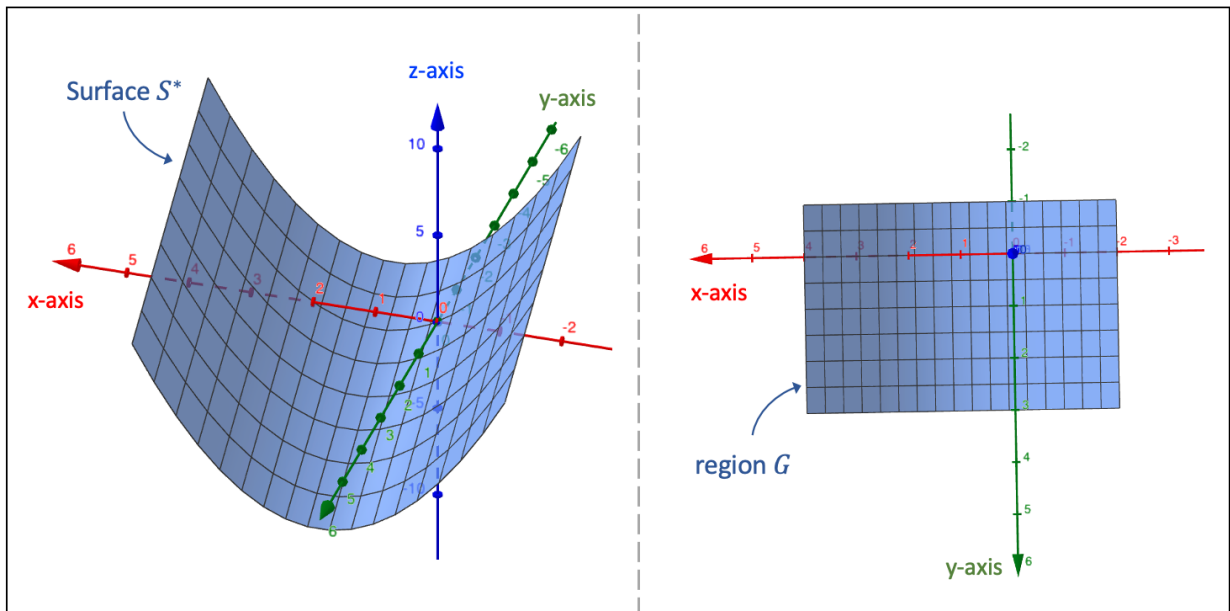


Figure 10.18: The surface S^*

Given that $\frac{\partial z}{\partial x} = 2(x - 1)$ and $\frac{\partial z}{\partial y} = -2$, we calculate the surface area of S^* as follows:

$$\begin{aligned}
\iint_{S^*} dS &= \iint_G \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dG \\
&= \int_{-1}^3 dy \int_{-2}^4 \sqrt{1 + [2(x-1)]^2 + [-2]^2} dx \\
&= [y]_{-1}^3 \int_{-2}^4 \sqrt{5} \sqrt{\frac{4}{5}(x-1)^2 + 1} dx \\
&= 10 \int_{s_1}^{s_2} \sqrt{s^2 + 1} ds \\
&= 10 \int_{\theta_1}^{\theta_2} \sec^3(\theta) d\theta \\
&= 10 \left[\frac{1}{2} \tan(\theta) \sec \theta \right]_{\theta_1}^{\theta_2} + 5 \int_{\theta_1}^{\theta_2} \sec(\theta) d\theta \\
&= 10 \left[\frac{1}{2} \tan(\theta) \sec \theta \right]_{\theta_1}^{\theta_2} + \frac{5}{2} \int_{u_1}^{u_2} \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\
&= 10 \left[\frac{1}{2} \tan(\theta) \sec \theta \right]_{\theta_1}^{\theta_2} + \frac{5}{2} \left[\ln \left| \frac{1+u}{1-u} \right| \right]_{u_1}^{u_2} \\
&= 5 \left[\frac{6}{\sqrt{5}} \cdot \frac{\sqrt{41}}{\sqrt{5}} - \left(-\frac{6}{\sqrt{5}} \right) \cdot \frac{\sqrt{41}}{\sqrt{5}} \right] + \frac{5}{2} \left[\ln \left| \frac{1 + \frac{6}{\sqrt{41}}}{1 - \frac{6}{\sqrt{41}}} \right| - \ln \left| \frac{1 - \frac{6}{\sqrt{41}}}{1 + \frac{6}{\sqrt{41}}} \right| \right] \\
&= 12\sqrt{41} + 5 \ln \left| \frac{77 + 12\sqrt{41}}{5} \right| \\
&\approx 93.97
\end{aligned}$$

whereby we implemented the substitution $s = \frac{2}{\sqrt{5}}(x-1)$ in line 4, the substitution $s = \tan \theta$ in line 5, and the substitution $u = \sin \theta$ in line 7. Going from line 5 to 6, we applied the reduction formula as discussed in Example 2 of the subsection *F. Integration of Irrational Functions* of section 9.1.2. We also applied partial fraction decomposition to the integral of line 7. Note furthermore that $\sin \alpha = \frac{\tan \alpha}{\sqrt{1+\tan^2 \alpha}}$, so that $\sin(\arctan(s)) = \frac{s}{\sqrt{1+s^2}}$.

- Spherical coordinates

If the surface S^* is spherically symmetrical, we can use the spherical coordinates to write up the vector function \vec{r} that describes S^* .

If we consider the sphere $x^2 + y^2 + z^2 = R^2$ and the vector $\vec{a} = (x_a, y_a, z_a)$, which represents a point on the sphere, we define the angle θ as the angle between the z -axis and the vector \vec{a} and the angle ϕ as the angle between the orthogonal projection of \vec{a} onto the xy -plane and the x -axis. These angles are shown in below Fig. 10.19.

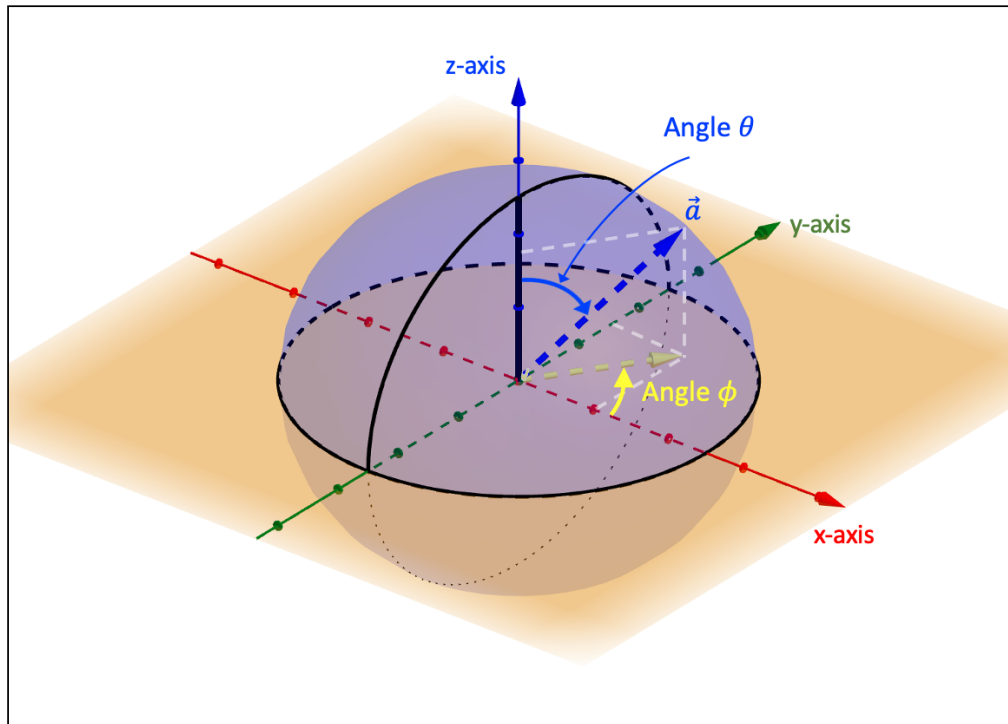


Figure 10.19: The spherical coordinate system

The spherical coordinates then take on the following form, whereby $\rho : 0 \rightarrow R$, $\theta : 0 \rightarrow \pi$, and $\phi : 0 \rightarrow 2\pi$:

$$\begin{cases} x = \rho \sin(\theta) \cos \phi \\ y = \rho \sin(\theta) \sin \phi \\ z = \rho \cos \theta \end{cases}$$

As an example, we wish to calculate the surface area of the surface S^* , which corresponds to the part of the sphere, with equation $x^2 + y^2 + z^2 = 25$, that lies in the second octant above the xy -plane and between the planes $x = 0$ and $\alpha : y = -\frac{2}{\sqrt{21}}x$, under the condition that the angle between the two planes is less than 90° . Fig. 10.20 depicts the surface S^* .

Given the spherical symmetry, we can describe S^* through the spherical coordinates whereby ρ remains constant, i.e., $\rho = 5$. The vector function of S^* is equal to $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (\theta, \phi) \mapsto \vec{r}(\theta, \phi) = (5 \sin(\theta) \cos \phi, 5 \sin(\theta) \sin \phi, 5 \cos \theta)$.

Regarding the domain G , we know that $\theta : 0 \rightarrow \frac{\pi}{2}$. Since the surface S^* is located in the second octant, the initial value of the angle ϕ is equal to $\phi_1 = \frac{\pi}{2}$. The final value ϕ_2 is determined by the plane $\alpha : y = -\frac{2}{\sqrt{21}}x$. Because the general form of a plane perpendicular to the xy -plane and going through the origin is equal to $y = \tan(\phi)x$, we have that $\tan \phi_2 = -\frac{2}{\sqrt{21}} \Leftrightarrow \phi_2 = \arctan(-\frac{2}{\sqrt{21}})$. As S^* lies within the second octant, we correct the value of ϕ_2 by an angle of π so that we have $\phi_2 = \arctan(-\frac{2}{\sqrt{21}}) + \pi$.

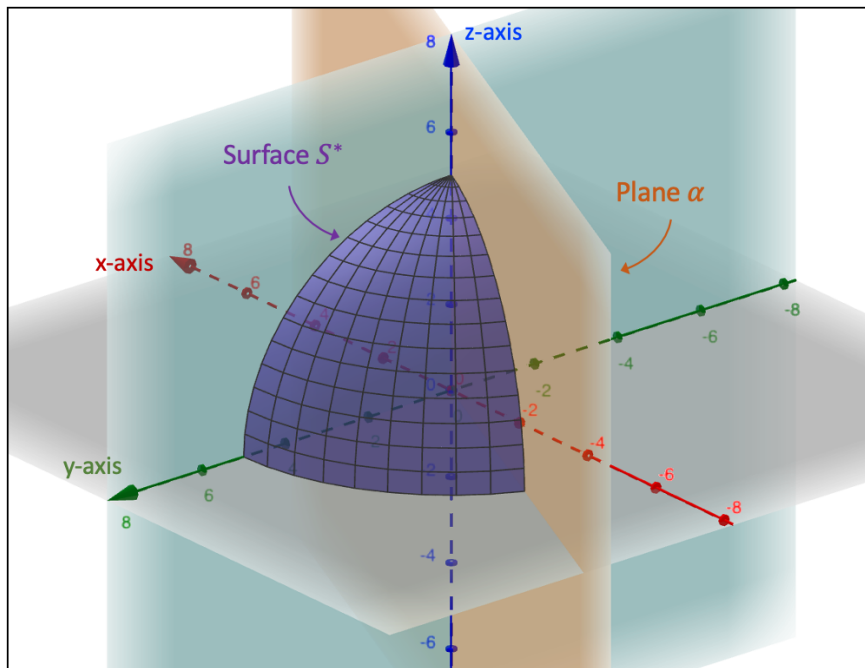


Figure 10.20: The surface of a part of a sphere

In a next step, we find the value of the norm of the curl between the partial derivatives of the vector function $\vec{r}(\theta, \phi)$:

$$\begin{aligned} \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| &= \left\| \begin{pmatrix} 5 \cos(\theta) \cos \phi \\ 5 \cos(\theta) \sin \phi \\ -5 \sin \theta \end{pmatrix} \times \begin{pmatrix} -5 \sin(\theta) \sin \phi \\ 5 \sin(\theta) \cos \phi \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 25 \sin^2(\theta) \cos \phi \\ 25 \sin^2(\theta) \sin \phi \\ 25 \sin(\theta) \cos \theta \end{pmatrix} \right\| \\ &= 25 \sin \theta \end{aligned}$$

The surface area of S^* is then calculated as follows:

$$\begin{aligned} \iint_{S^*} dS &= \iint_G \left\| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right\| dG = \int_{\frac{\pi}{2}}^{\phi_2} d\phi \int_0^{\frac{\pi}{2}} 25 \sin(\theta) d\theta = 25 [\theta]_{\frac{\pi}{2}}^{\phi_2} \cdot [-\cos \theta]_0^{\frac{\pi}{2}} \\ &= 25 \left[\arctan\left(-\frac{2}{\sqrt{21}}\right) + \frac{\pi}{2} \right] \end{aligned}$$

- Surface area of a solid of revolution

In section 9.2.3 we discussed how to calculate the volume of a solid of revolution. Now we wish to understand how to calculate the surface area S^* of such a solid.

Consider, for instance, the curve C in the yz -plane, which is described by the vector function $\vec{r}_C : [a, b] \rightarrow \mathbb{R}^3 : z \mapsto \vec{r}_C(z) = (0, f(z), z)$ whereby f is one-variable scalar function. Now, we let the curve C rotate one complete revolution of 360° about the z -axis to obtain a solid of revolution.

If we consider a point \vec{p} on the surface S^* of our solid of revolution, we use two parameters to construct the vector function \vec{r} to describe S^* : the parameter z and the parameter θ , which is the angle between the y -axis and the orthogonal projection of the vector \vec{p} onto the xy -plane. The vector function \vec{r} then becomes $\vec{r} : [a, b] \times [0, 2\pi[\rightarrow \mathbb{R}^3 : (z, \theta) \mapsto \vec{r}(z, \theta) = (f(z) \sin \theta, f(z) \cos \theta, z)$.

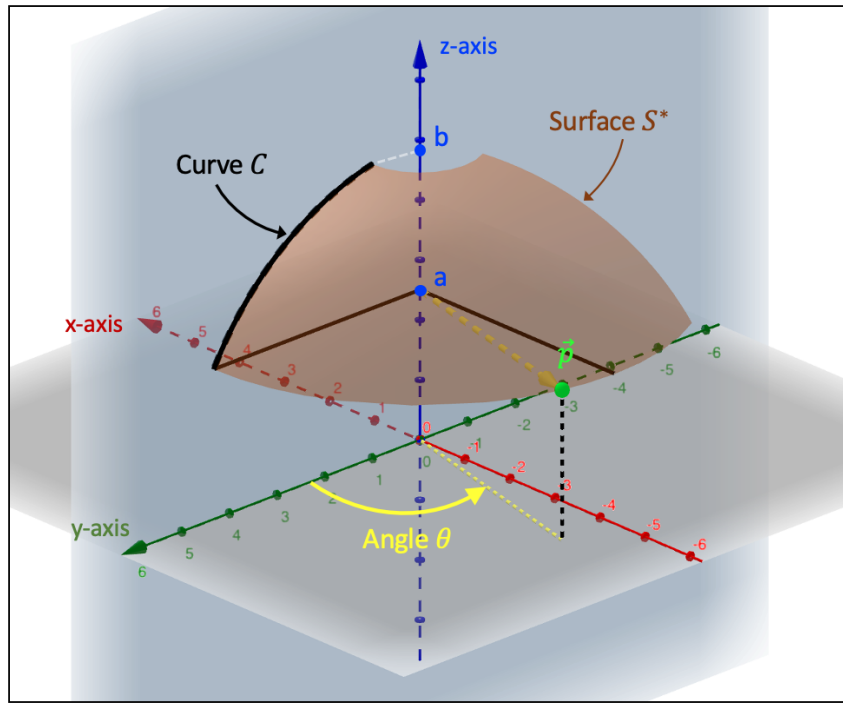


Figure 10.21: The surface of a solid of revolution

The norm of the curl between the partial derivatives of the vector function $\vec{r}(z, \theta)$ is equal to:

$$\begin{aligned} \left\| \frac{\partial \vec{r}}{\partial z} \times \frac{\partial \vec{r}}{\partial \theta} \right\| &= \left\| \begin{pmatrix} f'(z) \sin \theta \\ f'(z) \cos \theta \\ 1 \end{pmatrix} \times \begin{pmatrix} f(z) \cos \theta \\ -f(z) \sin \theta \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} f(z) \sin \theta \\ f(z) \cos \theta \\ -f(z)f'(z) \end{pmatrix} \right\| \\ &= f(z) \sqrt{1 + [f'(z)]^2} \end{aligned}$$

The surface area of S^* is then found as follows:

$$\begin{aligned} \iint_{S^*} dS &= \iint_G \left\| \frac{\partial \vec{r}}{\partial z} \times \frac{\partial \vec{r}}{\partial \theta} \right\| dG = \int_0^{2\pi} d\theta \int_a^b f(z) \sqrt{1 + [f'(z)]^2} dz \\ &= 2\pi \int_a^b f(z) \sqrt{1 + [f'(z)]^2} dz \end{aligned}$$

If instead of the variable z , we choose the variable y to describe the curve C and rotate the curve around the y -axis, the above expression changes slightly to:

$$\iint_{S^*} dS = 2\pi \int_a^b g(y) \sqrt{1 + [g'(y)]^2} dy$$

whereby $g(y)$ is a one-variable scalar function that is used to define the vector function $\vec{r}_C : [a, b] \rightarrow \mathbb{R}^3 : y \mapsto \vec{r}_C(y) = (0, y, g(y))$; note that the domain $[a, b]$ now refers to an interval in the y -direction.

As an example, we take the curve C , which is described by the vector function $\vec{r}_C : [\frac{1}{2}, \frac{7}{2}] \rightarrow \mathbb{R}^3 : y \mapsto \vec{r}_C(y) = (0, y, (y-2)^2 - y + 5)$, whereby $g(y) = (y-2)^2 - y + 5$, and rotate it about the z -axis for one full revolution. The surface S^* is then described by the vector function $\vec{r} : [\frac{1}{2}, \frac{7}{2}] \times [0, 2\pi[\rightarrow \mathbb{R}^3 : (y, \theta) \mapsto \vec{r}(y, \theta) = (y \sin \theta, y \cos \theta, (y-2)^2 - y + 5)$.

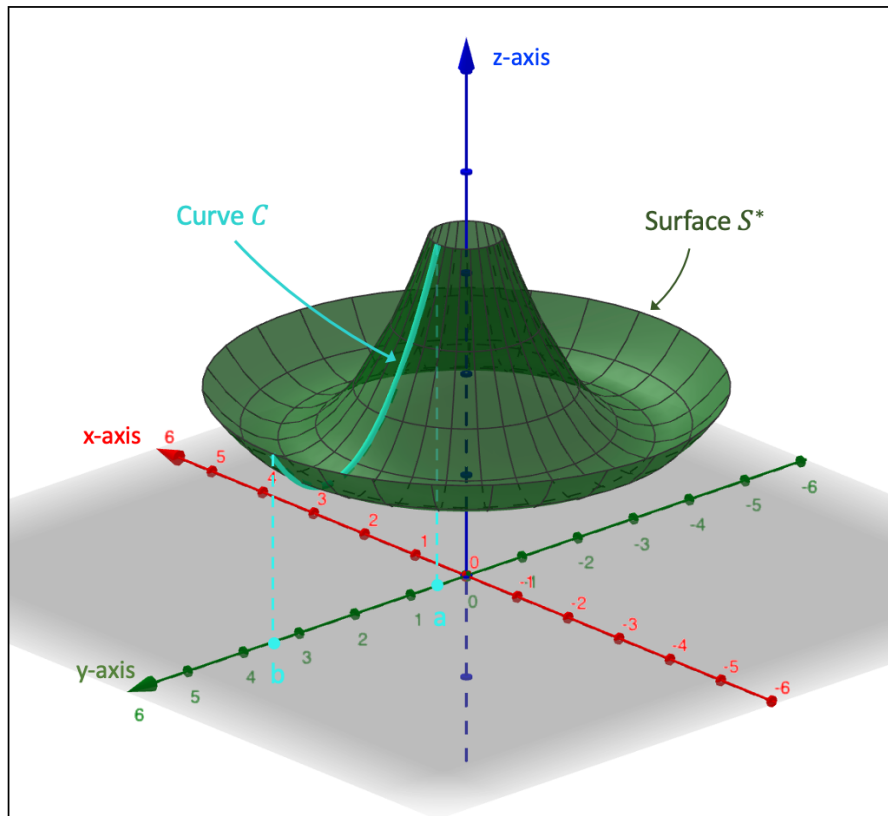


Figure 10.22: The surface of a solid of revolution: an example

Given that $g'(y) = 2y - 5$, the surface area of S^* is then equal to:

$$\begin{aligned}
 \iint_{S^*} dS &= 2\pi \int_a^b y \sqrt{1 + [g'(y)]^2} dy \\
 &= 2\pi \int_{\frac{1}{2}}^{\frac{7}{2}} y \sqrt{1 + (2y - 5)^2} dy \\
 &= 2\pi \int_{\frac{1}{2}}^{\frac{7}{2}} y \sqrt{4 \left(y - \frac{5}{2}\right)^2 + 1} dy \\
 &= \frac{\pi}{2} \int_{-4}^2 (s + 5) \sqrt{s^2 + 1} ds
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi}{2} \int_{-4}^2 s\sqrt{s^2+1} ds + \frac{5\pi}{2} \int_{-4}^2 \sqrt{s^2+1} ds \\
&= \frac{\pi}{6} \left[(s^2+1)^{\frac{3}{2}} \right]_{-4}^2 + \frac{5\pi}{2} \int_{\theta_1}^{\theta_2} \sec^3(\theta) d\theta \\
&= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 17^{\frac{3}{2}} \right) + \frac{5\pi}{4} \left[[\tan(\theta) \sec \theta]_{\theta_1}^{\theta_2} + \int_{\theta_1}^{\theta_2} \sec(\theta) d\theta \right] \\
&= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 17^{\frac{3}{2}} \right) + \frac{5\pi}{4} \left[[\tan(\theta) \sec \theta]_{\theta_1}^{\theta_2} + \int_{u_1}^{u_2} \frac{1}{1-u^2} du \right] \\
&= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 17^{\frac{3}{2}} \right) + \frac{5\pi}{4} \left[(2\sqrt{5} + 4\sqrt{17}) + \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right|_{u_1}^{u_2} \right] \\
&= \frac{\pi}{6} \left(5^{\frac{3}{2}} - 17^{\frac{3}{2}} \right) + \frac{5\pi}{4} \left[(2\sqrt{5} + 4\sqrt{17}) + \ln \left[(2 + \sqrt{5})(4 + \sqrt{17}) \right] \right] \\
&\approx 65.38
\end{aligned}$$

whereby we introduced the substitution $s = 2(y - \frac{5}{2})$ in line 4, the substitution $s = \tan \theta$ in line 6, and the substitution $u = \sin \theta$ in line 8. The respective boundaries of the variables θ and u are $\theta : \arctan(-4) \rightarrow \arctan(2)$ and $u : -\frac{4}{\sqrt{17}} \rightarrow \frac{2}{\sqrt{5}}$. Remember hereby that $\sin \alpha = \frac{\tan \alpha}{\sqrt{1+\tan^2 \alpha}}$, so that $\sin(\arctan(s)) = \frac{s}{\sqrt{1+s^2}}$. Note also that going from line 6 to 7, we applied the reduction formula as per Example 2 of the subsection *F. Integration of Irrational Functions* of section 9.1.2.

- Flux through the surface S^*

So far, we have considered the surface integral of a scalar function $f(x, y, z)$ across a surface S^* . In all of the above numerical examples, note that we have taken $f(x, y, z) = 1$ in order to calculate the surface area of S^* .

Apart from evaluating the surface integral of a scalar function f , we can equally integrate a vector field \vec{F} across the surface S^* . In that regard, the surface integral is an extension of the line integral (see section 10.4.1) to the situation of two variables: whereas the line integral of a vector field integrates a vector field along a curve, the surface integral evaluates a vector function across a surface.

What we are measuring when evaluating a vector field \vec{F} across the surface S^* , is the flow or *flux* of \vec{F} through S^* and is denoted by the symbol Φ . Given a vector function $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (u, v) \mapsto \vec{r}(u, v)$ that describes the surface S^* and a vector field $\vec{F} : H \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z)$, whereby $S^* \subset H$, the surface integral of the vector field \vec{F} across S^* is defined as follows:

$$\Phi = \iint_{S^*} \vec{F}(x, y, z) \cdot d\vec{S} = \iint_{S^*} \vec{F}(x, y, z) \cdot \vec{n} dS$$

whereby \vec{n} represents the unit normal vector to the surface S^* at a certain point. Let us have a closer look at the unit normal vector. Since the two vectors $\frac{\partial \vec{r}}{\partial u}(u, v)$ and $\frac{\partial \vec{r}}{\partial v}(u, v)$ at a certain point of the surface S^* lie within the tangent plane at that point (see Fig. 10.3 of section 10.2.1), we know that the curl of these two vectors is oriented perpendicular to the tangent plane, i.e., the curl represents a normal vector.

If we then turn the curl into a unit vector, i.e., we give it a length equal to 1 by dividing the vector through its norm, we can write the unit normal vector \vec{n} of the surface S^* as follows:

$$\vec{n} = \epsilon \cdot \frac{\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v)}{\left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\|}$$

whereby $\epsilon = \pm 1$. When one of the two values of ϵ is assigned to \vec{n} , we say that the surface S^* is *oriented*. The successful orientation of a surface implies that all the normal vectors are directed to the same side of the surface.

Inserting this expression for \vec{n} into the above definition of the flux Φ , we find that the surface integral of the vector field \vec{F} across S^* takes on the following form:

$$\begin{aligned} & \iint_{S^*} \vec{F}(x, y, z) \cdot \vec{n} \, dS \\ &= \iint_G \vec{F}(\vec{r}(u, v)) \cdot \vec{n} \left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\| \, dG \\ &= \iint_G \vec{F}(\vec{r}(u, v)) \cdot \left[\epsilon \cdot \frac{\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v)}{\left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\|} \right] \left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\| \, dG \\ &= \epsilon \iint_G \vec{F}(\vec{r}(u, v)) \cdot \left[\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right] \, dG \end{aligned}$$

whereby in line 2 we applied the definition of the surface integral as discussed at the beginning of this section.

As an example, we wish to calculate the flux of the vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z) = (-2x, y, -z^2)$ through the surface S^* , which is bounded by the scalar function $f :]0, +\infty[\times]0, +\infty[\rightarrow \mathbb{R}^3 : (x, y) \mapsto f(x, y) = \frac{\sqrt{x}}{y}$ and the three planes α , β , and γ , with equations $\alpha : y = 2x - 2$, $\beta : y = -2x + 10$, and $\gamma : y = \frac{1}{5}$. Note hereby that the unit normal vector \vec{n} is pointing to the outer side of the curvature.

The surface S^* is shown in below Fig. 10.23. To construct the vector function $\vec{r}(x, y)$ that describes S^* , we need to determine the boundaries of the region G , which is the projection of S^* onto the xy -plane. With respect to the x -direction, the lower and upper boundary is established by the plane α and β , respectively. That is, the x -variable runs from $x_1 = \frac{y}{2} + 1$ to $x_2 = 5 - \frac{y}{2}$.

In terms of the boundary of G in the y -direction, the variable y runs from $y_1 = \frac{1}{5}$ to the y -component of the point of intersection between plane α and β . Setting the respective equations equal to each other, we find that $2x - 2 = -2x + 10 \Leftrightarrow x = 3$, so that the y -component is $y_2 = 4$.

The vector function \vec{r} is then equal to $\vec{r}: G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3: (x, y) \mapsto \vec{r}(x, y) = (x, y, \frac{\sqrt{x}}{y})$. The vector field \vec{F} in terms of the parameters x and y is written as $\vec{F}(\vec{r}(x, y)) = (-2x, y, -\frac{x}{y^2})$.

Given that $\frac{\partial \vec{r}}{\partial x}(x, y) = (1, 0, \frac{1}{2y\sqrt{x}})$ and $\frac{\partial \vec{r}}{\partial y}(x, y) = (0, 1, -\frac{\sqrt{x}}{y^2})$, we find that the curl between these two vectors is equal to:

$$\frac{\partial \vec{r}}{\partial x}(x, y) \times \frac{\partial \vec{r}}{\partial y}(x, y) = (-\frac{1}{2y\sqrt{x}}, \frac{\sqrt{x}}{y^2}, 1)$$

As we know that the variables x and y take on only positive values and that the normal vector \vec{n} points to the outside of the curvature, i.e., into the negative y and z direction, the curl vector, which points into the direction of the normal, tells us that we must choose $\epsilon = -1$ in the definition of the normal.

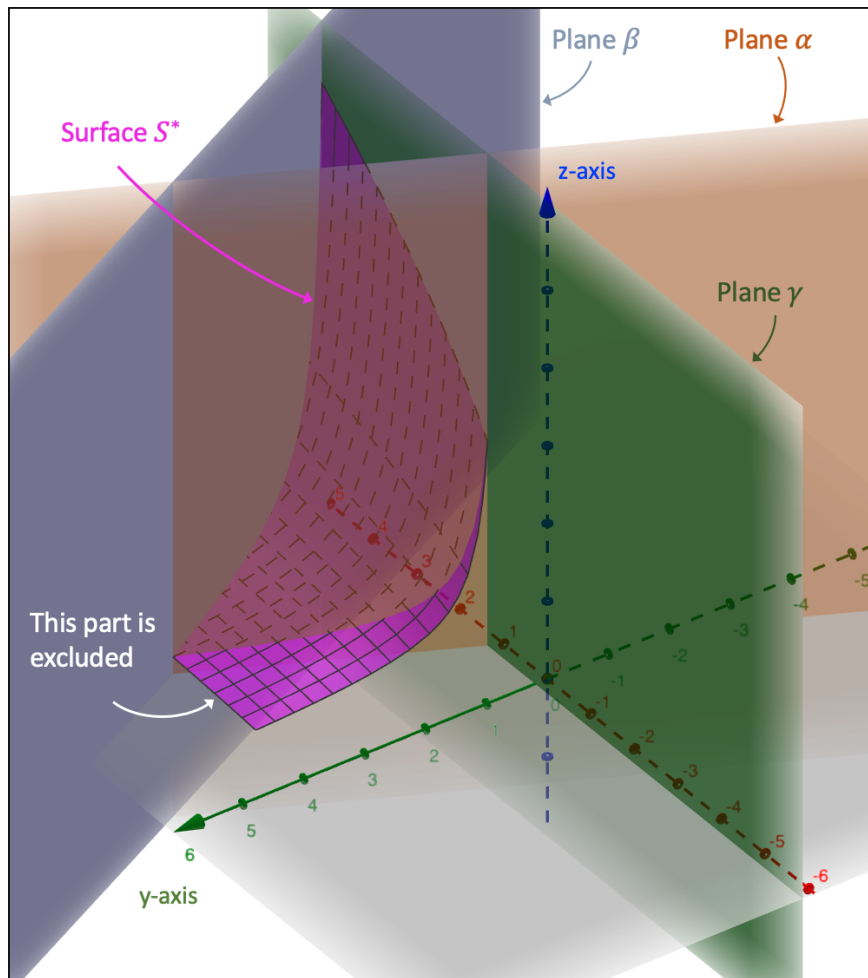


Figure 10.23: Calculating the flux through the surface S^*

We can now calculate the flux Φ through the surface S^* :

$$\begin{aligned}
\Phi &= \epsilon \iint_G \vec{F}(\vec{r}(x, y)) \cdot \left[\frac{\partial \vec{r}}{\partial x}(x, y) \times \frac{\partial \vec{r}}{\partial y}(x, y) \right] dG \\
&= - \int_{\frac{1}{5}}^4 dy \int_{\frac{y}{2}+1}^{5-\frac{y}{2}} (-2x, y, -\frac{x}{y^2}) \cdot \left(-\frac{1}{2y\sqrt{x}}, \frac{\sqrt{x}}{y^2}, 1 \right) dx \\
&= - \int_{\frac{1}{5}}^4 dy \int_{\frac{y}{2}+1}^{5-\frac{y}{2}} \left(\frac{2\sqrt{x}}{y} - \frac{x}{y^2} \right) dx \\
&= - \int_{\frac{1}{5}}^4 dy \left[\frac{4x^{\frac{3}{2}}}{3y} - \frac{x^2}{2y^2} \right]_{\frac{y}{2}+1}^{5-\frac{y}{2}} \\
&= - \int_{\frac{1}{5}}^4 \left[\frac{4}{3y} \left(5 - \frac{y}{2} \right) \sqrt{5 - \frac{y}{2}} - \frac{4}{3y} \left(\frac{y}{2} + 1 \right) \sqrt{\frac{y}{2} + 1} - \frac{1}{2y^2} \left(5 - \frac{y}{2} \right)^2 + \frac{1}{2y^2} \left(\frac{y}{2} + 1 \right)^2 \right] dy \\
&= - \int_{\frac{1}{5}}^4 \frac{20}{3y} \sqrt{5 - \frac{y}{2}} dy + \frac{2}{3} \int_{\frac{1}{5}}^4 \sqrt{5 - \frac{y}{2}} dy + \frac{2}{3} \int_{\frac{1}{5}}^4 \sqrt{\frac{y}{2} + 1} dy + \int_{\frac{1}{5}}^4 \frac{4}{3y} \sqrt{\frac{y}{2} + 1} dy \\
&\quad + 12 \int_{\frac{1}{5}}^4 \frac{1}{y^2} dy - 3 \int_{\frac{1}{5}}^4 \frac{1}{y} dy \\
&= \frac{40}{3} \int_{u_1}^{u_2} \left(-1 + \frac{1}{1 - \frac{u^2}{5}} \right) du - \frac{8}{9} \left[\left(\sqrt{5 - \frac{y}{2}} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 + \frac{8}{9} \left[\left(\sqrt{\frac{y}{2} + 1} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 \\
&\quad + \frac{8}{3} \int_{s_1}^{s_2} \left(1 + \frac{1}{s^2 - 1} \right) ds - 12 \left[\frac{1}{y} \right]_{\frac{1}{5}}^4 - 3 [\ln y]_{\frac{1}{5}}^4 \\
&= \frac{40}{3} \left[[-u]_{u_1}^{u_2} + \sqrt{5} \int_{t_1}^{t_2} \frac{1}{1 - t^2} dt \right] - \frac{8}{9} \left[\left(\sqrt{5 - \frac{y}{2}} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 + \frac{8}{9} \left[\left(\sqrt{\frac{y}{2} + 1} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 \\
&\quad + \frac{8}{3} \left[s - \frac{1}{2} \ln \left| \frac{s+1}{s-1} \right| \right]_{s_1}^{s_2} - 12 \left[\frac{1}{y} \right]_{\frac{1}{5}}^4 - 3 [\ln y]_{\frac{1}{5}}^4 \\
&= \frac{40}{3} \left[[-u]_{u_1}^{u_2} + \frac{\sqrt{5}}{2} \left[\ln \left| \frac{1+t}{1-t} \right| \right]_{t_1}^{t_2} \frac{1}{1-t^2} dt \right] - \frac{8}{9} \left[\left(\sqrt{5 - \frac{y}{2}} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 + \frac{8}{9} \left[\left(\sqrt{\frac{y}{2} + 1} \right)^{\frac{3}{2}} \right]_{\frac{1}{5}}^4 \\
&\quad + \frac{8}{3} \left[s - \frac{1}{2} \ln \left| \frac{s+1}{s-1} \right| \right]_{s_1}^{s_2} - 12 \left[\frac{1}{y} \right]_{\frac{1}{5}}^4 - 3 [\ln y]_{\frac{1}{5}}^4 \\
&= \frac{\sqrt{10}(2786 - 82\sqrt{11})}{225} - \frac{32\sqrt{3}}{3} + \frac{20\sqrt{5}}{3} \ln \left| \frac{4 + \sqrt{15}}{99 + 70\sqrt{2}} \right| - \frac{4}{3} \ln \left| \frac{2 + \sqrt{3}}{21 + 2\sqrt{110}} \right| + 57 - 3 \ln 20 \\
&\approx 20.03
\end{aligned}$$

whereby we introduced the substitution $u = \sqrt{5 - \frac{y}{2}}$ and $s = \sqrt{\frac{y}{2} + 1}$ in line 7 and the substitution $t = \frac{u}{\sqrt{5}}$ in line 8. The respective boundaries are equal to $u : \frac{7}{\sqrt{10}} \rightarrow \sqrt{3}$, $s : \sqrt{\frac{11}{10}} \rightarrow \sqrt{3}$, and $t : \frac{7}{5\sqrt{2}} \rightarrow \sqrt{\frac{3}{5}}$. We furthermore applied partial fraction decomposition twice in line 7.

- Stokes' Theorem

In section 10.4.2 we have seen how the line integral of a vector field along a two-dimensional curve can be linked to the double integral through Green's theorem.

Stokes' theorem links the line integral of a vector field along a three-dimensional curve C to the surface integral. More specifically, this theorem tells us that the amount of rotation of a continuous vector field $\vec{F} : H \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z)$ across the surface S^* , which is described by the vector function $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (u, v) \mapsto \vec{r}(u, v)$ and whereby $S^* \subset H$, is connected to the line integral of the curve C , which encloses the surface S^* :

$$\iint_{S^*} [(\nabla \times \vec{F}(x, y, z)) \cdot \vec{n}] dS = \oint_{C^+} \vec{F}(\vec{r}(u, v)) \cdot d\vec{r}(u, v)$$

Note hereby that the orientation of S^* must be aligned with the direction in which the line integral is evaluated across the curve C . That is, the unit normal vector \vec{n} is oriented according to the right-hand rule when applied to the direction of circulation along C . Remember that C^+ implies a positive direction of circulation and is taken to be counterclockwise.

The left-hand side of Stokes' theorem takes on the following form when the vector field \vec{F} is expressed in terms of the parameters u and v :

$$\begin{aligned} & \iint_{S^*} [(\nabla \times \vec{F}(x, y, z)) \cdot \vec{n}] dS \\ &= \iint_G [(\nabla \times \vec{F}(\vec{r}(u, v))) \cdot \vec{n}] \left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\| dG \\ &= \iint_G \left[(\nabla \times \vec{F}(\vec{r}(u, v))) \cdot \left(\epsilon \cdot \frac{\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v)}{\left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\|} \right) \right] \left\| \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right\| dG \\ &= \epsilon \iint_G (\nabla \times \vec{F}(\vec{r}(u, v))) \cdot \left[\frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v) \right] dG \end{aligned}$$

whereby G represents the projection of the surface S^* onto the uv -plane.

As an example, consider the vector field $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z) = (x^2 + z, x - y, -xy - z)$ and the surface S^* , which consists of the part of the paraboloid with equation $x^2 + y^2 - 4 = -z$ located above the xy -plane and bounded by the plane $\alpha : 4y + z = 4$ so that $y \geq 1 - \frac{z}{4}$. We will calculate both the left-hand side and the right-hand side of Stokes' theorem which should produce the same answer.

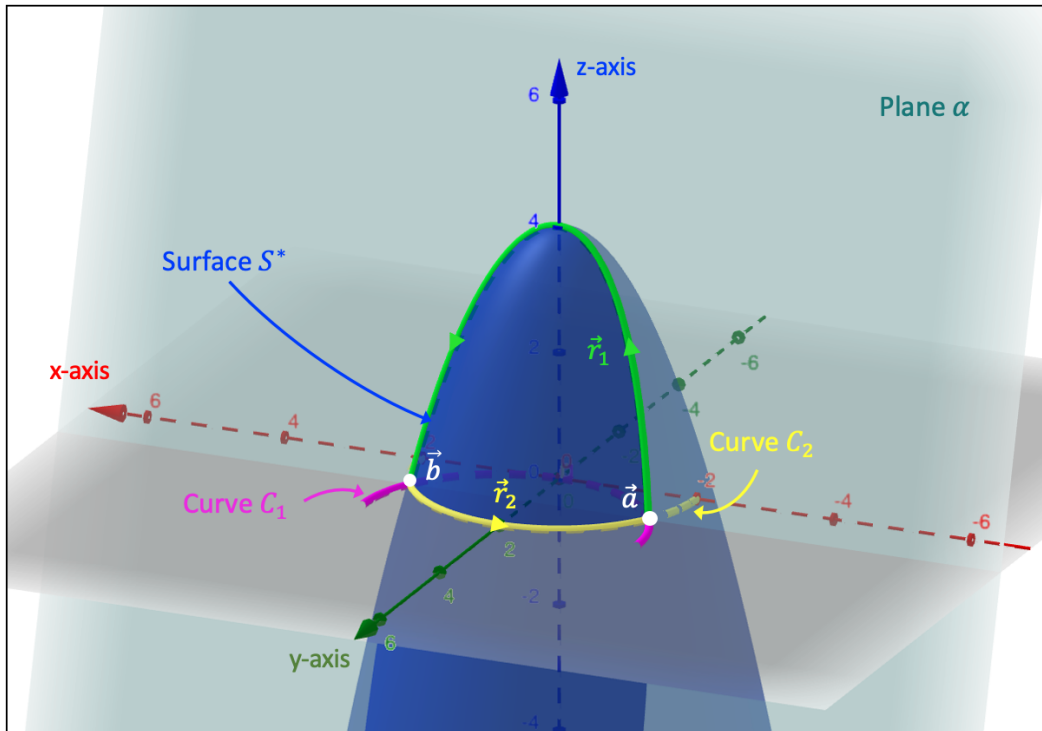


Figure 10.24: Stokes' theorem: an example

Before we start, we wish to construct the vector function $\vec{r}(x, y)$ that describes the surface S^* . The plane α intersects the xy -plane at $y = 1$ which gives two points of intersection with the paraboloid: $\vec{a} = (-\sqrt{3}, 1, 0)$ and $\vec{b} = (\sqrt{3}, 1, 0)$. The intersection between the plane α and the paraboloid produces an ellipse of which part of its projection onto the xy -plane is given by the pink circular curve C_1 with equation $C_1 : y = 2 - \sqrt{4 - x^2}$. The yellow circular curve C_2 represents the level curve of the paraboloid for $z = 0$ whose equation is equal to $C_2 : y = \sqrt{4 - x^2}$.

Based on the above information, we find that the region G , which is the projection of S^* onto the xy -plane, is bounded by the interval $[-\sqrt{3}, \sqrt{3}]$ in the x -direction and $[2 - \sqrt{4 - x^2}, \sqrt{4 - x^2}]$ in the y -direction. We can then describe the surface S^* by the vector function $\vec{r} : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (x, y) \mapsto \vec{r}(x, y) = (x, y, 4 - (x^2 + y^2))$.

We will first calculate the right-hand side of the Stokes' theorem, i.e., the line integral. The curve C that encloses the surface S^* consists of two parts: the green curve \vec{r}_1 that is part of the elliptical intersection between plane α and the paraboloid and the yellow segment \vec{r}_2 which lies on the curve C_2 . Given that the unit normal vector \vec{n} points outwards, the right-hand rule informs us that the circulation of C is positive, i.e., counterclockwise.

With respect to the green curve \vec{r}_1 , if we implement the parameterization $x = t$ and $y = 2 - \sqrt{4 - t^2}$, whereby t runs from $-\sqrt{3}$ to $\sqrt{3}$, and substitute these parameters for x and y in the vector function $\vec{r}(x, y)$ —after all, both \vec{r}_1 and \vec{r}_2 make up the edges of S^* —we can write \vec{r}_1 as $\vec{r}_1 : \mathbb{R} \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}_1(x(t), y(t)) = (t, 2 - \sqrt{4 - t^2}, 4\sqrt{4 - t^2} - 4)$.

Based on this parameterization of \vec{r}_1 , we can write its differential $d\vec{r}_1$ and the vector field $\vec{F}_1(\vec{r}_1(x(t), y(t)))$ as follows:

$$\begin{cases} d\vec{r}_1(x(t), y(t)) = \left(1, \frac{t}{\sqrt{4-t^2}}, -\frac{4t}{\sqrt{4-t^2}}\right) dt \\ \vec{F}_1(\vec{r}_1(x(t), y(t))) = (t^2 - 4 + 4\sqrt{4-t^2}, t - 2 + \sqrt{4-t^2}, -2t + 4 - (4-t)\sqrt{4-t^2}) \end{cases}$$

The line integral for the part of C described by \vec{r}_1 is equal to:

$$\begin{aligned} & \int_{C_1^+} \vec{F}_1(\vec{r}_1(x(t), y(t))) \cdot d\vec{r}_1(x(t), y(t)) \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} (t^2 - 4 + 4\sqrt{4-t^2}, t - 2 + \sqrt{4-t^2}, -2t + 4 - (4-t)\sqrt{4-t^2}) \cdot \left(1, \frac{t}{\sqrt{4-t^2}}, -\frac{4t}{\sqrt{4-t^2}}\right) dt \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \left[-3t^2 + 17t - 4 + 4\sqrt{4-t^2} + \frac{9t^2}{\sqrt{4-t^2}} - \frac{18t}{\sqrt{4-t^2}} \right] dt \\ &= \left[-t^3 + \frac{17}{2}t^2 - 4t \right]_{-\sqrt{3}}^{\sqrt{3}} + 16 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2(\theta) d\theta + 36 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2(\theta) d\theta + 18 \left[\sqrt{4-t^2} \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= (-14\sqrt{3}) + \left(\frac{16\pi}{3} + 4\sqrt{3} \right) + (12\pi - 9\sqrt{3}) \\ &= \frac{52\pi}{3} - 19\sqrt{3} \end{aligned}$$

whereby we introduced the substitution $t = 2 \sin \theta$ in line 4.

A similar reasoning as above for the yellow line segment \vec{r}_2 gives us the following expressions, whereby we used the parameterization $x = t$ and $y = \sqrt{4-t^2}$ and this time the parameter t runs from $\sqrt{3}$ to $-\sqrt{3}$, i.e., from point \vec{b} to \vec{a} :

$$\begin{cases} \vec{r}_2 : \mathbb{R} \rightarrow \mathbb{R}^3 : t \mapsto \vec{r}_2(x(t), y(t)) = (t, \sqrt{4-t^2}, 0) \\ d\vec{r}_2(x(t), y(t)) = \left(1, -\frac{t}{\sqrt{4-t^2}}, 0\right) dt \\ \vec{F}_2(\vec{r}_2(x(t), y(t))) = (t^2, t - \sqrt{4-t^2}, -t\sqrt{4-t^2}) \end{cases}$$

The line integral for the second part of the curve C described by \vec{r}_2 is then equal to:

$$\begin{aligned} & \int_{C_2^+} \vec{F}_2(\vec{r}_2(x(t), y(t))) \cdot d\vec{r}_2(x(t), y(t)) \\ &= \int_{\sqrt{3}}^{-\sqrt{3}} (t^2, t - \sqrt{4-t^2}, -t\sqrt{4-t^2}) \cdot \left(1, -\frac{t}{\sqrt{4-t^2}}, 0\right) dt \\ &= - \int_{-\sqrt{3}}^{\sqrt{3}} \left[t^2 + t - \frac{t^2}{\sqrt{4-t^2}} \right] dt \end{aligned}$$

$$\begin{aligned}
&= - \left[\frac{t^3}{3} + \frac{t^2}{2} \right]_{-\sqrt{3}}^{\sqrt{3}} + \int_{-\sqrt{3}}^{\sqrt{3}} \frac{t^2}{\sqrt{4-t^2}} dt \\
&= \left(-2\sqrt{3} \right) + 4 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2(\theta) d\theta \\
&= \left(-2\sqrt{3} \right) + \left(\frac{4\pi}{3} - \sqrt{3} \right) \\
&= \frac{4\pi}{3} - 3\sqrt{3}
\end{aligned}$$

whereby we implemented the substitution $t = 2 \sin \theta$ in line 5.

The total line integral is then equal to:

$$\begin{aligned}
\oint_{C^+} \vec{F}(\vec{r}(x(t), y(t))) \cdot d\vec{r}(x(t), y(t)) &= \int_{C_1^+} \vec{F}_1(\vec{r}_1(x(t), y(t))) \cdot d\vec{r}_1(x(t), y(t)) \\
&\quad + \int_{C_2^+} \vec{F}_2(\vec{r}_2(x(t), y(t))) \cdot d\vec{r}_2(x(t), y(t)) \\
&= \left(\frac{52\pi}{3} - 19\sqrt{3} \right) + \left(\frac{4\pi}{3} - 3\sqrt{3} \right) \\
&= \frac{56\pi}{3} - 22\sqrt{3}
\end{aligned}$$

In second step, we calculate the left-hand side of Stokes' theorem, i.e., the surface integral. The curl of the vector field $\vec{F}(x, y, z)$ is equal to $\text{rot}(\vec{F}(x, y, z)) = (-x, 1 + y, 1)$. Note that in terms of the x and y parameterization, the curl remains unchanged. The partial derivatives of the vector function $\vec{r}(x, y)$ are $\frac{\partial \vec{r}}{\partial x} = (1, 0, -2x)$ and $\frac{\partial \vec{r}}{\partial y} = (0, 1, -2y)$. Their curl is equal to the vector $(2x, 2y, 1)$. Since the unit normal vector \vec{n} , which is parallel to this curl, points into the positive z -direction at every point of S^* , we have that $\epsilon = +1$.

The left-hand side of Stokes' theorem is then equal to:

$$\begin{aligned}
&\iint_{S^*} \left[(\nabla \times \vec{F}(x, y, z)) \cdot \vec{n} \right] dS \\
&= \epsilon \iint_G \left(\nabla \times \vec{F}(\vec{r}(x, y)) \right) \cdot \left[\frac{\partial \vec{r}}{\partial x}(x, y) \times \frac{\partial \vec{r}}{\partial y}(x, y) \right] dG \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{2-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x, 1 + y, 1) \cdot (2x, 2y, 1) dy \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{2-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-2x^2 + 2y + 2y^2 + 1) dy \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} dx \left[-2x^2y + y^2 + \frac{2}{3}y^3 + y \right]_{2-\sqrt{4-x^2}}^{\sqrt{4-x^2}}
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\sqrt{3}}^{\sqrt{3}} dx \left[-\frac{16}{3}x^2\sqrt{4-x^2} + 8x^2 + \frac{58}{3}\sqrt{4-x^2} - \frac{82}{3} \right] dx \\
&= -\frac{256}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2(\theta) \cos^2(\theta) d\theta + \frac{8}{3} [x^3]_{-\sqrt{3}}^{\sqrt{3}} + \frac{232}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2(\theta) d\theta - \frac{82}{3} [x]_{-\sqrt{3}}^{\sqrt{3}} \\
&= \left(-\frac{32}{3} \left[\theta - \frac{\sin(4\theta)}{4} \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \right) + (16\sqrt{3}) + \left(\frac{116}{3} \left[\theta + \frac{\sin(2\theta)}{2} \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \right) + \left(-\frac{164\sqrt{3}}{3} \right) \\
&= \left(-\frac{64\pi}{9} - \frac{8\sqrt{3}}{3} \right) + (16\sqrt{3}) + \left(\frac{232\pi}{9} + \frac{58\sqrt{3}}{3} \right) + \left(-\frac{164\sqrt{3}}{3} \right) \\
&= \frac{56\pi}{3} - 22\sqrt{3}
\end{aligned}$$

whereby we implemented the substitution $x = 2 \sin \theta$ in line 7.

10.4.4 The Triple Integral

In section 10.4.2 we have seen how the double integral allows us to calculate the volume of the region between the graph of a function $f : S \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ and the surface S . The **triple integral** is a further generalization as it enables us to calculate the volume not necessarily of regions below the graph of f but also of regions comprised between any bounded surfaces.

If $D \subset \mathbb{R}^3$ is a region in three-dimensional space, the volume of D can be calculated through the following triple integral:

$$\iiint_D dD = \iiint_D dx dy dz$$

- Example

We wish to calculate the volume of the region D that is bounded by the graph of the scalar function $f : \mathbb{R}^2 \rightarrow]-\infty, 4[: (x, y) \mapsto f(x, y) = -\frac{1}{2e^y} + 4$, the surface W described by the vector function $\vec{r}_W : [-8, 4] \times [-2, 6] \rightarrow \mathbb{R}^3 : (x, z) \mapsto \vec{r}_W(x, z) = (x, 4 + \ln \left[\frac{2}{3}(x+5) \right], z)$, and by the planes $x = 0$, $x = -4$ and $\alpha : 2x - 3z = -10$. The region D is shown in below Fig. 10.25.

With respect to the integral boundaries in the x -direction, we have that $x : -4 \rightarrow 0$. Regarding the y -direction, the parameter y runs from the intersection between the graph of f and the plane α to the surface W . In other words, we have that $y : y_1 = \ln \left[\frac{3}{4(1-x)} \right] \rightarrow y_2 = 4 + \ln \left[\frac{2}{3}(x+5) \right]$. Finally, the z -variable runs from the plane α to the graph of f . That is, $z : z_1 = \frac{2}{3}(x+5) \rightarrow z_2 = -\frac{1}{2e^y} + 4$.

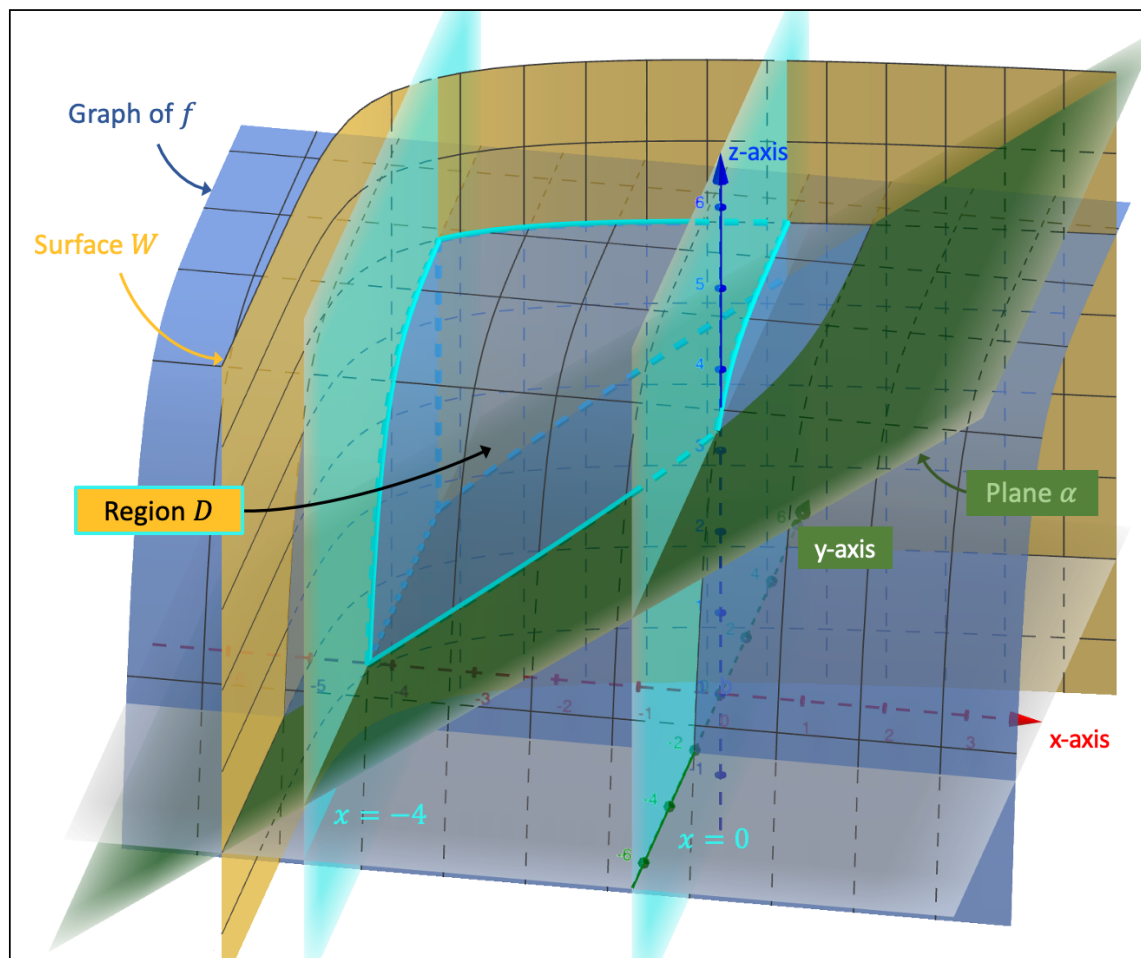


Figure 10.25: Calculating the volume with the triple integral

The volume of the region D is then found as follows:

$$\begin{aligned}
 V &= \iiint_D dx dy dz \\
 &= \int_{-4}^0 dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} dz \\
 &= \int_{-4}^0 dx \int_{y_1}^{y_2} [z]_{z_1=\frac{2}{3}(x+5)}^{z_2=-\frac{1}{2e^y}+4} dy \\
 &= \int_{-4}^0 dx \int_{y_1}^{y_2} \left(-\frac{1}{2e^y} + \frac{2}{3} - \frac{2x}{3} \right) dy \\
 &= \int_{-4}^0 \left[\frac{1}{2e^y} + \frac{2y}{3} - \frac{2xy}{3} \right]_{y_1=\ln\left[\frac{3}{4(1-x)}\right]}^{y_2=4+\ln\left[\frac{2}{3}(x+5)\right]} dx \\
 &= \int_{-4}^0 \frac{3}{4e^4(x+5)} dx + \int_{-4}^0 \frac{2}{3} \left(4 + \ln \left[\frac{2}{3}(x+5) \right] \right) dx - \int_{-4}^0 \frac{2x}{3} \left(4 + \ln \left[\frac{2}{3}(x+5) \right] \right) dx \\
 &\quad - \int_{-4}^0 \frac{2(1-x)}{3} dx - \int_{-4}^0 \frac{2}{3} \left(\ln \left[\frac{3}{4(1-x)} \right] \right) dx + \int_{-4}^0 \frac{2x}{3} \left(\ln \left[\frac{3}{4(1-x)} \right] \right) dx
 \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4e^4} [\ln(x+5)]_{-4}^0 + \frac{8}{3} [x]_{-4}^0 + \frac{2}{3} \left[x \ln \left[\frac{2}{3}(x+5) \right] - x + 5 \ln(x+5) \right]_{-4}^0 - \frac{4}{3} [x^2]_{-4}^0 \\
&\quad - \frac{2}{3} \left[\frac{x^2}{2} \ln \left[\frac{2}{3}(x+5) \right] - \frac{x^2}{4} + \frac{5x}{2} - \frac{25}{2} \ln(x+5) \right]_{-4}^0 - \frac{2}{3} [x]_{-4}^0 + \frac{1}{3} [x^2]_{-4}^0 \\
&\quad - \frac{2}{3} \left[x \ln \left[\frac{3}{4} \right] - x \ln(1-x) + x + \ln(1-x) \right]_{-4}^0 \\
&\quad + \frac{2}{3} \left[\frac{x^2}{2} \ln \left[\frac{3}{4} \right] - \frac{x^2}{2} \ln(1-x) + \frac{x^2}{4} + \frac{x}{2} + \frac{1}{2} \ln(1-x) \right]_{-4}^0 \\
&= \left(\frac{3}{4e^4} + 20 \right) \ln(5) - 16 \ln(3) + 24 \ln(2) + 8 \\
&\approx 39.27
\end{aligned}$$

whereby we applied partial integration and/or partial fraction decomposition to the second, third, fifth and sixth integral of line 6, when going from line 6 to line 7.

- Density function f

If the scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, z) \mapsto f(x, y, z)$ represents the density function, the triple integral across a volume D calculates the mass within that volume and is given by:

$$\iiint_D f(x, y, z) dD = \iiint_D f(x, y, z) dx dy dz$$

- Change of coordinates: Spherical coordinates

If our problem has spherical symmetry, we can opt to change from the usual (x, y, z) system to a coordinate system expressed in spherical coordinates. As already introduced in section 10.4.3, the spherical coordinates are defined as follows, whereby $\rho : 0 \rightarrow R$, $\theta : 0 \rightarrow \pi$, and $\phi : 0 \rightarrow 2\pi$:

$$\begin{cases} x = \rho \sin(\theta) \cos \phi \\ y = \rho \sin(\theta) \sin \phi \\ z = \rho \cos \theta \end{cases}$$

The triple integral then obtains the following form:

$$\iiint_D f(x, y, z) dx dy dz = \iiint_D f(x(\rho, \theta, \phi), y(\rho, \theta, \phi), z(\rho, \theta, \phi)) \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \right| d\rho d\theta d\phi$$

The Jacobian determinant is equal to:

$$\begin{aligned}
 \det(\mathcal{L}) &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
 &= \begin{vmatrix} \sin(\theta) \cos \phi & \rho \cos(\theta) \cos \phi & -\rho \sin(\theta) \sin \phi \\ \sin(\theta) \sin \phi & \rho \cos(\theta) \sin \phi & \rho \sin(\theta) \cos \phi \\ \cos \theta & -\rho \sin \theta & 0 \end{vmatrix} \\
 &= \sin(\theta) \cos(\phi) [\rho^2 \sin^2(\theta) \cos \phi] + \rho \cos(\theta) \cos(\phi) [\rho \sin(\theta) \cos(\theta) \cos \phi] \\
 &\quad - \rho \sin(\theta) \sin(\phi) [-\rho \sin^2(\theta) \sin \phi - \rho \cos^2(\theta) \sin \phi] \\
 &= \rho^2 \sin \theta
 \end{aligned}$$

- Example

We would like to calculate the volume of the region D that consists of the region outside the cylinder L with radius $r = \sqrt{6}$ but within the sphere $S : x^2 + y^2 + z^2 = 36$. The cylinder L lies entirely within the sphere so that its upper and lower edge are tangent to the sphere. This means that the region D is made up of the region D_1 and two regions D_2 : D_1 refers to the region that extends radially and horizontally beyond the cylinder and within the sphere, and the two regions D_2 are the regions right above and below the cylinder.

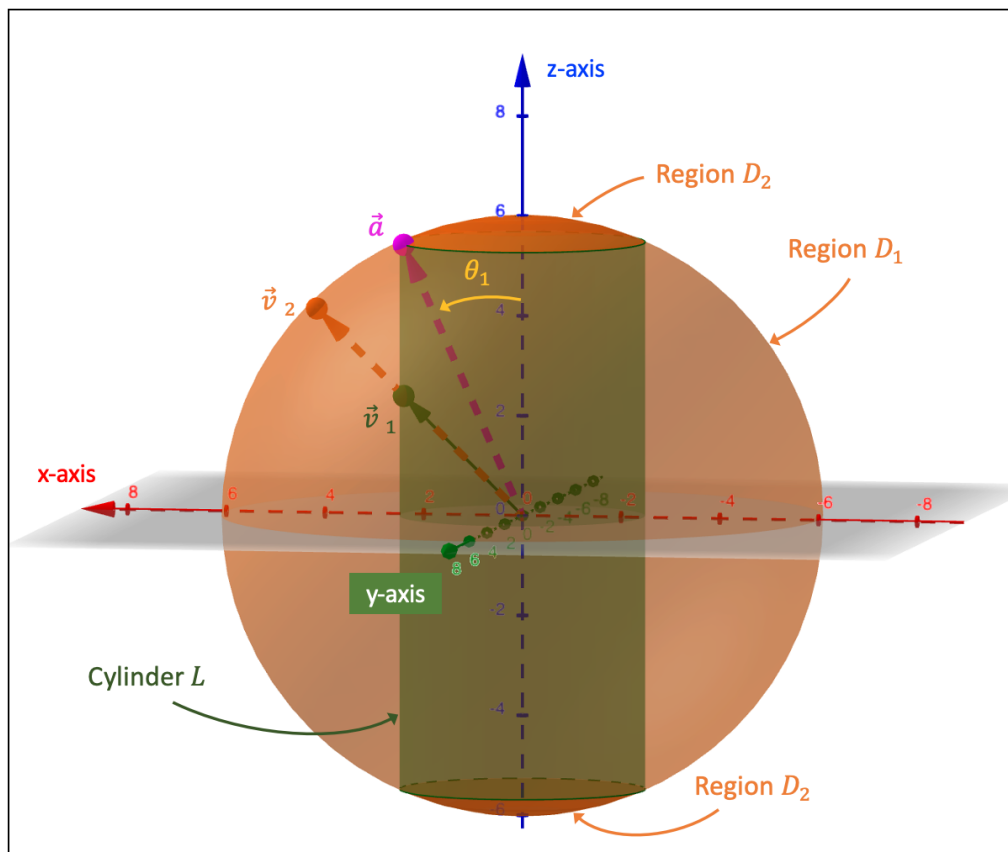


Figure 10.26: Calculating the volume with spherical coordinates

Let us begin with region D_1 . In terms of the spherical coordinates, the domain of the parameter ϕ is equal to $[0, 2\pi[$. With respect to the angle θ , we want it to have an initial value of θ_1 (indicated in yellow in Fig. 10.26) and a final value of $\theta_2 = \pi - \theta_1$. Given that the radius of the cylinder is $r = \sqrt{6}$, we find that the height of the cylinder above the xy -plane—if we set, for instance, $x = \sqrt{6}$ and $y = 0$ in the equation of the sphere—is equal to $z = \sqrt{30}$. This means that θ_1 is equal to $\tan \theta_1 = \frac{\sqrt{6}}{\sqrt{30}} \Leftrightarrow \theta_1 = \arctan\left(\frac{\sqrt{5}}{5}\right)$.

The parameter ρ has to be taken into account only from the surface of the cylinder onwards, extending radially towards the edge of the sphere. In other words, we have to find an equation for the cylinder and express it in spherical coordinates. The cylinder can be described by the equation $x^2 + y^2 = 6$, so that $(\rho_L \sin(\theta) \cos \phi)^2 + (\rho_L \sin(\theta) \sin \phi)^2 = 6 \Leftrightarrow \rho_L = \frac{\sqrt{6}}{\sin \theta}$. Since ρ_L represents the distance from the origin to a point on the cylinder, we have that $\rho : \rho_L \rightarrow \rho_S$, whereby ρ_S is the radius of the sphere S , i.e., $\rho_S = 6$.

The volume V_1 of region D_1 is then calculated as follows:

$$\begin{aligned}
 V_1 &= \iiint_{D_1} dx dy dz = \iiint_{D_1} \rho^2 \sin(\theta) d\rho d\theta d\phi \\
 &= \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} d\theta \int_{\rho_L}^{\rho_S} \rho^2 \sin(\theta) d\rho \\
 &= \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} \left[\frac{\rho^3}{3} \right]_{\rho_L = \frac{\sqrt{6}}{\sin \theta}}^{\rho_S = 6} \sin(\theta) d\theta \\
 &= \int_0^{2\pi} d\phi \int_{\theta_1}^{\theta_2} \left(72 \sin \theta - \frac{2\sqrt{6}}{\sin^2 \theta} \right) d\theta \\
 &= \int_0^{2\pi} \left(72 \int_{\theta_1}^{\theta_2} \sin(\theta) d\theta - 2\sqrt{6} \int_{t_1}^{t_2} \frac{1}{t^2} dt \right) d\phi \\
 &= \int_0^{2\pi} \left(72 [-\cos(\theta)]_{\theta_1 = \arctan(\frac{\sqrt{5}}{5})}^{\theta_2 = \pi - \arctan(\frac{\sqrt{5}}{5})} - 2\sqrt{6} \left[-\frac{1}{t} \right]_{t_1 = \frac{\sqrt{5}}{5}}^{t_2 = -\frac{\sqrt{5}}{5}} \right) d\phi \\
 &= \int_0^{2\pi} \left(24\sqrt{30} - 4\sqrt{30} \right) d\phi \\
 &= 40\pi\sqrt{30}
 \end{aligned}$$

whereby we introduced the substitution $t = \tan \theta$ in line 5. Also remember that $\cos \theta = \frac{1}{\sqrt{1+\tan^2 \theta}}$ and that $\tan(\pi - \alpha) = -\tan \alpha$.

We now focus on the two regions D_2 . For the parameters ϕ and θ , we have that $\phi : 0 \rightarrow 2\pi$ and $\theta : 0 \rightarrow \theta_1$. With respect to the upper region D_2 , the parameter ρ can only run from a point on the upper horizontal surface of the cylinder L to the edge of the sphere. Again, we need to find an expression for the distance ρ_U from the origin to the upper horizontal surface and express it in spherical coordinates. Since the height is equal to $z = \sqrt{30}$, the Pythagorean theorem tells us that $\rho_U = \sqrt{x^2 + y^2 + 30}$. From the definition of the spherical

coordinates, we also see that $x^2 + y^2 = \rho_U^2 \sin^2 \theta$. Combining both equations, we find that $\rho_U = \frac{\sqrt{30}}{\cos \theta}$. Therefore, we have that $\rho : \rho_U \rightarrow \rho_S$.

Given the symmetry, we will calculate the volume of the upper region D_2 and double it to find the total volume V_2 of the two regions D_2 together:

$$\begin{aligned}
 V_2 &= 2 \iiint_{D_2} dx dy dz = 2 \iiint_{D_2} \rho^2 \sin(\theta) d\rho d\theta d\phi \\
 &= 2 \int_0^{2\pi} d\phi \int_0^{\theta_1} d\theta \int_{\rho_U}^{\rho_S} \rho^2 \sin(\theta) d\rho \\
 &= 2 \int_0^{2\pi} d\phi \int_0^{\theta_1} \left[\frac{\rho^3}{3} \right]_{\rho_U = \frac{\sqrt{30}}{\cos \theta}}^{\rho_S = 6} \sin(\theta) d\theta \\
 &= 2 \int_0^{2\pi} d\phi \int_0^{\theta_1} \left(72 \sin \theta - \frac{10\sqrt{30} \sin \theta}{\cos^3 \theta} \right) d\theta \\
 &= 2 \int_0^{2\pi} \left(72 \int_0^{\theta_1} \sin(\theta) d\theta + 10\sqrt{30} \int_{t_1}^{t_2} \frac{1}{t^3} dt \right) d\phi \\
 &= 2 \int_0^{2\pi} \left(72 [-\cos(\theta)]_0^{\theta_1 = \arctan(\frac{\sqrt{5}}{5})} + 10\sqrt{30} \left[-\frac{1}{2t^2} \right]_{t_1=1}^{t_2 = \frac{\sqrt{5}}{6}} \right) d\phi \\
 &= 2 \int_0^{2\pi} \left(72 - 12\sqrt{30} - \sqrt{30} \right) d\phi \\
 &= 288\pi - 52\pi\sqrt{30}
 \end{aligned}$$

whereby we introduced the substitution $t = \cos \theta$ in line 5.

The total volume V of region D is then equal to:

$$V = V_1 + V_2 = (40\pi\sqrt{30}) + (288\pi - 52\pi\sqrt{30}) = 288\pi - 12\pi\sqrt{30}$$

We can check this result by calculating the volume of the sphere and subtracting from it the volume of the cylinder: $\frac{4}{3}\pi R^3 - \pi r^2 h = \frac{4}{3}\pi 6^3 - \pi(\sqrt{6})^2(2\sqrt{30}) = 288\pi - 12\pi\sqrt{30}$.

- Divergence theorem

Green's theorem connects the line integral of a vector field \vec{F} to the double integral (see section 10.4.2) and Stokes' theorem enables a link between the line integral of \vec{F} and the surface integral (see section 10.4.3).

The divergence theorem or Ostrogradsky's theorem makes a connection between the surface integral and the triple integral. Given a region $D \subset \mathbb{R}^3$ and a continuous vector field

$\vec{F} : H \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z)$, whereby the surface S^* that encloses the region D is defined within H , the divergence theorem states that the surface integral of the vector field \vec{F} across the surface S^* —this is the flux of \vec{F} through S^* —is equal to the triple integral of the divergence of the vector field \vec{F} :

$$\iint_{S^*} \vec{F}(x, y, z) \cdot \vec{n} \, dS = \iiint_D \vec{\nabla} \cdot \vec{F}(x, y, z) \, dx \, dy \, dz$$

whereby the unit normal vector \vec{n} is oriented towards the outer side of the surface S^* .

The divergence theorem says that if the vector field \vec{F} is constant, its divergence is zero, so that no flux is measured across the surface S^* . What is more, if the divergence of \vec{F} is equal to 1, we can see from this theorem that the surface integral calculates the volume of the region D .

- Example

Consider the region D , which is bounded by the graph of the scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto f(x, y) = -x^2y + 4y - y^2$ and the plane $z = 0$, and the continuous vector field $\vec{F} : H \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 : (x, y, z) \mapsto \vec{F}(x, y, z) = (\cosh(2x), xy, z + y^2)$. We will calculate both the left-hand and right-hand side of Ostrogradsky's theorem which should produce the same answer.

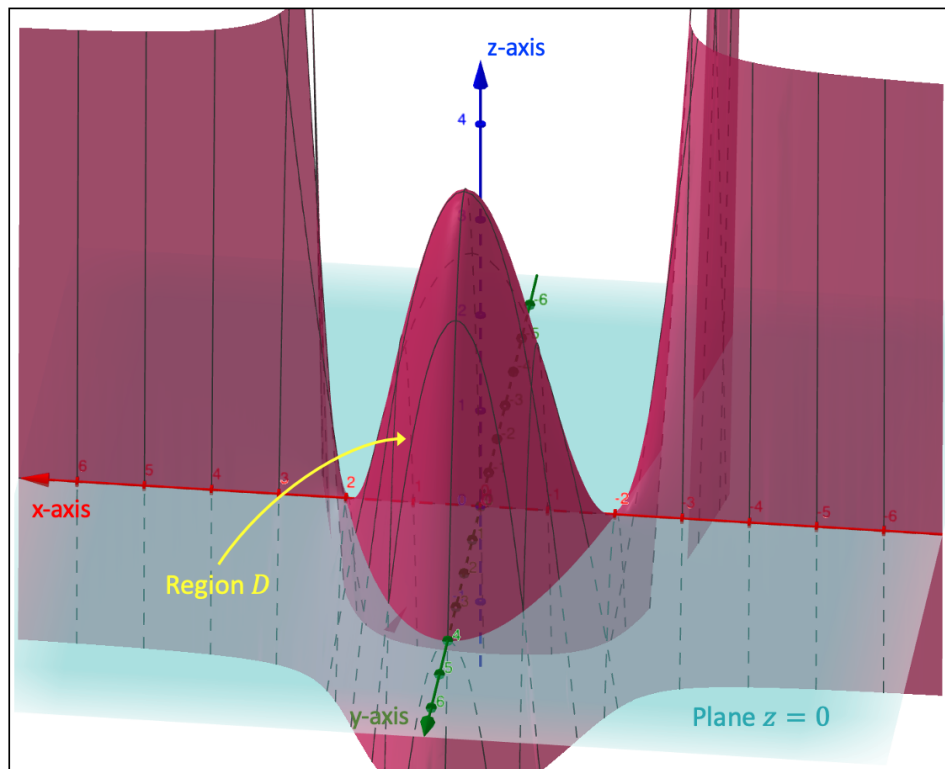


Figure 10.27: An example of Ostrogradsky's theorem

The surface S^* that encloses D consists of two parts: the surface S_1^* , which corresponds to the surface above the xy -plane, and the surface S_2^* , which describes the bottom of region D .

We first calculate the left-hand side of the divergence theorem and we start with the surface S_1^* .

The surface S_1^* can be described by the vector function $\vec{r}_1 : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (x, y) \mapsto \vec{r}_1(x, y) = (x, y, -x^2y + 4y - y^2)$ and the vector field \vec{F}_1 expressed in this x and y parameterization is equal to $\vec{F}_1(\vec{r}_1(x, y)) = (\cosh(2x), xy, -x^2y + 4y)$. With respect to the domain of G , we let the x parameter run from -2 to 2 , whereas for the y variable we have that $y : 0 \rightarrow 4 - x^2$, whose right boundary we obtained by considering the level curve $z = 0$ of the graph of f .

The partial derivatives of the vector function $\vec{r}_1(x, y)$ are equal to $\frac{\partial \vec{r}_1}{\partial x} = (1, 0, -2xy)$ and $\frac{\partial \vec{r}_1}{\partial y} = (0, 1, -x^2 - 2y + 4)$, so that their curl is the vector $(2xy, x^2 + 2y - 4, 1)$. Given that the unit normal vector \vec{n} , which is parallel to this curl, points into the positive z -direction at every point across the surface S_1^* , we have that $\epsilon = +1$.

The left-hand side of the divergence theorem with respect to the surface S_1^* is equal to:

$$\begin{aligned}
& \iint_{S_1^*} \vec{F}_1(x, y, z) \cdot \vec{n} \, dS_1 \\
&= \epsilon \iint_G \vec{F}_1(\vec{r}_1(x, y)) \cdot \left[\frac{\partial \vec{r}_1}{\partial x}(x, y) \times \frac{\partial \vec{r}_1}{\partial y}(x, y) \right] \, dG \\
&= (+1) \int_{-2}^2 dx \int_0^{4-x^2} (\cosh(2x), xy, -x^2y + 4y) \cdot (2xy, x^2 + 2y - 4, 1) \, dy \\
&= \int_{-2}^2 dx \int_0^{4-x^2} (2xy \cosh(2x) + x^3y + 2xy^2 - 4xy - x^2y + 4y) \, dy \\
&= \int_{-2}^2 \left[xy^2 \cosh(2x) + \frac{x^3y^2}{2} + \frac{2xy^3}{3} - 2xy^2 - \frac{x^2y^2}{2} + 2y^2 \right]_0^{4-x^2} dx \\
&= \int_{-2}^2 \left(x(4-x^2)^2 \cosh(2x) + \frac{x^3(4-x^2)^2}{2} + \frac{2x(4-x^2)^3}{3} - 2x(4-x^2)^2 - \frac{x^2(4-x^2)^2}{2} + 2(4-x^2)^2 \right) dx \\
&= \int_{-2}^2 \left(16x \cosh(2x) - 8x^3 \cosh(2x) + x^5 \cosh(2x) - \frac{x^7}{6} - \frac{x^6}{2} + 2x^5 + 6x^4 - 8x^3 - 24x^2 + \frac{32x}{3} + 32 \right) dx \\
&= \int_{-2}^2 (16x \cosh(2x) - 8x^3 \cosh(2x)) \, dx + \left(\left[\frac{x^5}{2} \sinh(2x) - \frac{5x^4}{4} \cosh(2x) \right]_{-2}^2 \right. \\
&\quad \left. + 5 \int_{-2}^2 x^3 \cosh(2x) \, dx \right) + \left[-\frac{x^8}{48} - \frac{x^7}{14} + \frac{x^6}{3} + \frac{6x^5}{5} - 2x^4 - 8x^3 + \frac{16x^2}{3} + 32x \right]_{-2}^2 \\
&= \int_{-2}^2 (16x \cosh(2x) - 3x^3 \cosh(2x)) \, dx + [0] + \frac{2048}{35}
\end{aligned}$$

$$\begin{aligned}
&= \int_{-2}^2 16x \cosh(2x) dx - 3 \left(\left[\frac{x^3}{2} \sinh(2x) - \frac{3x^2}{4} \cosh(2x) \right]_{-2}^2 + \frac{3}{2} \int_{-2}^2 x \cosh(2x) dx \right) + \frac{2048}{35} \\
&= \frac{23}{2} \int_{-2}^2 x \cosh(2x) dx - 3[0] + \frac{2048}{35} \\
&= \frac{23}{2} \left[\frac{x}{2} \sinh(2x) - \frac{1}{4} \cosh(2x) \right]_{-2}^2 + \frac{2048}{35} \\
&= \frac{2048}{35}
\end{aligned}$$

whereby we applied the below reduction formula in line 8 and 10 and partial integration in line 12.

$$\int x^{a+2} \cosh(bx) dx = \frac{x^{a+2}}{b} \sinh(bx) - \frac{(a+2)}{b^2} x^{a+1} \cosh(bx) + \frac{(a+1)(a+2)}{b^2} \int x^a \cosh(bx) dx$$

We now consider the bottom of region D , i.e., the surface S_2^* . This surface is described by the vector function $\vec{r}_2 : G \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 : (x, y) \mapsto \vec{r}_2(x, y) = (x, y, 0)$ and the vector field \vec{F}_2 expressed in this x and y parameterization is equal to $\vec{F}_2(\vec{r}_2(x, y)) = (\cosh(2x), xy, y^2)$.

The partial derivatives of the vector function $\vec{r}_2(x, y)$ are equal to $\frac{\partial \vec{r}_2}{\partial x} = (1, 0, 0)$ and $\frac{\partial \vec{r}_2}{\partial y} = (0, 1, 0)$, so that their curl is the vector $(0, 0, 1)$. Given that the unit normal vector \vec{n} , which is parallel to this curl, points into the negative z -direction at every point across the surface S_2^* , we have that $\epsilon = -1$. Remember that the unit normal vector \vec{n} points towards the outside of region D .

The left-hand side of the divergence theorem with respect to the surface S_2^* is calculated as follows:

$$\begin{aligned}
\iint_{S_2^*} \vec{F}_2(x, y, z) \cdot \vec{n} dS_2 &= \epsilon \iint_G \vec{F}_2(\vec{r}_2(x, y)) \cdot \left[\frac{\partial \vec{r}_2}{\partial x}(x, y) \times \frac{\partial \vec{r}_2}{\partial y}(x, y) \right] dG \\
&= (-1) \int_{-2}^2 dx \int_0^{4-x^2} (\cosh(2x), xy, y^2) \cdot (0, 0, 1) dy \\
&= (-1) \int_{-2}^2 dx \int_0^{4-x^2} y^2 dy \\
&= - \int_{-2}^2 \left[\frac{y^3}{3} \right]_0^{4-x^2} dx \\
&= - \int_{-2}^2 \left[\frac{(4-x^2)^3}{3} \right] dx
\end{aligned}$$

$$\begin{aligned}
&= \int_{-2}^2 \left[-\frac{64}{3} + 16x^2 - 4x^4 + \frac{x^6}{3} \right] dx \\
&= \left[-\frac{64x}{3} + \frac{16x^3}{3} - \frac{4x^5}{5} + \frac{x^7}{21} \right]_{-2}^2 \\
&= -\frac{4096}{105}
\end{aligned}$$

The left-hand side of the divergence theorem with respect to the surface $S^* = S_1^* + S_2^*$ is then equal to:

$$\begin{aligned}
\iint_{S^*} \vec{F}(x, y, z) \cdot \vec{n} dS &= \iint_{S_1^*} \vec{F}_1(x, y, z) \cdot \vec{n} dS_1 + \iint_{S_2^*} \vec{F}_2(x, y, z) \cdot \vec{n} dS_2 \\
&= \frac{2048}{35} - \frac{4096}{105} \\
&= \frac{2048}{105}
\end{aligned}$$

The right-hand side of Ostrogradsky's theorem is calculated as follows:

$$\begin{aligned}
&\iiint_D \vec{\nabla} \cdot \vec{F}(x, y, z) dx dy dz \\
&= \int_{-2}^2 dx \int_0^{4-x^2} dy \int_0^{-x^2y+4y-y^2} \vec{\nabla} \cdot (\cosh(2x), xy, z + y^2) dz \\
&= \int_{-2}^2 dx \int_0^{4-x^2} dy \int_0^{-x^2y+4y-y^2} (2 \sinh(2x) + x + 1) dz \\
&= \int_{-2}^2 dx \int_0^{4-x^2} (2 \sinh(2x) + x + 1) [z]_0^{-x^2y+4y-y^2} dy \\
&= \int_{-2}^2 dx \int_0^{4-x^2} (2 \sinh(2x) + x + 1) (-x^2y + 4y - y^2) dy \\
&= \int_{-2}^2 dx \int_0^{4-x^2} (-2x^2y \sinh(2x) + 8y \sinh(2x) - 2y^2 \sinh(2x) - x^3y + 4xy - xy^2 - x^2y + 4y - y^2) dy \\
&= \int_{-2}^2 dx \left[-x^2y^2 \sinh(2x) + 4y^2 \sinh(2x) - \frac{2y^3}{3} \sinh(2x) - \frac{x^3y^2}{2} + 2xy^2 - \frac{xy^3}{3} - \frac{x^2y^2}{2} \right. \\
&\quad \left. + 2y^2 - \frac{y^3}{3} \right]_0^{4-x^2} \\
&= \int_{-2}^2 \left(-x^2(4-x^2)^2 \sinh(2x) + 4(4-x^2)^2 \sinh(2x) - \frac{2(4-x^2)^3}{3} \sinh(2x) - \frac{x^3(4-x^2)^2}{2} \right.
\end{aligned}$$

$$\begin{aligned}
& +2x(4-x^2)^2 - \frac{x(4-x^2)^3}{3} - \frac{x^2(4-x^2)^2}{2} + 2(4-x^2)^2 - \frac{(4-x^2)^3}{3} \Big) dx \\
& = \int_{-2}^2 \left(-\frac{x^6}{3} \sinh(2x) + 4x^4 \sinh(2x) - 16x^2 \sinh(2x) + \frac{64}{3} \sinh(2x) - \frac{x^7}{6} - \frac{x^6}{6} + 2x^5 \right. \\
& \quad \left. + 2x^4 - 8x^3 - 8x^2 + \frac{32x}{3} + \frac{32}{3} \right) dx \\
& = \left(\left[-\frac{x^6}{6} \cosh(2x) + \frac{3x^5}{2} \sinh(2x) \right]_{-2}^2 - \frac{5}{2} \int_{-2}^2 x^4 \sinh(2x) dx \right) + 4 \int_{-2}^2 x^4 \sinh(2x) dx \\
& \quad - 16 \int_{-2}^2 x^2 \sinh(2x) dx + \frac{64}{3} \left[\frac{\cosh(2x)}{2} \right]_{-2}^2 + \left[-\frac{x^8}{48} - \frac{x^7}{42} + \frac{x^6}{3} + \frac{2x^5}{5} - 2x^4 - \frac{8x^3}{3} \right. \\
& \quad \left. + \frac{16x^2}{3} + \frac{32x}{3} \right]_{-2}^2 \\
& = [0] + \frac{3}{2} \int_{-2}^2 x^4 \sinh(2x) dx - 16 \int_{-2}^2 x^2 \sinh(2x) dx + \frac{64}{3} [0] + \frac{2048}{105} \\
& = \left(\left[\frac{3x^4}{4} \cosh(2x) - \frac{3x^3}{2} \sinh(2x) \right]_{-2}^2 + \frac{9}{2} \int_{-2}^2 x^2 \sinh(2x) dx \right) + \frac{2048}{105} \\
& = -\frac{23}{2} \int_{-2}^2 x^2 \sinh(2x) dx + \frac{2048}{105} \\
& = \left(\left[-\frac{23x^2}{4} \cosh(2x) + \frac{23x}{4} \sinh(2x) \right]_{-2}^2 - \frac{23}{4} \int_{-2}^2 \sinh(2x) dx \right) + \frac{2048}{105} \\
& = [0] - \frac{23}{4} \left[\frac{\cosh(2x)}{2} \right]_{-2}^2 + \frac{2048}{105} \\
& = -\frac{23}{4} [0] + \frac{2048}{105} \\
& = \frac{2048}{105}
\end{aligned}$$

whereby we applied the below reduction formula in line 10, 12 and 14. The reduction formula is the same as the one we used earlier except that the hyperbolic sine and cosine have swapped places.

$$\int x^{a+2} \sinh(bx) dx = \frac{x^{a+2}}{b} \cosh(bx) - \frac{(a+2)}{b^2} x^{a+1} \sinh(bx) + \frac{(a+1)(a+2)}{b^2} \int x^a \sinh(bx) dx$$

As we expected, both the left-hand side and right-hand side of the divergence theorem give the same result.

11 Methods for Solving Differential Equations

11.1 Definitions

A **differential equation** is an equation that contains the variable x , the function $f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto f(x) = y$, as well as the derivatives of f . Recall that the n^{th} derivative of f is written as $y^{(n)}$ or $\frac{d^n y}{dx^n}$.

We can express the differential equation as the following function g :

$$g : \mathbb{R}^{n+2} \rightarrow \mathbb{R} : (x, y, y', y'', \dots, y^{(n)}) \mapsto g(x, y, y', y'', \dots, y^{(n)}) = 0$$

The **order** of a differential equation is equal to the number that corresponds to the highest derivative present within the equation. The **degree** of a differential equation refers to the number that is equal to the exponent of the variable $y^{(n)}$.

For example, the differential equation $-xy'' + x^2y'^3 + y^2y''^2 = 0$ is a differential equation of order 3 and degree 2. Another example is the differential equation $5y' + 2xy^{(4)} + x^4y^2 = 0$, which is a differential equation of order 4 and degree 1.

In the sections 11.3, 11.4, 11.5 and 11.6, we discuss how differential equations can be solved.

11.2 Change of Variables

Given is a differential equation $g(x, y, y', y'', \dots, y^{(n)}) = 0$, whereby y is a function of x . We would like to rewrite the differential equation in terms of two new variables u and v , if we know that u is a function of v . The equations whereby we transform from one set of variables to the other are the following:

$$\begin{cases} x = \phi_1(u, v) \\ y = \phi_2(u, v) \end{cases}$$

Since u is a function of v , we start with differentiating the above equations with respect to v :

$$\begin{cases} \frac{dx}{dv} = \frac{\partial \phi_1}{\partial u} \frac{du}{dv} + \frac{\partial \phi_1}{\partial v} \frac{dv}{dv} = \frac{\partial \phi_1}{\partial u} \frac{du}{dv} + \frac{\partial \phi_1}{\partial v} \\ \frac{dy}{dv} = \frac{\partial \phi_2}{\partial u} \frac{du}{dv} + \frac{\partial \phi_2}{\partial v} \frac{dv}{dv} = \frac{\partial \phi_2}{\partial u} \frac{du}{dv} + \frac{\partial \phi_2}{\partial v} \end{cases}$$

We can now find the following expression for the first derivative of y (with respect to x):

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx} = \frac{dy}{dv} \left(\frac{dx}{dv} \right)^{-1} = \left(\frac{\partial \phi_2}{\partial u} \frac{du}{dv} + \frac{\partial \phi_2}{\partial v} \right) \left(\frac{\partial \phi_1}{\partial u} \frac{du}{dv} + \frac{\partial \phi_1}{\partial v} \right)^{-1}$$

Regarding the second derivative of y , we first write out the second-order derivatives of the two equations of transformation, using the chain rule:

$$\begin{cases} \frac{d^2x}{dv^2} = \frac{\partial^2\phi_1}{\partial u^2} \left(\frac{du}{dv}\right)^2 + 2\frac{\partial^2\phi_1}{\partial u\partial v} \frac{du}{dv} + \frac{\partial\phi_1}{\partial u} \frac{d^2u}{dv^2} + \frac{\partial^2\phi_1}{\partial v^2} \\ \frac{d^2y}{dv^2} = \frac{\partial^2\phi_2}{\partial u^2} \left(\frac{du}{dv}\right)^2 + 2\frac{\partial^2\phi_2}{\partial u\partial v} \frac{du}{dv} + \frac{\partial\phi_2}{\partial u} \frac{d^2u}{dv^2} + \frac{\partial^2\phi_2}{\partial v^2} \end{cases}$$

An expression for the second derivative of y (with respect to x) is found as follows:

$$\begin{aligned} \frac{d}{dv} \left(\frac{dy}{dv} \right) &= \frac{d}{dv} \left(\frac{dy}{dx} \frac{dx}{dv} \right) \Leftrightarrow \frac{d^2y}{dv^2} = \frac{d^2y}{dx^2} \left(\frac{dx}{dv} \right)^2 + \frac{dy}{dx} \frac{d^2x}{dv^2} \\ &\Leftrightarrow \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dv^2} - \frac{dy}{dx} \frac{d^2x}{dv^2} \right) \left(\frac{dx}{dv} \right)^{-2} \end{aligned}$$

The derivations of higher derivatives are found in a similar way. When we substitute these expressions for the derivatives of y (with respect to x) in our original differential equation $g(x, y, y', y'', \dots, y^{(n)}) = 0$, we find a differential equation expressed in terms of u and v , whereby u is a function of v .

- Example 1

For a function y of x , we are given the differential equation $y'' = 0$. We would like to write this equation in terms of the variables u and v , considering that u is a function of v , with the following transformation between the variables:

$$\begin{cases} x = \cos u + \sin v \\ y = \sin u + \cos v \end{cases}$$

The first and second derivatives of these equations with respect to v are:

$$\begin{cases} \frac{dx}{dv} = -\sin(u) \frac{du}{dv} + \cos v \\ \frac{dy}{dv} = \cos(u) \frac{du}{dv} - \sin v \\ \frac{d^2x}{dv^2} = -\cos(u) \left(\frac{du}{dv}\right)^2 - \sin(u) \frac{d^2u}{dv^2} - \sin v \\ \frac{d^2y}{dv^2} = -\sin(u) \left(\frac{du}{dv}\right)^2 + \cos(u) \frac{d^2u}{dv^2} - \cos v \end{cases}$$

The first derivative of y (with respect to x) is equal to:

$$\frac{dy}{dx} = \frac{dy}{dv} \left(\frac{dx}{dv} \right)^{-1} = \frac{\cos(u) \frac{du}{dv} - \sin v}{-\sin(u) \frac{du}{dv} + \cos v}$$

The second derivative of y (with respect to x) is found as follows:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \left(\frac{d^2y}{dv^2} - \frac{dy}{dx} \frac{d^2x}{dv^2} \right) \left(\frac{dx}{dv} \right)^{-2} \\ &= \frac{d^2y}{dv^2} \left(\frac{dx}{dv} \right)^{-2} - \frac{dy}{dx} \frac{d^2x}{dv^2} \left(\frac{dx}{dv} \right)^{-2} \\ &= \left[-\sin(u) \left(\frac{du}{dv} \right)^2 + \cos(u) \frac{d^2u}{dv^2} - \cos v \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-2} \\ &\quad - \left[\frac{\cos(u) \frac{du}{dv} - \sin v}{-\sin(u) \frac{du}{dv} + \cos v} \right] \left[-\cos(u) \left(\frac{du}{dv} \right)^2 - \sin(u) \frac{d^2u}{dv^2} - \sin v \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-2} \\ &= \left[-\sin(u) \left(\frac{du}{dv} \right)^2 + \cos(u) \frac{d^2u}{dv^2} - \cos v \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-3} \\ &\quad - \left[\cos(u) \frac{du}{dv} - \sin v \right] \left[-\cos(u) \left(\frac{du}{dv} \right)^2 - \sin(u) \frac{d^2u}{dv^2} - \sin v \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-3} \\ &= \left[\sin^2(u) \left(\frac{du}{dv} \right)^3 - \sin(u) \cos(v) \left(\frac{du}{dv} \right)^2 - \sin(u) \cos(u) \left(\frac{du}{dv} \right) \left(\frac{d^2u}{dv^2} \right) \right. \\ &\quad \left. + \cos(u) \cos(v) \left(\frac{d^2u}{dv^2} \right) + \sin(u) \cos(v) \left(\frac{du}{dv} \right) - \cos^2(v) + \cos^2(u) \left(\frac{du}{dv} \right)^3 \right. \\ &\quad \left. + \sin(u) \cos(u) \left(\frac{du}{dv} \right) \left(\frac{d^2u}{dv^2} \right) + \sin(v) \cos(u) \left(\frac{du}{dv} \right) - \sin(v) \cos(u) \left(\frac{du}{dv} \right)^2 \right. \\ &\quad \left. - \sin(u) \sin(v) \left(\frac{d^2u}{dv^2} \right) - \sin^2(v) \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-3} \\ &= \left[\left(\frac{du}{dv} \right)^3 + \sin(u+v) \left(\frac{du}{dv} \right) \left(1 - \frac{du}{dv} \right) + \cos(u+v) \left(\frac{d^2u}{dv^2} \right) - 1 \right] \left[-\sin(u) \frac{du}{dv} + \cos v \right]^{-3} \end{aligned}$$

The differential equation $y'' = 0$ in terms of the new variables u and v then becomes:

$$\left(\frac{du}{dv} \right)^3 + \sin(u+v) \left(\frac{du}{dv} \right) \left(1 - \frac{du}{dv} \right) + \cos(u+v) \left(\frac{d^2u}{dv^2} \right) - 1 = 0$$

- Example 2

We would like to write the differential equation $2yy' - xy'^2 + xyy'' = 0$ in terms of the variables u and v , given that y is a function of x and v is a function of u with the following relations between the two sets of variables:

$$\begin{cases} u = e^{-2x} \\ v = 2x \ln y \end{cases}$$

In a first step, we consider the derivative of v with respect to x :

$$\frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx} \Leftrightarrow 2 \ln y + \frac{2x}{y} \frac{dy}{dx} = \frac{dv}{du} (-2e^{-2x})$$

If we differentiate the above equation again with respect to the variable x , we obtain the following expression:

$$\begin{aligned} \frac{d}{dx} \left[2 \ln y + \frac{2x}{y} \frac{dy}{dx} \right] &= \frac{d}{dx} \left[\frac{dv}{du} (-2e^{-2x}) \right] \\ \Leftrightarrow \frac{2}{y} \frac{dy}{dx} + \left(\frac{2y - 2x \frac{dy}{dx}}{y^2} \right) \frac{dy}{dx} + \frac{2x}{y} \frac{d^2y}{dx^2} &= \frac{d^2v}{du^2} \frac{du}{dx} (-2e^{-2x}) + \frac{dv}{du} (4e^{-2x}) \\ \Leftrightarrow \frac{2y'}{y} + \frac{2y'}{y} - \frac{2xy'^2}{y^2} + \frac{2xy''}{y} &= \frac{d^2v}{du^2} (-2e^{-2x}) (-2e^{-2x}) + \frac{dv}{du} (4e^{-2x}) \\ \Leftrightarrow 4yy' - 2xy'^2 + 2xxy'' &= 4y^2 e^{-4x} \frac{d^2v}{du^2} + 4y^2 e^{-2x} \frac{dv}{du} \end{aligned}$$

Given that the differential equation is equal to $2yy' - xy'^2 + xyy'' = 0$, we find that the differential equation in terms of the variables u and v has the following form:

$$4y^2 e^{-4x} \frac{d^2v}{du^2} + 4y^2 e^{-2x} \frac{dv}{du} = 0 \Leftrightarrow u \frac{d^2v}{du^2} + \frac{dv}{du} = 0 \Leftrightarrow uu'' + u' = 0$$

- Example 3

In a final example, we wish to explore how a differential equation changes when we switch the role of the variables x and y . Consider the differential equation $5^x y^2 y' - 5xy'' + yy''' \ln x = 0$ whereby y is a function of x . We implement the following transformations, given that u is a function of v :

$$\begin{cases} x = u \\ y = v \end{cases}$$

The first derivative of y (with respect to x) is calculated as follows:

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx} = \left(\frac{dx}{dv} \right)^{-1}$$

The second derivative of y (with respect to x) is found by differentiating the first derivative with respect to v :

$$\begin{aligned} \frac{d}{dv} \left[\frac{dy}{dx} \right] &= \frac{d}{dv} \left[\left(\frac{dx}{dv} \right)^{-1} \right] \Leftrightarrow \frac{d^2 y}{dx^2} \frac{dx}{dv} = - \left(\frac{dx}{dv} \right)^{-2} \frac{d^2 x}{dv^2} \\ &\Leftrightarrow \frac{d^2 y}{dx^2} = - \frac{d^2 x}{dv^2} \left(\frac{dx}{dv} \right)^{-3} \end{aligned}$$

Differentiating the second derivative with respect to v gives us the third derivative of y (with respect to x):

$$\begin{aligned} \frac{d}{dv} \left[\frac{d^2 y}{dx^2} \right] &= \frac{d}{dv} \left[- \frac{d^2 x}{dv^2} \left(\frac{dx}{dv} \right)^{-3} \right] \Leftrightarrow \frac{d^3 y}{dx^3} \frac{dx}{dv} = - \frac{d^3 x}{dv^3} \left(\frac{dx}{dv} \right)^{-3} + 3 \frac{d^2 x}{dv^2} \left(\frac{dx}{dv} \right)^{-4} \frac{d^2 x}{dv^2} \\ &\Leftrightarrow \frac{d^3 y}{dx^3} = - \frac{d^3 x}{dv^3} \left(\frac{dx}{dv} \right)^{-4} + 3 \left(\frac{d^2 x}{dv^2} \right)^2 \left(\frac{dx}{dv} \right)^{-5} \end{aligned}$$

The differential equation then obtains the following form:

$$\begin{aligned} 5^x y^2 y' - 5x y'' + y y''' \ln x &= 0 \\ \Leftrightarrow 5^u v^2 \left(\frac{dx}{dv} \right)^{-1} + 5u \frac{d^2 x}{dv^2} \left(\frac{dx}{dv} \right)^{-3} - v \ln(u) \frac{d^3 x}{dv^3} \left(\frac{dx}{dv} \right)^{-4} + 3v \ln(u) \left(\frac{d^2 x}{dv^2} \right)^2 \left(\frac{dx}{dv} \right)^{-5} &= 0 \\ \Leftrightarrow 5^u v^2 \left(\frac{dx}{dv} \right)^4 + 5u \frac{d^2 x}{dv^2} \left(\frac{dx}{dv} \right)^2 - v \ln(u) \frac{d^3 x}{dv^3} \frac{dx}{dv} + 3v \ln(u) \left(\frac{d^2 x}{dv^2} \right)^2 &= 0 \\ \Leftrightarrow 5^x y^2 x'^4 + 5x x'' x'^2 - y \ln(x) x''' x' + 3y \ln(x) x''^2 &= 0 \end{aligned}$$

whereby in the last line we switched back to the x and y variables. Note that, for instance, the expression x' represents the first derivative of x with respect to y , so we have effectively swapped the roles of x and y .

11.3 Differential Equations of Order 1 and Degree 1

In this section, we will discuss differential equations of the general form $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ that are linear with respect to the variable y' , i.e., the degree of the differential equation is equal to 1.

11.3.1 Total Differential Equations

Consider a scalar function $f : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y) \mapsto z = f(x, y)$. As we discussed in section 10.2.2, the total differential of f is equal to:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Recall that, whereas a partial derivative of f can be used to linearly approximate f in one particular dimension, the total differential can provide us with such linear approximation across all the respective dimensions taken together (in our case, the dimensions x and y).

In the reverse case, whereby we are given the expression $P_1(x, y)dx + P_2(x, y)dy$, this expression represents a total differential if and only if:

$$\frac{\partial P_1}{\partial y} = \frac{\partial P_2}{\partial x}$$

In this case, we call the differential equation $P_1(x, y)dx + P_2(x, y)dy = 0$ the **total differential equation**. The solution to this total differential equation is then equal to:

$$f(x, y) = a \quad \text{with } a \in \mathbb{R}$$

- Example

Consider the differential equation $(x^3y^2 - xye^{x^2}) dx + \left(\frac{x^4y}{2} - \frac{e^{x^2}}{2}\right) dy = 0$. If this differential equation is a total differential equation, we wish to find its solution.

In a first step, we determine whether the condition $\frac{\partial P_1}{\partial y} = \frac{\partial P_2}{\partial x}$ is satisfied:

$$\begin{cases} \frac{\partial P_1}{\partial y} = 2yx^3 - xe^{x^2} \\ \frac{\partial P_2}{\partial x} = 2x^3y - xe^{x^2} \end{cases}$$

Since the condition has been met, we are dealing with a total differential equation. In a next step, let us take, for instance, $P_2(x, y)$ and integrate this scalar function with respect to y to obtain a general expression for the function $z = f(x, y)$:

$$\frac{\partial z}{\partial y} = P_2(x, y) = \frac{x^4y}{2} - \frac{e^{x^2}}{2} \Leftrightarrow z = \int \left(\frac{x^4y}{2} - \frac{e^{x^2}}{2} \right) dy$$

$$= \frac{x^4 y^2}{4} - \frac{y e^{x^2}}{2} + c(x)$$

Note that the integration constant $c(x)$ is a function of x as we are integrating with respect to y ; as $c(x)$ does not depend on y , differentiating this constant with respect to y gives zero as a result, which is the definition of integrating a constant.

Next, we differentiate the obtained expression for z with respect to the other variable x :

$$\frac{\partial z}{\partial x} = x^3 y^2 - x y e^{x^2} + c'(x)$$

Since $\frac{\partial z}{\partial x}$ is also equal to $P_1(x, y)$ we can set both expressions equal to each other:

$$\begin{aligned} \frac{\partial z}{\partial x} = P_1(x, y) &\Leftrightarrow x^3 y^2 - x y e^{x^2} + c'(x) = x^3 y^2 - x y e^{x^2} \\ &\Leftrightarrow c'(x) = 0 \\ &\Leftrightarrow c(x) = c \quad \text{with } c \in \mathbb{R} \end{aligned}$$

In a final step, we can write down the solution $z = f(x, y) = a$:

$$\frac{x^4 y^2}{4} - \frac{y e^{x^2}}{2} + c(x) = a \Leftrightarrow \frac{x^4 y^2}{4} - \frac{y e^{x^2}}{2} + c = a$$

11.3.2 Separation of Variables

If the differential equation $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ can be written in the following way:

$$p_1(x) dx = p_2(y) dy$$

then the solution to $g(x, y, y') = 0$ is found by directly integrating both sides:

$$\int p_1(x) dx = \int p_2(y) dy$$

- Example

Consider the following differential equation:

$$\left(\frac{\tanh x + 1}{\coth y + 1} \right) y' - \frac{\operatorname{sech} x}{\operatorname{csch} y} = 0$$

We solve this differential equation as follows:

$$\begin{aligned}
\left(\frac{\tanh x + 1}{\coth y + 1}\right) y' - \frac{\operatorname{sech} x}{\operatorname{csch} y} = 0 &\Leftrightarrow \frac{\operatorname{csch} y}{\coth y + 1} dy = \frac{\operatorname{sech} x}{\tanh x + 1} dx \\
&\Leftrightarrow \frac{1}{\cosh y + \sinh y} dy = \frac{1}{\cosh x + \sinh x} dx \\
&\Leftrightarrow \int e^{-y} dy = \int e^{-x} dx \\
&\Leftrightarrow -e^{-y} + c_1 = -e^{-x} + c_2 \\
&\Leftrightarrow e^y = \frac{e^x}{1 + (c_1 - c_2)e^x} \\
&\Leftrightarrow y = x - \ln |1 + (c_1 - c_2)e^x|
\end{aligned}$$

whereby the integration constants c_1 and c_2 are real numbers.

11.3.3 Homogenous Differential Equations

Consider two homogeneous polynomials $H_1(x, y)$ and $H_2(x, y)$ of degree m —remember that in a homogeneous polynomial every term has the same degree (see section 2.3). A **homogeneous differential equation** has the following form:

$$H_1(x, y) dx - H_2(x, y) dy = 0$$

To solve this differential equation, we follow three steps:

1. Divide the homogeneous differential equation by x^m
2. Implement the substitution $u = \frac{y}{x}$, whereby $dy = u dx + x du$
3. Solve the new differential equation through the method of separation of variables

- Example

Let us consider the homogeneous differential equation $(3x^3y^2 - xy^4) dx + (x^2y^3 + 2x^5) dy = 0$, whereby the polynomials $H_1(x, y)$ and $H_2(x, y)$ are homogeneous polynomials of degree $m = 5$.

We find the solution as follows:

$$\begin{aligned}
(3x^3y^2 - xy^4) dx + (x^2y^3 + 2x^5) dy &= 0 \\
(\text{Step 1}) \quad &\Leftrightarrow \left(\frac{3y^2}{x^2} - \frac{y^4}{x^4}\right) dx + \left(\frac{y^3}{x^3} + 2\right) dy = 0 \\
(\text{Step 2}) \quad &\Leftrightarrow (3u^2 - u^4) dx + (u^3 + 2)(u dx + x du) = 0 \\
&\Leftrightarrow (3u^2 + 2u) dx + x(u^3 + 2) du = 0
\end{aligned}$$

$$\begin{aligned}
(\text{Step 3}) \quad &\Leftrightarrow - \int \frac{1}{x} dx = \int \frac{u^3 + 2}{3u^2 + 2u} du \\
&\Leftrightarrow \ln \left| \frac{1}{x} \right| + c_1 = \int \left(\frac{u}{3} - \frac{2}{9} + \frac{\frac{4}{9}u + 2}{3u^2 + 2u} \right) du \\
&\Leftrightarrow \ln \left| \frac{1}{x} \right| + c_1 = \int \left(\frac{u}{3} - \frac{2}{9} + \frac{6u + 2}{3u^2 + 2u} - \frac{50}{9} \frac{1}{(3u + 2)} \right) du \\
&\Leftrightarrow \ln \left| \frac{1}{x} \right| + c_1 = \frac{u^2}{6} - \frac{2u}{9} + \ln |3u^2 + 2u| - \frac{50}{27} \ln |3u + 2| + c_2 \\
&\Leftrightarrow \ln \left| \frac{1}{x} \right| + c_1 = \frac{y^2}{6x^2} - \frac{2y}{9x} + \ln \left| \frac{3y^2}{x^2} + \frac{2y}{x} \right| - \frac{50}{27} \ln \left| \frac{3y}{x} + 2 \right| + c_2
\end{aligned}$$

whereby we applied partial fraction decomposition in line 6 and we have split in line 7 the last term of the integral of line 6 into two fractions.

Apart from a general solution, a differential equation can also have **singular solutions**, which are unique solutions that satisfy the differential equation but cannot be obtained through the general solution regardless of the value of the integration constant(s). In our example, the homogeneous differential equation has three singular solutions: $x = 0$, $y = 0$, and $y = -\frac{2x}{3}$.

11.3.4 Reducible to Homogenous Differential Equations

If a differential equation is of the form $S_1(x, y) dx + S_2(x, y) dy = 0$ whereby $S_1(x, y) = 0$ and $S_2(x, y) = 0$ represent the equation of a straight line in two-dimensional space and at least one of these two lines does not go through the origin of the coordinate system, we can reduce this differential equation to a homogeneous differential equation.

To solve this type of differential equation we distinguish two cases: the lines S_1 and S_2 intersect or the lines are parallel to each other.

Case A. S_1 and S_2 intersect

Suppose that the straight lines S_1 and S_2 intersect at the point $\vec{a} = (a_1, a_2)$. To solve the differential equation, we introduce the following transformation of coordinates, which represents a translation of both lines in such a way that they now go through the origin and become homogeneous:

$$\begin{cases} x = u + a_1 \\ y = v + a_2 \end{cases}$$

After this change of variables, we can apply the method for homogeneous differential equations, as discussed in the previous subsection.

- Example

Let us consider the differential equation $(5x - 2y + 2) dx + (2x + 5y - 34) dy = 0$. Since the two straight lines $5x - 2y + 2 = 0$ and $2x + 5y - 34 = 0$ intersect at the point $\vec{a} = (2, 6)$, we implement the following transformation of coordinates:

$$\begin{cases} x = u + 2 \\ y = v + 6 \end{cases}$$

We now solve this differential equation as follows:

$$\begin{aligned} & (5x - 2y + 2) dx + (2x + 5y - 34) dy = 0 \\ \Leftrightarrow & [5(u + 2) - 2(v + 6) + 2] du + [2(u + 2) + 5(v + 6) - 34] dv = 0 \\ \Leftrightarrow & (5u - 2v) du + (2u + 5v) dv = 0 \\ \Leftrightarrow & \left(5 - 2\frac{v}{u}\right) du + \left(2 + 5\frac{v}{u}\right) dv = 0 \\ \Leftrightarrow & (5 - 2t) du + (2 + 5t)(t du + u dt) = 0 \\ \Leftrightarrow & 5(1 + t^2) du + u(2 + 5t) dt = 0 \\ \Leftrightarrow & \frac{2 + 5t}{5(t^2 + 1)} dt = -\frac{1}{u} du \\ \Leftrightarrow & \frac{2}{5} \int \frac{1}{t^2 + 1} dt + \int \frac{t}{t^2 + 1} dt = \ln \left| \frac{1}{u} \right| + c_1 \\ \Leftrightarrow & \frac{2}{5} \arctan(t) + \frac{1}{2} \ln(t^2 + 1) + c_2 = \ln \left| \frac{1}{u} \right| + c_1 \\ \Leftrightarrow & \frac{2}{5} \arctan \left(\frac{y - 6}{x - 2} \right) + \frac{1}{2} \ln \left[\left(\frac{y - 6}{x - 2} \right)^2 + 1 \right] + c_2 = \ln \left| \frac{1}{x - 2} \right| + c_1 \end{aligned}$$

whereby we applied the method for solving homogeneous differential equations in line 5 with the corresponding substitutions $t = \frac{v}{u}$ and $dv = t du + u dt$. Note that the point $\vec{a} = (2, 6)$ is a singular solution.

Case B. S_1 and S_2 are parallel

Suppose that $S_1(x, y) = a_1x + b_1y + d_1$ and $S_2(x, y) = a_2x + b_2y + d_2$. If the lines $S_1(x, y) = 0$ and $S_2(x, y) = 0$ are parallel, it means that their slope is equal so that we can write:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

We now follow these two steps to solve this differential equation:

1. Using the above equation, we write $S_1(x, y)$ as $S_1(x, y) = a_1x + b_1y + d_1 = \frac{a_1}{a_2}(a_2x + \frac{a_2b_1}{a_1}y) + d_1 = \frac{a_1}{a_2}(a_2x + b_2y) + d_1$.
2. Introduce the substitution $t = a_2x + b_2y$ in both $S_1(x, y)$ and $S_2(x, y)$.

- Example

Consider the differential equation $(-6x + y - 9) dx + (12x - 2y - 70) dy = 0$. Given that the lines $-6x + y - 9 = 0$ and $12x - 2y - 70 = 0$ are parallel and that $\frac{a_1}{a_2} = -\frac{1}{2}$, we solve this differential equation in the following way:

$$\begin{aligned}
 & (-6x + y - 9) dx + (12x - 2y - 70) dy = 0 \\
 \Leftrightarrow & \left[-\frac{1}{2}(12x - 2y) - 9 \right] dx + (12x - 2y - 70) dy = 0 \\
 \Leftrightarrow & \left(-\frac{t}{2} - 9 \right) dx + (t - 70) \left(6 dx - \frac{dt}{2} \right) = 0 \\
 \Leftrightarrow & \left(\frac{11t}{2} - 429 \right) dx - \frac{1}{2}(t - 70) dt = 0 \\
 \Leftrightarrow & -2 dx = \frac{t - 70}{\frac{11t}{2} - 429} dt \\
 \Leftrightarrow & -2 \int dx = \int \left(\frac{2}{11} + \frac{8}{\frac{11t}{2} - 429} \right) dt \\
 \Leftrightarrow & -2x + c_1 = \frac{2t}{11} + \frac{16}{11} \ln \left| \frac{11t}{2} - 429 \right| + c_2 \\
 \Leftrightarrow & -2x + c_1 = \frac{2}{11}(12x - 2y) + \frac{16}{11} \ln \left| \frac{11}{2}(12x - 2y) - 429 \right| + c_2
 \end{aligned}$$

whereby we introduced the substitutions $t = 12x - 2y$ and $dy = 6 dx - \frac{dt}{2}$ in line 3 and applied long division in line 6. This differential equation also has the singular solution $y = 6x - 39$.

11.3.5 Linear Differential Equations

A differential equation $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ that is linear with respect to both the variable y and y' is called a **linear differential equation** and has the following general form:

$$a(x)y' + b(x)y = k(x)$$

If we are unable to appeal to the method of separation of variables, we can rely on another method whereby we make use of the fact that we can write the general solution y of the linear differential equation as the sum of a homogeneous solution y_h and a particular solution y_p : $y = y_h + y_p$.

This method involves two steps:

1. Turn the original differential equation into a homogeneous differential equation by setting $k(x) = 0$ and solve the new differential equation through the method of separation of variables. The homogeneous solution y_h will be of the form $y_h = cf(x)$, with $c \in \mathbb{R}$ the integration constant.
2. The particular solution y_p is found through the *method of variation of constants*, which says that y_p has the form $y_p = c(x)f(x)$, whereby $c(x) = \int \frac{k(x)}{a(x)f(x)} dx$.

- Example

Consider the differential equation $(y \sin x + 2x + 1) dx - \cos(x) dy = 0$ for which we wish to find all the solutions.

Since the method of separation of variables cannot be used here, we first check whether this differential equation is a linear differential equation:

$$(y \sin x + 2x + 1) dx - \cos(x) dy = 0 \Leftrightarrow \cos(x)y' - \sin(x)y = 2x + 1$$

As we are indeed dealing with a linear differential equation, we apply the method as discussed above. In a first step, we solve the corresponding homogeneous differential equation:

$$\begin{aligned} \cos(x)y' - \sin(x)y = 0 &\Leftrightarrow \frac{1}{y} dy = \frac{\sin x}{\cos x} dx \Leftrightarrow \int \frac{1}{y} dy = \int \frac{\sin x}{\cos x} dx \\ &\Leftrightarrow \ln(y) + c_1 = - \int \frac{1}{u} du \\ &\Leftrightarrow \ln(y) + c_1 = \ln \left| \frac{1}{u} \right| + c_2 \\ &\Leftrightarrow \ln(y) + c_1 = \ln |\sec x| + c_2 \\ &\Leftrightarrow y_h = c \sec x \end{aligned}$$

whereby we introduced the substitution $u = \cos x$ in line 2 and in the last line we replaced the constant $e^{c_2 - c_1}$ by the constant c .

In a next step, we wish to find the particular solution $y_p = c(x)f(x)$. First, we calculate the term $c(x)$, whereby $f(x) = \sec x$:

$$c(x) = \int \frac{k(x)}{a(x)f(x)} dx = \int \frac{2x+1}{\cos(x)\sec(x)} dx = \int (2x+1) dx = x^2 + x$$

The particular solution y_p is then equal to:

$$y_p = c(x)f(x) = (x^2 + x) \sec x$$

So that the general solution is found to be:

$$y = y_h + y_p = c \sec x + (x^2 + x) \sec x = (x^2 + x + c) \sec x$$

Note that this linear differential equation also has an infinite number of singular solutions equal to $x = \frac{\pi}{2} + k\pi$ with $k \in \mathbb{Z}$.

11.3.6 Bernoulli Differential Equations

A **Bernoulli differential equation** has the following form:

$$a(x)y' + b(x)y = d(x)y^r$$

If $r = 0$ or $r = 1$ the differential equation is linear and we can apply the method discussed in the previous subsection. If $r \in \mathbb{Z}_0 \setminus \{1\}$ we follow these two steps to solve the Bernoulli differential equation:

1. Linearize the Bernoulli differential equation by implementing the substitution $t = y^{1-r}$.
2. Apply the method discussed in the previous subsection for solving the new (linear) differential equation.

- Example

Consider the Bernoulli differential equation $y' + [1 + \ln(3x)]y = (3x)^{2x}y^3$. In a first step, we introduce the substitution $t = y^{1-r} = y^{1-3} = y^{-2}$, whereby $dy = -\frac{1}{2t\sqrt{t}} dt$, so that the differential equation becomes:

$$\begin{aligned} y' + [1 + \ln(3x)]y = (3x)^{2x}y^3 &\Leftrightarrow -\frac{1}{2t\sqrt{t}}t' + [1 + \ln(3x)]\frac{1}{\sqrt{t}} = (3x)^{2x}\frac{1}{t\sqrt{t}} \\ &\Leftrightarrow -\frac{t'}{2} + [1 + \ln(3x)]t = (3x)^{2x} \end{aligned}$$

We now are dealing with a linear differential equation for which the solution has the form $t = t_h + t_p$ according to the method as discussed in section 11.3.5. We find the homogeneous solution $t_h = cf(x)$ as follows:

$$\begin{aligned}
-\frac{t'}{2} + [1 + \ln(3x)]t = 0 &\Leftrightarrow [1 + \ln(3x)] dx = \frac{1}{2t} dt \\
&\Leftrightarrow \int [1 + \ln(3x)] dx = \int \frac{1}{2t} dt \\
&\Leftrightarrow \int dx + \int \ln(3x) dx = \frac{1}{2} \ln |t| + c_1 \\
&\Leftrightarrow x + c_2 + \left(x \ln(3x) - \int dx \right) = \frac{1}{2} \ln |t| + c_1 \\
&\Leftrightarrow x + c_2 + x \ln(3x) - x = \frac{1}{2} \ln |t| + c_1 \\
&\Leftrightarrow 2x \ln(3x) + 2(c_2 - c_1) = \ln |t| \\
&\Leftrightarrow e^{2x \ln(3x)} e^{2(c_2 - c_1)} = t_h \\
&\Leftrightarrow (3x)^{2x} c = t_h
\end{aligned}$$

whereby we applied partial integration in line 4 and we replaced in the last line the constant expression $e^{2(c_2 - c_1)}$ by the integration constant c . Also remember that $x^x = e^{\ln(x^x)} = e^{x \ln x}$.

The particular solution t_p has the form $t_p = c(x)f(x)$. Let us now find the term $c(x)$:

$$c(x) = \int \frac{k(x)}{a(x)f(x)} dx = \int \frac{(3x)^{2x}}{-\frac{1}{2}(3x)^{2x}} dx = -2 \int dx = -2x$$

The particular solution is therefore equal to $t_p = c(x)f(x) = -2x(3x)^{2x}$ so that the general solution becomes:

$$t = t_h + t_p = (3x)^{2x} c - 2x(3x)^{2x} = (c - 2x)(3x)^{2x}$$

If we change back to the original y variable, the solution is equal to:

$$t = y^{-2} \Leftrightarrow y = \pm \frac{(3x)^{-x}}{\sqrt{c - 2x}}$$

Note that we only find solutions if $0 < x < \frac{c}{2}$ with $c \in \mathbb{R}_0^+$. Finally, the Bernoulli differential equation also has the singular solution $y = 0$.

11.4 Differential Equations of Order 1 and Degree n

In this section, we discuss differential equations of the general form $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ that are not necessarily linear with respect to the variable y' , i.e., we allow the power of y' to be greater than 1.

11.4.1 Factoring Differential Equations

If the differential equation $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ can be factored and written as:

$$g(x, y, y') = 0 \Leftrightarrow [y' - f_1(x, y)] [y' - f_2(x, y)] [y' - f_3(x, y)] \dots [y' - f_n(x, y)] = 0$$

whereby $n \in \mathbb{N}_0$, the solution of the differential equation has the form of:

$$F_1(x, y, d_1) F_2(x, y, d_2) F_3(x, y, d_3) \dots F_n(x, y, d_n) = 0$$

whereby d_i is the aggregate integration constant of the solution F_i (with $i \in \{1, 2, \dots, n\}$).

- Example

Let us consider the following differential equation:

$$\begin{aligned} &2y^4 y'^2 - x^3 y^2 y' + 2y^5 y' \sin(x) \cos(2x) - x^3 y^3 \sin(x) \cos(2x) - 2xy^2 y'^3 + x^4 y'^2 \\ &- 2xy^3 y'^2 \sin(x) \cos(2x) + x^4 y y' \sin(x) \cos(2x) = 0 \end{aligned}$$

We can factor this equation as follows:

$$\begin{aligned} &2y^4 y'^2 - x^3 y^2 y' + 2y^5 y' \sin(x) \cos(2x) - x^3 y^3 \sin(x) \cos(2x) - 2xy^2 y'^3 + x^4 y'^2 \\ &- 2xy^3 y'^2 \sin(x) \cos(2x) + x^4 y y' \sin(x) \cos(2x) = 0 \\ \Leftrightarrow &2y^2 y'^2 (y^2 - xy') - x^3 y' (y^2 - xy') + y(2y^4 y' - x^3 y^2 - 2xy^2 y'^2 + x^4 y') \sin(x) \cos(2x) = 0 \\ \Leftrightarrow &y'(2y^2 y' - x^3)(y^2 - xy') + y[2y^2 y'(y^2 - xy') - x^3(y^2 - xy')] \sin(x) \cos(2x) = 0 \\ \Leftrightarrow &(y^2 - xy') [y'(2y^2 y' - x^3) + y(2y^2 y' - x^3) \sin(x) \cos(2x)] = 0 \\ \Leftrightarrow &(y^2 - xy')(2y^2 y' - x^3) [y' + y \sin(x) \cos(2x)] = 0 \\ \Leftrightarrow &g_1(x, y, y') g_2(x, y, y') g_3(x, y, y') = 0 \end{aligned}$$

whereby in the last line we defined the three individual differential equations g_1 , g_2 , and g_3 as $g_1(x, y, y') = y^2 - xy'$, $g_2(x, y, y') = 2y^2 y' - x^3$, and $g_3(x, y, y') = y' + y \sin(x) \cos(2x)$.

With respect to $g_1(x, y, y') = y^2 - xy'$, we solve this equation as follows:

$$\begin{aligned}
 y^2 - xy' = 0 &\Leftrightarrow \frac{1}{x} dx = \frac{1}{y^2} dy \Leftrightarrow \int \frac{1}{x} dx = \int \frac{1}{y^2} dy \\
 &\Leftrightarrow \ln|x| + c_1 = -\frac{1}{y} + c_2 \\
 &\Leftrightarrow \frac{1}{y} = c_2 - c_1 - \ln|x| \\
 &\Leftrightarrow y = \frac{1}{d_1 - \ln|x|}
 \end{aligned}$$

whereby in the last line we introduced the substitution $d_1 = c_2 - c_1$ with $d_1 \in \mathbb{R}$ and whereby $d_1 \neq \ln|x|$. This differential equation has the singular solutions $y = 0$ and the point $(0, 0)$.

Regarding $g_2(x, y, y') = 2y^2y' - x^3$, we find the following solution:

$$\begin{aligned}
 2y^2y' - x^3 = 0 &\Leftrightarrow x^3 dx = 2y^2 dy \Leftrightarrow \int x^3 dx = 2 \int y^2 dy \\
 &\Leftrightarrow \frac{x^4}{4} + c_1 = \frac{2y^3}{3} + c_2 \\
 &\Leftrightarrow y = \sqrt[3]{\frac{3}{2} \left(\frac{x^4}{4} + d_2 \right)}
 \end{aligned}$$

whereby we implemented the substitution $d_2 = c_1 - c_2$ in the last line.

Finally, we find the solution of the third differential equation $g_3(x, y, y') = y' + y \sin(x) \cos(2x)$ in the following way:

$$\begin{aligned}
 y' + y \sin(x) \cos(2x) = 0 &\Leftrightarrow -\frac{1}{y} dy = \sin(x) \cos(2x) dx \\
 &\Leftrightarrow -\int \frac{1}{y} dy = \int \sin(x) \cos(2x) dx \\
 &\Leftrightarrow -\ln|y| + c_1 = \frac{1}{2} \int \sin(-x) dx + \frac{1}{2} \int \sin(3x) dx \\
 &\Leftrightarrow -\ln|y| + c_1 = \frac{1}{2} \cos(x) - \frac{1}{6} \cos(3x) + c_2 \\
 &\Leftrightarrow y = e^{-\frac{1}{2} \cos(x)} e^{\frac{1}{6} \cos(3x)} e^{c_1 - c_2} \\
 &\Leftrightarrow y = d_3 e^{-\frac{1}{2} \cos(x)} e^{\frac{1}{6} \cos(3x)}
 \end{aligned}$$

whereby we implemented the substitution $d_3 = e^{c_1 - c_2}$ in the last line. This differential equation has the singular solution $y = 0$.

The general solution of this differential equation is then equal to:

$$\left(y - \frac{1}{d_1 - \ln|x|}\right) \left(y - \sqrt[3]{\frac{3}{2} \left(\frac{x^4}{4} + d_2\right)}\right) \left(y - d_3 e^{-\frac{1}{2} \cos(x)} e^{\frac{1}{6} \cos(3x)}\right) = 0$$

11.4.2 Parameterization of Differential Equations

If the differential equation $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ cannot be factored, we can check whether g falls within four specific cases: the variable x is missing from the equation, the variable y is absent, none of the variables are missing and g is homogeneous in both x and y , or g can be written in the form of $y = \psi(x, y')$.

If the differential equation $g(x, y, y') = 0$ pertains to one of these four categories, we can solve g via an adequate parameterization.

Case A. The variable x is missing

In this case, the differential equation takes on the form $g : \mathbb{R}^2 \rightarrow \mathbb{R} : (y, y') \mapsto g(y, y') = 0$ which we solve by following these three steps:

1. Write y and y' in parametric form:

$$\begin{cases} y = \phi_1(t) \\ y' = \phi_2(t) \end{cases}$$

2. In step 1, take the differential of the first equation and rewrite the second one as follows:

$$\begin{cases} dy = \phi_1'(t) dt \\ dy = \phi_2(t) dx \end{cases}$$

3. Combine the two equations in step 2 and write down the solution of $g(y, y') = 0$ in parametric form:

$$\begin{cases} x = \int \frac{\phi_1'(t)}{\phi_2(t)} dt \\ y = \phi_1(t) \end{cases}$$

- Example

Consider the following differential equation:

$$7(y' - 2)^2 - 4(y' - 1)^2 + 4y(y' - 3) + y'(1 - 2y) + y(11 - 2y') + 12y' - 23 = 0$$

In a first step, we set $y' = \phi_2(t) = t$, so that the differential equation obtains the following form:

$$7(t-2)^2 - 4(t-1)^2 + 4y(t-3) + t(1-2y) + y(11-2t) + 12t - 23 = 0$$

Simplifying this equation gives us the following expression for $y = \phi_1(t)$:

$$\begin{aligned} & 7(t-2)^2 - 4(t-1)^2 + 4y(t-3) + t(1-2y) + y(11-2t) + 12t - 23 = 0 \\ \Leftrightarrow & 7(t^2 - 4t + 4) - 4(t^2 - 2t + 1) + 4yt - 12y + t - 2yt + 11y - 2yt + 12t - 23 = 0 \\ \Leftrightarrow & 7t^2 - 4t^2 - 28t + 8t + t + 12t + 4yt - 2yt - 2yt - 12y + 11y + 28 - 4 - 23 = 0 \\ \Leftrightarrow & 3t^2 - 7t + 1 = y \end{aligned}$$

Step 2 gives us the following two equations:

$$\begin{cases} y = \phi_1(t) = 3t^2 - 7t + 1 \\ y' = \phi_2(t) = t \end{cases} \Leftrightarrow \begin{cases} dy = \phi_1'(t) dt = (6t - 7) dt \\ dy = \phi_2(t) dx = t dx \end{cases}$$

In a final step, we write down the parametric solution of the differential equation:

$$\begin{cases} x = \int \frac{\phi_1'(t)}{\phi_2(t)} dt = \int \frac{6t-7}{t} dt = \int \left(6 - \frac{7}{t}\right) dt = 6t - 7 \ln |t| + c \\ y = \phi_1(t) = 3t^2 - 7t + 1 \end{cases}$$

Case B. The variable y is missing

In the second case, the differential equation has the form $g : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y') \mapsto g(x, y') = 0$ which is solved in a way similar to Case A:

1. Write x and y' in parametric form:

$$\begin{cases} x = \phi_1(t) \\ y' = \phi_2(t) \end{cases}$$

2. In step 1, take the differential of the first equation and rewrite the second one as follows:

$$\begin{cases} dx = \phi_1'(t) dt \\ dy = \phi_2(t) dx \end{cases}$$

3. Combine the two equations in step 2 and write down the solution of $g(x, y') = 0$ in parametric form:

$$\begin{cases} x = \phi_1(t) \\ y = \int \phi_2(t)\phi_1'(t) dt \end{cases}$$

• Example

Let us consider the differential equation $5^x y' - 5^{2x} y'^2 - 25^x + 5 = 0$. In a first step, we introduce the substitution $t = 5^x y'$, so that the differential equation obtains the following form:

$$5^x y' - 5^{2x} y'^2 - 25^x + 5 = 0 \Leftrightarrow t - t^2 - 25^x + 5 = 0$$

The expression $x = \phi_1(t)$ is found as follows:

$$\begin{aligned} t - t^2 - 25^x + 5 = 0 &\Leftrightarrow t - t^2 + 5 = 25^x \\ &\Leftrightarrow t - t^2 + 5 = 5^{2x} \\ &\Leftrightarrow \frac{1}{2} \log_5(t - t^2 + 5) = x \end{aligned}$$

Step 2 gives us the following two equations:

$$\begin{cases} dx = \phi_1'(t) dt = \frac{d}{dt} \left[\frac{1}{2} \log_5(t - t^2 + 5) \right] dt = \frac{(-2t + 1)}{2 \ln(5)(t - t^2 + 5)} dt \\ dy = \phi_2(t) dx = \frac{t}{5^x} dx = \frac{t}{\sqrt{t - t^2 + 5}} dx \end{cases}$$

The solution for the variable y in parametric form is then found as follows:

$$\begin{aligned} dy &= \frac{t}{\sqrt{t - t^2 + 5}} dx \\ &= \frac{t}{\sqrt{t - t^2 + 5}} \left[\frac{(-2t + 1)}{2 \ln(5)(t - t^2 + 5)} dt \right] \\ \Leftrightarrow y &= \frac{1}{2 \ln(5)} \int \frac{t(-2t + 1)}{(t - t^2 + 5)^{\frac{3}{2}}} dt \\ &= \frac{1}{2 \ln(5)} \int \frac{t(-2t + 1)}{\left[\frac{21}{4} - \left(t - \frac{1}{2}\right)^2 \right]^{\frac{3}{2}}} dt \\ &= -\frac{1}{\ln(5)} \int \frac{u(u + 1)}{(21 - u^2)^{\frac{3}{2}}} du \\ &= -\frac{1}{\ln(5)} \int \tan^2(\theta) d\theta - \frac{1}{\sqrt{21} \ln(5)} \int \tan(\theta) \sec(\theta) d\theta \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\ln(5)} \int [\sec^2 \theta - 1] d\theta + \frac{1}{\sqrt{21} \ln(5)} \int \frac{1}{s^2} ds \\
&= -\frac{1}{\ln(5)} [\tan \theta - \theta] + \frac{1}{\sqrt{21} \ln(5)} \left(-\frac{1}{s} \right) + c \\
&= \frac{(-2t + 1)}{2 \ln(5)(t - t^2 + 5)} + \frac{1}{\ln(5)} \arcsin \left(\frac{2t - 1}{\sqrt{21}} \right) - \frac{1}{2 \ln(5) \sqrt{t - t^2 + 5}} + c \\
&= \frac{-t}{\ln(5) \sqrt{t - t^2 + 5}} + \frac{1}{\ln(5)} \arcsin \left(\frac{2t - 1}{\sqrt{21}} \right) + c
\end{aligned}$$

whereby we implemented the substitution $u = 2(t - \frac{1}{2})$ in line 5, the substitution $u = \sqrt{21} \sin \theta$ in line 6, and the substitution $s = \cos \theta$ in line 7. Remember also that $\tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$, so that $\tan \theta = \frac{u}{\sqrt{21 - u^2}} = \frac{2t - 1}{2\sqrt{-t^2 + t + 5}}$.

To summarize, the solution to this integral in parametric form is equal to:

$$\begin{cases} x = \frac{1}{2} \log_5(t - t^2 + 5) \\ y = \frac{-t}{\ln(5) \sqrt{t - t^2 + 5}} + \frac{1}{\ln(5)} \arcsin \left(\frac{2t - 1}{\sqrt{21}} \right) + c \end{cases}$$

Note that a solution only exists if $t \in]\frac{1 - \sqrt{21}}{2}, \frac{1 + \sqrt{21}}{2}[$.

Case C. Homogeneous in both x and y

In the final case, the differential equation is written in the form of $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ whereby g is homogeneous of degree m in both x and y . The differential equations that fall under this category are solved in a similar way to the other two cases:

1. Divide each term by x^m and write $\frac{y}{x}$ and y' in parametric form:

$$\begin{cases} \frac{y}{x} = \phi_1(t) \\ y' = \phi_2(t) \end{cases}$$

2. In step 1, take the differential of the first equation and rewrite the second one as follows:

$$\begin{cases} dy = x \phi_1'(t) dt + \phi_1(t) dx \\ dy = \phi_2(t) dx \end{cases}$$

3. Combine the two equations in step 2 and write down the solution of $g(x, y, y') = 0$ in parametric form:

$$\begin{cases} \frac{dx}{x} = \frac{\phi_1'(t)}{\phi_2(t) - \phi_1(t)} dt & \Leftrightarrow x = c \exp \left[\int \frac{\phi_1'(t)}{\phi_2(t) - \phi_1(t)} dt \right] \\ y = x\phi_1(t) \end{cases}$$

- Example

Consider the differential equation $y^2(y' - 1)^2 - x^2y'^2 + 2y^2y' = 0$, which is homogeneous of degree $m = 2$ in both x and y .

In a first step, we divide each term by $x^m = x^2$ and we introduce the substitution $\frac{y}{x} = \phi_1(t) = t$, so that we can find an expression for $y' = \phi_2(t)$:

$$\begin{aligned} y^2(y' - 1)^2 - x^2y'^2 + 2y^2y' = 0 & \Leftrightarrow \left(\frac{y}{x}\right)^2 (y' - 1)^2 - y'^2 + 2\left(\frac{y}{x}\right)^2 y' = 0 \\ & \Leftrightarrow t^2(y' - 1)^2 - y'^2 + 2t^2y' = 0 \\ & \Leftrightarrow t^2y'^2 - 2t^2y' + t^2 - y'^2 + 2t^2y' = 0 \\ & \Leftrightarrow t^2y'^2 + t^2 - y'^2 = 0 \\ & \Leftrightarrow y' = \frac{t}{\sqrt{1 - t^2}} \end{aligned}$$

whereby we have arbitrarily chosen to work with the positive root of y'^2 for the remainder of this exercise.

In a next step, we write down the following two equations for the differential of the variable y :

$$\begin{cases} y = xt & \Leftrightarrow dy = x dt + t dx \\ dy = \phi_2(t) dx = \frac{t}{\sqrt{1 - t^2}} dx \end{cases}$$

Setting the above two expressions for dy equal to each other, we find a parametric solution for x :

$$\begin{aligned} x dt + t dx &= \frac{t}{\sqrt{1 - t^2}} dx \\ \Leftrightarrow \frac{dx}{x} &= \frac{\sqrt{1 - t^2}}{t(1 - \sqrt{1 - t^2})} dt \\ \Leftrightarrow \int \frac{dx}{x} &= \int \frac{\sqrt{1 - t^2}}{t(1 - \sqrt{1 - t^2})} dt \\ \Leftrightarrow \ln|x| &= \int \frac{\cos^2(\theta)}{\sin(\theta)(1 - \cos(\theta))} d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{(1-u^2)^2}{u^3(1+u^2)} du \\
&= \frac{1}{2} \int \left(\frac{4u}{1+u^2} - \frac{3}{u} + \frac{1}{u^3} \right) du \\
&= \int \frac{2u}{1+u^2} du - \frac{3}{2} \int \frac{1}{u} du + \frac{1}{2} \int \frac{1}{u^3} du \\
&= \ln|1+u^2| - \frac{3}{2} \ln|u| - \frac{1}{4u^2} + c \\
&= \ln \left| \frac{1+u^2}{u\sqrt{u}} \right| - \frac{1}{4u^2} + c \\
&= \ln \left| \frac{2}{\sqrt{t(1-\sqrt{1-t^2})}} \right| - \frac{t^2}{4(1-\sqrt{1-t^2})^2} + c \\
\Leftrightarrow x &= \frac{2}{\sqrt{t(1-\sqrt{1-t^2})}} \exp \left[-\frac{t^2}{4(1-\sqrt{1-t^2})^2} \right] e^c
\end{aligned}$$

whereby we implemented the substitution $t = \sin \theta$ in line 4 and the substitution $u = \tan(\frac{\theta}{2})$ in line 5. We furthermore applied partial fraction decomposition in line 6.

The parametric solution of this differential equation is then equal to:

$$\left\{ \begin{array}{l} x = \frac{2}{\sqrt{t(1-\sqrt{1-t^2})}} \exp \left[-\frac{t^2}{4(1-\sqrt{1-t^2})^2} \right] e^c \\ y = x\phi_1(t) = xt = \frac{2\sqrt{t}}{\sqrt{(1-\sqrt{1-t^2})}} \exp \left[-\frac{t^2}{4(1-\sqrt{1-t^2})^2} \right] e^c \end{array} \right.$$

Note that a solution only exists for the interval $t \in]0, 1[$. This differential equation has also the singular solution $x = y = 0$.

Case D. Differential equations in the form of $y = \psi(x, y')$

If the differential equation $g : \mathbb{R}^3 \rightarrow \mathbb{R} : (x, y, y') \mapsto g(x, y, y') = 0$ can be written in the form of $y = \psi(x, y')$, we can implement the following three steps to solve the equation:

1. Set the variable y' equal to $y' = t$, so that we can write the variable y in parametric form $y = \psi(x, t)$.

2. In step 1, take the derivative of y with respect to x and set subsequently $y' = t$:

$$y' = \frac{\partial \psi}{\partial x}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \cdot t' \Leftrightarrow t = \frac{\partial \psi}{\partial x}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \cdot t'$$

3. The new differential equation in step 2 is of order 1 and degree 1. If we can integrate this equation, we obtain a solution of the form $\Psi(x, t, c) = 0$, so that the parametric solution of the original differential equation is equal to:

$$\begin{cases} \Psi(x, t, c) = 0 \\ y = \psi(x, t) \end{cases}$$

- Example

Let us consider the differential equation $y'^2 + yy' - x - 1 = 0$. In a first step, we write the equation in the form of $y = \psi(x, t)$ with $t = y'$:

$$\begin{aligned} y'^2 + yy' - x - 1 = 0 &\Leftrightarrow y = \frac{x+1}{y'} - y' \\ &= \frac{x+1}{t} - t \end{aligned}$$

Next, we take the derivative of y with respect to x to obtain a linear differential equation:

$$\begin{aligned} \frac{d}{dx}(y) = \frac{d}{dx} \left[\frac{x+1}{t} - t \right] &\Leftrightarrow y' = \frac{t - (1+x)t'}{t^2} - t' \\ &\Leftrightarrow t = \frac{1}{t} - \left[\frac{(1+x)}{t^2} + 1 \right] \frac{dt}{dx} \\ &\Leftrightarrow t(1-t^2) \frac{dx}{dt} - x = 1 + t^2 \end{aligned}$$

Note that in the above obtained linear differential equation the variable t is the independent variable and x the dependent variable. The solution will therefore be of the form $x = x_h + x_p$.

The homogeneous solution x_h is found by solving the following differential equation:

$$\begin{aligned} t(1-t^2) \frac{dx}{dt} - x = 0 &\Leftrightarrow \frac{dx}{x} = \frac{dt}{t(1-t^2)} \\ &\Leftrightarrow \int \frac{1}{x} dx = \int \frac{1}{t(1-t^2)} dt \\ &\Leftrightarrow \ln|x| = \int \frac{1}{t} dt + \frac{1}{2} \int \frac{1}{1-t} dt - \frac{1}{2} \int \frac{1}{1+t} dt \\ &= \ln|t| - \frac{1}{2} (\ln|1-t| + \ln|1+t|) + c \\ &= \ln|t| - \frac{1}{2} \ln|1-t^2| + c \end{aligned}$$

$$= \ln |t| + \ln \left| \frac{1}{\sqrt{1-t^2}} \right| + c$$

$$\Leftrightarrow x_h = \frac{te^c}{\sqrt{1-t^2}}$$

whereby we applied partial fraction decomposition in line 3.

The particular solution x_p has the form $x_p = c(t)f(t)$ with $f(t) = \frac{t}{\sqrt{1-t^2}}$. Given the general differential equation $a(t)x' + b(t)x = k(t)$, we first find an expression for $c(t)$:

$$c(t) = \int \frac{k(t)}{a(t)f(t)} dt = \int \frac{(1+t^2)\sqrt{1-t^2}}{t^2(1-t^2)} dt$$

$$= \int \frac{1+t^2}{t^2\sqrt{1-t^2}} dt$$

$$= \int \frac{1}{t^2\sqrt{1-t^2}} dt + \int \frac{1}{\sqrt{1-t^2}} dt$$

$$= \int \csc^2(\theta) d\theta + \arcsin(t)$$

$$= -\cot(\theta) + \arcsin(t)$$

$$= -\frac{\sqrt{1-t^2}}{t} + \arcsin(t)$$

whereby we introduced the substitution $t = \sin \theta$ in line 4. Also remember that $\cot \theta = \frac{\sqrt{1-\sin^2 \theta}}{\sin \theta}$.

The particular solution x_p is then equal to:

$$x_p = c(t)f(t) = \left[-\frac{\sqrt{1-t^2}}{t} + \arcsin(t) \right] \frac{t}{\sqrt{1-t^2}} = -1 + \frac{t \arcsin(t)}{\sqrt{1-t^2}}$$

For $x = x_h + x_p$ we then find:

$$x = x_h + x_p = \frac{te^c}{\sqrt{1-t^2}} - 1 + \frac{t \arcsin(t)}{\sqrt{1-t^2}} = -1 + \frac{t[e^c + \arcsin(t)]}{\sqrt{1-t^2}}$$

To summarize, the parametric solution to the original differential equation is equal to:

$$\begin{cases} x = -1 + \frac{t[e^c + \arcsin(t)]}{\sqrt{1-t^2}} \\ y = \frac{x+1}{t} - t \end{cases}$$

whereby $t \in]0, 1[$. The differential equation also has the singular solution $y = x$.

11.5 Differential Equations of Order n and Degree n

In this section, we explore differential equations of the general form $g : \mathbb{R}^{n+2} \rightarrow \mathbb{R} : (x, y, y', y'', \dots, y^{(n)}) \mapsto g(x, y, y', y'', \dots, y^{(n)}) = 0$ whereby the order is not necessarily restricted to $n = 1$.

In particular, we discuss differential equations that fall under four categories: only the variables x and $y^{(n)}$ are present, the variable y is missing, the variable x is missing, or the equation is homogeneous in the variables $y, y', \dots, y^{(n)}$.

Case A. Only the variables x and $y^{(n)}$ are present

The differential equation has the form of $g : \mathbb{R}^2 \rightarrow \mathbb{R} : (x, y^{(n)}) \mapsto g(x, y^{(n)}) = 0$ and can be rewritten as $y^{(n)} = \phi(x)$.

If we integrate this expression we obtain:

$$\int y^{(n)} dx = \int \phi(x) dx \Leftrightarrow y^{(n-1)} + c_1 = \int \phi(x) dx$$

We continue to integrate until we have reduced the order to $n = 0$, at which point we have found a solution for the differential equation.

- Example

Consider the differential equation $x^2 y^{(4)} - x^3 - 2 = 0$. If we first rewrite the equation in the form of $y^{(4)} = \phi(x)$ and subsequently integrate this expression four times, we obtain a solution for this differential equation:

$$\begin{aligned} x^2 y^{(4)} - x^3 - 2 = 0 &\Leftrightarrow y^{(4)} = x + \frac{2}{x^2} \\ &\Leftrightarrow \int y^{(4)} dx = \int \left(x + \frac{2}{x^2} \right) dx \\ &\Leftrightarrow y''' = \frac{x^2}{2} - \frac{2}{x} + c_1 \\ &\Leftrightarrow \int y''' dx = \int \left(\frac{x^2}{2} - \frac{2}{x} + c_1 \right) dx \\ &\Leftrightarrow y'' = \frac{x^3}{6} - 2 \ln |x| + c_1 x + c_2 \\ &\Leftrightarrow \int y'' dx = \int \left(\frac{x^3}{6} - 2 \ln |x| + c_1 x + c_2 \right) dx \\ &\Leftrightarrow y' = \frac{x^4}{24} - 2(x \ln |x| - x) + \frac{c_1 x^2}{2} + c_2 x + c_3 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \int y' dx &= \int \left[\frac{x^4}{24} - 2(x \ln |x| - x) + \frac{c_1 x^2}{2} + c_2 x + c_3 \right] dx \\ \Leftrightarrow y &= \frac{x^5}{120} - 2 \left[\left(\frac{x^2}{2} \ln |x| - \frac{x^2}{4} \right) - \frac{x^2}{2} \right] + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \\ &= \frac{x^5}{120} + \frac{c_1 x^3}{6} + \frac{x^2}{2} (-2 \ln |x| + 3 + c_2) + c_3 x + c_4 \end{aligned}$$

whereby $x \neq 0$.

Case B. The variable y is missing

As the variable y is missing, the differential equation has the form of $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (x, y', y'', \dots, y^{(n)}) \mapsto g(x, y', y'', \dots, y^{(n)}) = 0$.

The idea is to reduce the order of the differential equation and the following three steps help us solve the differential equation:

1. Introduce the substitution $t = y'$, so that the newly obtained differential equation is of the form $G : \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (x, t, t', \dots, t^{(n-1)}) \mapsto G(x, t, t', \dots, t^{(n-1)}) = 0$.
2. If possible, determine an expression for t .
3. Integrate t to find an expression for the solution y .

- Example

Let us consider the differential equation $\csc x - \sin(x)y'' - \cos(x)y' = 0$. In a first step, we implement the substitution $t = y'$, so that we obtain the following differential equation:

$$\begin{aligned} \csc x - \sin(x)y'' - \cos(x)y' = 0 &\Leftrightarrow \csc x - \sin(x)t' - \cos(x)t = 0 \\ &\Leftrightarrow \sin(x)t' + \cos(x)t = \csc x \end{aligned}$$

Since this differential equation is linear, we can apply the method discussed in section 11.3.5 whereby the solution is of the form $t = t_h + t_p$. In a first instance, we determine the homogeneous solution t_h :

$$\begin{aligned} \sin(x)t' + \cos(x)t = 0 &\Leftrightarrow \frac{dt}{t} = -\cot(x) dx \\ &\Leftrightarrow \int \frac{dt}{t} = -\int \cot(x) dx \\ &\Leftrightarrow \ln |t| = \ln |\csc x| + c_1 \\ &\Leftrightarrow t_h = e^{c_1} \csc x \end{aligned}$$

In a next step, we calculate the particular solution t_p , which has the form $t_p = c(x)f(x)$ with $f(x) = \csc x$. Given the general differential equation $a(x)t' + b(x)t = k(x)$, we first find an expression for $c(x)$:

$$\begin{aligned} c(x) &= \int \frac{k(x)}{a(x)f(x)} dx = \int \frac{\csc(x)}{\sin(x)\csc(x)} dx = \int \frac{1}{\sin(x)} dx = - \int \frac{1}{1-u^2} du \\ &= -\operatorname{arctanh}(u) \\ &= -\frac{1}{2} \ln \left| \frac{1+u}{1-u} \right| \\ &= -\frac{1}{2} \ln \left| \frac{1+\cos x}{1-\cos x} \right| \\ &= -\ln |\csc x + \cot x| \end{aligned}$$

whereby we implemented the substitution $u = \cos x$ in the last integral of line 1. The particular solution t_p is then equal to:

$$t_p = c(x)f(x) = -\ln |\csc x + \cot x| \csc x$$

The total solution $t = t_h + t_p$ then becomes:

$$t = t_h + t_p = e^{c_1} \csc x - \ln |\csc x + \cot x| \csc x$$

We find the solution y for our original differential equation by integrating the above expression for t , given that at the beginning we introduced the substitution $t = y'$:

$$\begin{aligned} y &= \int t dx = \int (e^{c_1} \csc x - \ln |\csc x + \cot x| \csc x) dx \\ &= e^{c_1} \int \csc(x) dx - \int \ln |\csc x + \cot x| \csc(x) dx \\ &= -e^{c_1} \ln |\csc x + \cot x| + \frac{1}{2} \ln^2 |\csc x + \cot x| + c_2 \end{aligned}$$

with $x \neq 2k\pi$ ($k \in \mathbb{Z}$) and whereby the first integral has already been calculated earlier in this exercise and the second integral is found by applying partial integration in the following way (note that we have set $u = \ln |\csc x + \cot x|$ and $dv = \csc(x) dx$ in the formula of partial integration):

$$\begin{aligned} \int \ln |\csc x + \cot x| \csc(x) dx &= -\ln^2 |\csc x + \cot x| - \int \ln |\csc x + \cot x| \csc(x) dx \\ \Leftrightarrow 2 \int \ln |\csc x + \cot x| \csc(x) dx &= -\ln^2 |\csc x + \cot x| \\ \Leftrightarrow \int \ln |\csc x + \cot x| \csc(x) dx &= -\frac{1}{2} \ln^2 |\csc x + \cot x| \end{aligned}$$

Case C. The variable x is missing

If the variable x is missing from the differential equation, it has the form of $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (y, y', y'', \dots, y^{(n)}) \mapsto g(y, y', y'', \dots, y^{(n)}) = 0$.

The below two steps help us solve the differential equation:

1. Reverse the roles of the variables x and y : treat y as the independent variable so that differential equation obtains the form of $G : \mathbb{R}^{n+1} \rightarrow \mathbb{R} : (y, x', x'', \dots, x^{(n)}) \mapsto G(y, x', x'', \dots, x^{(n)}) = 0$.
2. Follow the procedure as explained under Case B to solve the differential equation whereby x and y have swapped roles.

During the first step, we have to transform $y', y'', \dots, y^{(n)}$ into an expression in terms of $x', x'', \dots, x^{(n)}$. Let us look at the first three derivatives of y to see how this works.

For y' , we find that:

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'}$$

With respect to y'' , we find the following expression:

$$y'' = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{\frac{dx}{dy}} \right) = \frac{d}{dy} \left(\frac{1}{\frac{dx}{dy}} \right) \left(\frac{dx}{dy} \right)^{-1} = \left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^2} \right] \left(\frac{dx}{dy} \right)^{-1} = -\frac{x''}{x'^3}$$

Finally, we transform y''' as follows:

$$\begin{aligned} y''' &= \frac{d}{dx} \left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^3} \right] = \frac{d}{dy} \left[-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^3} \right] \left(\frac{dx}{dy} \right)^{-1} \\ &= \left[\frac{-\frac{d^3x}{dy^3} \left(\frac{dx}{dy} \right)^3 + 3 \frac{d^2x}{dy^2} \left(\frac{dx}{dy} \right)^2 \frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^6} \right] \left(\frac{dx}{dy} \right)^{-1} \\ &= \left[\frac{-\frac{d^3x}{dy^3} \left(\frac{dx}{dy} \right) + 3 \frac{d^2x}{dy^2} \frac{d^2x}{dy^2}}{\left(\frac{dx}{dy} \right)^4} \right] \left(\frac{dx}{dy} \right)^{-1} \\ &= \frac{-x'''x' + 3x''^2}{x'^5} \end{aligned}$$

- Example

Consider the differential equation $6y'' - 7yy' = 0$. In a first step, we reverse the roles of the variables x and y :

$$6y'' - 7yy' = 0 \Leftrightarrow 6 \left(-\frac{x''}{x'^3} \right) - 7y \left(\frac{1}{x'} \right) = 0 \Leftrightarrow -6x'' - 7yx'^2 = 0$$

Now, we follow the procedure as explained under Case B whereby we set $t = x'$ and find an expression for t :

$$\begin{aligned} -6x'' - 7yx'^2 = 0 &\Leftrightarrow -6t' - 7yt^2 = 0 \Leftrightarrow -7yt^2 = 6 \frac{dt}{dy} \Leftrightarrow -7y dy = \frac{6}{t^2} dt \\ &\Leftrightarrow -7 \int y dy = 6 \int \frac{1}{t^2} dt \\ &\Leftrightarrow -\frac{7y^2}{2} = -\frac{6}{t} + c_1 \\ &\Leftrightarrow t = \frac{12}{7y^2 + 2c_1} \end{aligned}$$

In a final step, we replace t by x' and integrate both sides to find a solution y of the original differential equation:

$$\begin{aligned} t = x' = \frac{12}{7y^2 + 2c_1} &\Leftrightarrow \frac{dx}{dy} = \frac{12}{7y^2 + 2c_1} \Leftrightarrow dx = \frac{12}{7y^2 + 2c_1} dy \\ &\Leftrightarrow \int dx = \int \frac{12}{7y^2 + 2c_1} dy \\ &\Leftrightarrow x = \frac{12}{2c_1} \int \frac{1}{\left(y\sqrt{\frac{7}{2c_1}}\right)^2 + 1} dy \\ &= \frac{12}{\sqrt{14c_1}} \int \frac{1}{u^2 + 1} du \\ &= \frac{12}{\sqrt{14c_1}} \arctan(u) + c_2 \\ &= \frac{12}{\sqrt{14c_1}} \arctan\left(y\sqrt{\frac{7}{2c_1}}\right) + c_2 \\ &\Leftrightarrow y = \sqrt{\frac{2c_1}{7}} \tan\left[\frac{\sqrt{14c_1}}{12}(x - c_2)\right] \end{aligned}$$

whereby we implemented the substitution $u = y\sqrt{\frac{7}{2c_1}}$ in line 4.

Case D. Homogeneous in the variables $y, y', \dots, y^{(n)}$

If the differential equation $g : \mathbb{R}^{n+2} \rightarrow \mathbb{R} : (x, y, y', y'', \dots, y^{(n)}) \mapsto g(x, y, y', y'', \dots, y^{(n)}) = 0$ is homogeneous in the variables $y, y', \dots, y^{(n)}$, we can reduce the order by introducing the substitution $t = \frac{y'}{y}$.

Given that $y' = yt$, we find the expressions for, for instance, the second and third derivative of y (with respect to x) as follows:

$$\begin{cases} y'' = (yt)' = y't + yt' = (yt)t + yt' = y(t^2 + t') \\ y''' = y'(t^2 + t') + y(2tt' + t'') = (yt)(t^2 + t') + y(2tt' + t'') = y(t^3 + 3tt' + t'') \end{cases}$$

Once the new differential equation is found, we can try to apply any of the known methods to solve the differential equation.

- Example

Let us consider the differential equation $xyy'' - xy'^2 - 9y^2(x^2 - 1) = 0$, which is homogeneous of degree 2 in the variables y, y' , and y'' . In a first instance, we implement the substitution $t = \frac{y'}{y}$:

$$\begin{aligned} xyy'' - xy'^2 - 9y^2(x^2 - 1) = 0 &\Leftrightarrow xy^2(t^2 + t') - xy^2t^2 - 9y^2(x^2 - 1) = 0 \\ &\Leftrightarrow xt' - 9(x^2 - 1) = 0 \end{aligned}$$

This newly obtained differential equation is of order 1 and degree 1, which we can solve by the method of separation of variables:

$$\begin{aligned} xt' - 9(x^2 - 1) = 0 &\Leftrightarrow dt = \left(9x - \frac{9}{x}\right) dx \\ &\Leftrightarrow \int dt = \int \left(9x - \frac{9}{x}\right) dx \\ &\Leftrightarrow t = \frac{9x^2}{2} - 9 \ln |x| + c_1 \end{aligned}$$

If we now replace t by $t = \frac{y'}{y}$, we find the solution y for the original differential equation as follows:

$$\begin{aligned} t = \frac{y'}{y} = \frac{9x^2}{2} - 9 \ln |x| + c_1 &\Leftrightarrow \frac{dy}{y} = \left(\frac{9x^2}{2} - 9 \ln |x| + c_1\right) dx \\ &\Leftrightarrow \int \frac{1}{y} dy = \int \left(\frac{9x^2}{2} - 9 \ln |x| + c_1\right) dx \end{aligned}$$

$$\Leftrightarrow \ln |y| = \frac{3x^3}{2} - 9(x \ln |x| - x) + c_1x + c_2$$

$$\Leftrightarrow y = e^{\frac{3x^3}{2}} x^{-9x} e^{9x} e^{c_1x} e^{c_2}$$

whereby we applied partial integration in line 3. Note that this differential equation has the singular solutions $y = 0$ and the point $(0, 0)$.

11.6 Differential Equations with Constant Coefficients

In this section, we study the differential equation $g : \mathbb{R}^{n+2} \rightarrow \mathbb{R} : (x, y, y', y'', \dots, y^{(n)}) \mapsto g(x, y, y', y'', \dots, y^{(n)}) = 0$ that is linear in all the variables $y, y', y'', \dots, y^{(n)}$ and whereby the coefficients of these variables are real constants with the constant of $y^{(n)}$ equal to 1. This differential equation has therefore the following form:

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \dots + a_{n-2}y'' + a_{n-1}y' + a_ny = b(x)$$

11.6.1 The General Method

The general solution y of this differential equation consists of a homogeneous solution y_h and a particular solution y_p and can thus be written as $y = y_h + y_p$.

- The homogeneous solution

The homogeneous solution is the solution of the differential equation whereby we set $b(x) = 0$. The linear homogeneous differential equation is thus of the form:

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \dots + a_{n-2}y'' + a_{n-1}y' + a_ny = 0$$

Before we propose a solution to the general homogeneous differential equation let us first consider the case $n = 1$. We can solve the differential equation $y' + a_1y = 0$ through the method of separation of variables, so that its solution is equal to:

$$y' + a_1y = 0 \Leftrightarrow \int \frac{1}{y} dy = -a_1 \int dx \Leftrightarrow \ln |y| = -a_1x + c_1 \Leftrightarrow y = e^{c_1} e^{-a_1x}$$

For the general case, it would thus make sense to propose a solution of the form $y = e^{\lambda x}$. If we remember that $y^{(n)} = (e^{\lambda x})^{(n)} = \lambda^n e^{\lambda x}$, we can write the homogeneous differential equation in the following way:

$$\begin{aligned}
& y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-2} y'' + a_{n-1} y' + a_n y = 0 \\
\Leftrightarrow & \lambda^n e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + a_2 \lambda^{n-2} e^{\lambda x} + \dots + a_{n-2} \lambda^2 e^{\lambda x} + a_{n-1} \lambda e^{\lambda x} + a_n e^{\lambda x} = 0 \\
\Leftrightarrow & e^{\lambda x} [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-2} \lambda^2 + a_{n-1} \lambda + a_n] = 0 \\
\Leftrightarrow & \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_{n-2} \lambda^2 + a_{n-1} \lambda + a_n = 0
\end{aligned}$$

This last equation is referred to as the **characteristic equation**. Suppose that this polynomial has r roots $\lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_r$ whereby the root λ_i has multiplicity m_i . Note that $r \leq n$, $m_i \leq n$, and $\sum_{k=1}^r m_k = n$. The characteristic equation becomes:

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_{r-1})^{m_{r-1}} (\lambda - \lambda_r)^{m_r} = 0$$

The general solution for the root λ_i has the form:

$$y = x^p e^{\lambda_i x}$$

whereby $p \in \{0, 1, \dots, m_i - 1\}$. The n solutions for the linear homogeneous differential equation are thus equal to:

$$\begin{array}{cccccc}
e^{\lambda_1 x} & x e^{\lambda_1 x} & x^2 e^{\lambda_1 x} & \dots & x^{m_1-2} e^{\lambda_1 x} & x^{m_1-1} e^{\lambda_1 x} \\
e^{\lambda_2 x} & x e^{\lambda_2 x} & x^2 e^{\lambda_2 x} & \dots & x^{m_2-2} e^{\lambda_2 x} & x^{m_2-1} e^{\lambda_2 x} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e^{\lambda_{(r-1)} x} & x e^{\lambda_{(r-1)} x} & x^2 e^{\lambda_{(r-1)} x} & \dots & x^{m_{(r-1)}-2} e^{\lambda_{(r-1)} x} & x^{m_{(r-1)}-1} e^{\lambda_{(r-1)} x} \\
e^{\lambda_r x} & x e^{\lambda_r x} & x^2 e^{\lambda_r x} & \dots & x^{m_r-2} e^{\lambda_r x} & x^{m_r-1} e^{\lambda_r x}
\end{array}$$

The homogeneous solution y_h is equal to a linear combination of all these individual solutions:

$$y_h = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} + \dots + c_{n-1} x^{m_r-2} e^{\lambda_r x} + c_n x^{m_r-1} e^{\lambda_r x}$$

with $c_1, c_2, \dots, c_n \in \mathbb{R}$.

In case that, let's say, the root λ_1 is complex, i.e., $\lambda_1 = \alpha + i\beta$, the individual solution is written as follows:

$$y = x^p e^{\lambda_1 x} = x^p e^{(\alpha+i\beta)x} = x^p e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)]$$

with $p \in \{0, 1, \dots, m_1 - 1\}$ and whereby we made use of the definition of the complex exponential function (see section 8.4).

It follows automatically that the complex conjugate $\bar{\lambda}_1$ of the root λ_1 is also a root of the characteristic equation, whose solution is equal to:

$$y = x^p e^{\bar{\lambda}_1 x} = x^p e^{(\alpha - i\beta)x} = x^p e^{\alpha x} [\cos(\beta x) - i \sin(\beta x)]$$

In other words, for a complex root its solution is a linear combination of the terms $x^p e^{\alpha x} \cos(\beta x)$ and $x^p e^{\alpha x} \sin(\beta x)$ whereby the coefficients might be complex.

- The particular solution

We write the full differential equation as follows, whereby we set $b(x) = P(x)e^{qx}$ with $P(x)$ a polynomial function of degree p and $q \in \mathbb{C}$ (remember that $\mathbb{R} \subset \mathbb{C}$):

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-2} y'' + a_{n-1} y' + a_n y = P(x)e^{qx}$$

If q is a root of the corresponding characteristic equation with multiplicity m , the particular solution y_p of the linear differential equation is equal to:

$$y_p = x^m P_s(x) e^{qx}$$

whereby $P_s(x)$ is a polynomial function of degree p , i.e., the same degree as $P(x)$. If q is not a root of the characteristic equation, then we set $m = 0$.

11.6.2 Examples

- Example 1

Let us consider the differential equation $y''' - 5y'' + 8y' - 4y = x^3 - 2x + 1$. In a first step, we look for the solution y_h of the following homogeneous differential equation:

$$y''' - 5y'' + 8y' - 4y = 0$$

If we write the general solution as $y = e^{\lambda x}$, the characteristic equation becomes the following:

$$\begin{aligned} y''' - 5y'' + 8y' - 4y = 0 &\Leftrightarrow \lambda^3 e^{\lambda x} - 5\lambda^2 e^{\lambda x} + 8\lambda e^{\lambda x} - 4e^{\lambda x} = 0 \\ &\Leftrightarrow e^{\lambda x} (\lambda^3 - 5\lambda^2 + 8\lambda - 4) = 0 \\ &\Leftrightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \\ &\Leftrightarrow (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0 \\ &\Leftrightarrow (\lambda - 1)(\lambda - 2)^2 = 0 \end{aligned}$$

In other words, the root $\lambda_1 = 1$ has the multiplicity $m_1 = 1$ and the root $\lambda_2 = 2$ has the multiplicity $m_2 = 2$. With respect to $\lambda_1 = 1$, its solution is equal to $y = x^0 e^{\lambda_1 x} = e^x$. Regarding $\lambda_2 = 2$, it has two solutions: $y = x^0 e^{\lambda_2 x} = e^{2x}$ and $y = x^1 e^{\lambda_2 x} = x e^{2x}$.

The homogeneous solution y_h of this differential equation is therefore equal to a linear combination of these three individual solutions:

$$y_h = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} = c_1 e^x + (c_2 + c_3 x) e^{2x}$$

In a next step, we determine the particular solution y_p . If we look at the right-hand side of the differential equation and compare it with the general expression $b(x) = P(x)e^{qx}$, we can see that $q = 0$ and that $P(x) = x^3 - 2x + 1$ with degree $p = 3$. Since $q = 0$ is not a root of the characteristic equation, we can set $m = 0$ in the general form of the particular solution $y_p = x^m P_s(x)e^{qx}$.

As a result, the particular solution y_p of our differential equation will be of the form $y_p = P_s(x) = Ax^3 + Bx^2 + Cx + D$. If we insert this particular solution into the left-hand side of the differential equation, we obtain the following expression:

$$\begin{aligned} & y_p''' - 5y_p'' + 8y_p' - 4y_p \\ &= (6A) - 5(6Ax + 2B) + 8(3Ax^2 + 2Bx + C) - 4(Ax^3 + Bx^2 + Cx + D) \\ &= -4Ax^3 + (24A - 4B)x^2 + (-30A + 16B - 4C)x + (6A - 10B + 8C - 4D) \end{aligned}$$

If we compare the above expression with the right-hand side of the differential equation, i.e., $x^3 - 2x + 1$, we find a system of four equations and four unknown variables:

$$\begin{cases} -4A = 1 \\ 24A - 4B = 0 \\ -30A + 16B - 4C = -2 \\ 6A - 10B + 8C - 4D = 1 \end{cases} \Leftrightarrow \begin{cases} A = -\frac{1}{4} \\ B = -\frac{3}{2} \\ C = -\frac{29}{8} \\ D = -\frac{33}{8} \end{cases}$$

The particular solution y_p of the differential equation is therefore equal to:

$$y_p = -\frac{x^3}{4} - \frac{3x^2}{2} - \frac{29x}{8} - \frac{33}{8}$$

The total general solution $y = y_h + y_p$ then becomes:

$$y = y_h + y_p = c_1 e^x + (c_2 + c_3 x) e^{2x} - \frac{x^3}{4} - \frac{3x^2}{2} - \frac{29x}{8} - \frac{33}{8}$$

- Example 2

Consider the following differential equation:

$$y^{(5)} - 15y^{(4)} + 92y''' - 288y'' + 460y' - 300y = e^{3x} \cos x$$

In a first step, we determine the roots of the characteristic equation of the corresponding homogeneous differential equation:

$$\begin{aligned} & y^{(5)} - 15y^{(4)} + 92y''' - 288y'' + 460y' - 300y = 0 \\ \Leftrightarrow & e^\lambda [\lambda^5 - 15\lambda^4 + 92\lambda^3 - 288\lambda^2 + 460\lambda - 300] = 0 \\ \Leftrightarrow & \lambda^5 - 15\lambda^4 + 92\lambda^3 - 288\lambda^2 + 460\lambda - 300 = 0 \\ \Leftrightarrow & (\lambda^4 - 12\lambda^3 + 56\lambda^2 - 120\lambda + 100)(\lambda - 3) = 0 \\ \Leftrightarrow & [(\lambda^4 - 12\lambda^3 + 36\lambda^2) + (20\lambda^2 - 120\lambda) + 100](\lambda - 3) = 0 \\ \Leftrightarrow & [(\lambda^2 - 6\lambda)^2 + 20(\lambda^2 - 6\lambda) + 100](\lambda - 3) = 0 \\ \Leftrightarrow & (\lambda^2 - 6\lambda + 10)^2(\lambda - 3) = 0 \\ \Leftrightarrow & [\lambda - (3 + i)]^2 [\lambda - (3 - i)]^2 (\lambda - 3) = 0 \end{aligned}$$

For the root $\lambda_1 = 3$ with multiplicity $m_1 = 1$, we have the solution $y = x^0 e^{\lambda_1 x} = e^{3x}$.

With respect to the complex root $\lambda_2 = \alpha + i\beta = 3 + i$ with multiplicity $m_2 = 2$, its solutions are equal to $y = x^0 e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)] = e^{3x} [\cos(x) + i \sin(x)]$ and $y = x^1 e^{\alpha x} [\cos(\beta x) + i \sin(\beta x)] = x e^{3x} [\cos(x) + i \sin(x)]$.

We find two similar solutions for the complex root $\lambda_3 = \alpha + i\beta = 3 - i$ with multiplicity $m_3 = 2$: $y = e^{3x} [\cos(x) - i \sin(x)]$ and $y = x e^{3x} [\cos(x) - i \sin(x)]$.

With respect to the complex roots, the homogeneous solution entails a linear combination of the terms $e^{3x} \cos x$, $e^{3x} \sin x$, $x e^{3x} \cos x$, and $x e^{3x} \sin x$.

The homogeneous solution y_h of the differential equation is equal to:

$$\begin{aligned} y_h &= c_1 e^{3x} + c_2 e^{3x} \cos x + c_3 e^{3x} \sin x + c_4 x e^{3x} \cos x + c_5 x e^{3x} \sin x \\ &= [c_1 + (c_2 + c_4 x) \cos x + (c_3 + c_5 x) \sin x] e^{3x} \end{aligned}$$

Regarding the particular solution y_p , we want it to have the form $y_p = x^m P_s(x) e^{qx}$. Recall from section 8.4 that the cosine in complex terms can be written as $\cos x = \frac{e^{ix} + e^{-ix}}{2}$. The right-hand side of the differential equation then becomes:

$$P(x) e^{qx} = e^{3x} \left[\frac{e^{ix} + e^{-ix}}{2} \right] = \frac{e^{(3+i)x} + e^{(3-i)x}}{2}$$

Given that $q_1 = 3 + i$ and $q_2 = 3 - i$ are both a root of the characteristic equation with multiplicity $m = 2$, the particular solution has the following form:

$$\begin{aligned} y_p &= x^2 P_{s_1}(x) e^{(3+i)x} + x^2 P_{s_2}(x) e^{(3-i)x} = x^2 C_1 e^{(3+i)x} + x^2 C_2 e^{(3-i)x} \\ &= x^2 C_1 e^{3x} (\cos x + i \sin x) + x^2 C_2 e^{3x} (\cos x - i \sin x) \\ &= x^2 e^{3x} [(C_1 + C_2) \cos x + i(C_1 - C_2) \sin x] \\ &= x^2 e^{3x} (A \cos x + B \sin x) \end{aligned}$$

Since in the right-hand side of the original differential equation $P(x) = \frac{1}{2}$, which has a degree equal to $p = 0$, we have set $P_{s_1}(x)$ and $P_{s_2}(x)$ equal to the constants C_1 and C_2 .

Next, we calculate the first five derivatives of y_p and insert them into the left-hand side of the differential equation. These derivatives are the following:

- $y'_p = ([2Ax + (3A + B)x^2] \cos x + [2Bx + (-A + 3B)x^2] \sin x) e^{3x}$
- $y''_p = ([2A + (12A + 4B)x + (8A + 6B)x^2] \cos x + [2B + (-4A + 12B)x + (-6A + 8B)x^2] \sin x) e^{3x}$
- $y'''_p = ([(18A + 6B) + (48A + 36B)x + (18A + 26B)x^2] \cos x + [(-6A + 18B) + (-36A + 48B)x + (-26A + 18B)x^2] \sin x) e^{3x}$
- $y^{(4)}_p = ([(96A + 72B) + (144A + 208B)x + (28A + 96B)x^2] \cos x + [(-72A + 96B) + (-208A + 144B)x + (-96A + 28B)x^2] \sin x) e^{3x}$
- $y^{(5)}_p = ([(360A + 520B) + (280A + 960B)x + (-12A + 316B)x^2] \cos x + [(-520A + 360B) + (-960A + 280B)x + (-316A - 12B)x^2] \sin x) e^{3x}$

If we insert these expressions into the left-hand side of the differential equation, we obtain the following result:

$$y_p^{(5)} - 15y_p^{(4)} + 92y_p''' - 288y_p'' + 460y_p' - 300y_p = (-8B \cos x + 8A \sin x) e^{3x}$$

Given that the right-hand side of the original differential equation is equal to $e^{3x} \cos x$, we have that $A = 0$ and $B = -\frac{1}{8}$. Therefore, the particular solution y_p becomes:

$$y_p = x^2 e^{3x} (A \cos x + B \sin x) = -\frac{x^2}{8} e^{3x} \sin x$$

The general solution $y = y_h + y_p$ to this differential equation is then equal to:

$$\begin{aligned} y &= y_h + y_p = [c_1 + (c_2 + c_4x) \cos x + (c_3 + c_5x) \sin x] e^{3x} - \frac{x^2}{8} e^{3x} \sin x \\ &= \left[c_1 + (c_2 + c_4x) \cos x + \left(c_3 + c_5x - \frac{x^2}{8} \right) \sin x \right] e^{3x} \end{aligned}$$

- Example 3

Let us consider the differential equation $15y''' - y'' - 11y' - 3y = 15xe^x$. First, we find the roots of the characteristic equation of the corresponding homogeneous differential equation:

$$\begin{aligned}
 15y''' - y'' - 11y' - 3y = 0 &\Leftrightarrow y''' - \frac{y''}{15} - \frac{11y'}{15} - \frac{y}{5} = 0 \\
 &\Leftrightarrow e^\lambda \left[\lambda^3 - \frac{\lambda^2}{15} - \frac{11\lambda}{15} - \frac{1}{5} \right] = 0 \\
 &\Leftrightarrow \lambda^3 - \frac{\lambda^2}{15} - \frac{11\lambda}{15} - \frac{1}{5} = 0 \\
 &\Leftrightarrow (\lambda - 1) \left(\lambda^2 + \frac{14\lambda}{15} + \frac{1}{5} \right) = 0 \\
 &\Leftrightarrow (\lambda - 1) \left(\lambda + \frac{1}{3} \right) \left(\lambda + \frac{3}{5} \right) = 0
 \end{aligned}$$

The root $\lambda_1 = 1$ with multiplicity $m_1 = 1$ produces the individual solution $y = e^x$. The next root $\lambda_2 = -\frac{1}{3}$ with multiplicity $m_2 = 1$ has the solution $y = e^{-\frac{x}{3}}$. Finally, the third root $\lambda_3 = -\frac{3}{5}$ with multiplicity $m_3 = 1$ gives the individual solution $y = e^{-\frac{3x}{5}}$.

The homogeneous solution y_h of the differential equation is a linear combination of these three individual solutions:

$$y_h = c_1 e^x + c_2 e^{-\frac{x}{3}} + c_3 e^{-\frac{3x}{5}}$$

With respect to the particular solution $y_p = x^m P_s(x) e^{qx}$, we find that $q = 1$ and $m = 1$ because $\lambda_1 = 1$ is a root of the characteristic equation with multiplicity $m_1 = 1$. In the right-hand side of the original differential equation, we see that $P(x) = 15x$, which has a degree $p = 1$. Therefore, $P_s(x)$ constitutes a polynomial with an equal degree $p = 1$ and of the form $P_s(x) = Ax + B$. The particular solution y_p thus looks as follows:

$$y_p = x^m P_s(x) e^{qx} = x(Ax + B)e^x$$

Next, we calculate its first three derivatives and insert them into the left-hand side of the differential equation. These three derivatives are the following:

$$\begin{aligned}
 \bullet \quad y'_p &= [Ax^2 + (2A + B)x + B] e^x \\
 \bullet \quad y''_p &= [Ax^2 + (4A + B)x + 2(A + B)] e^x \\
 \bullet \quad y'''_p &= [Ax^2 + (6A + B)x + 3(2A + B)] e^x
 \end{aligned}$$

If we insert these expressions into the left-hand side of the differential equations, we find the following result:

$$\begin{aligned}
& 15 \left([Ax^2 + (6A + B)x + 3(2A + B)] e^x \right) - \left([Ax^2 + (4A + B)x + 2(A + B)] e^x \right) \\
& - 11 \left([Ax^2 + (2A + B)x + B] e^x \right) - 3 \left([Ax^2 + Bx] e^x \right) = 15xe^x \\
\Leftrightarrow & [64Ax + 88A + 32B] e^x = 15xe^x
\end{aligned}$$

Comparing both sides gives us the following set of equations:

$$\begin{cases} 64A = 15 \\ 88A + 32B = 0 \end{cases} \Leftrightarrow \begin{cases} A = \frac{15}{64} \\ B = -\frac{165}{256} \end{cases}$$

The particular solution then becomes:

$$y_p = x(Ax + B)e^x = \left[\frac{15x^2}{64} - \frac{165x}{256} \right] e^x$$

The general solution $y = y_h + y_p$ of the differential equation is therefore equal to:

$$\begin{aligned}
y = y_h + y_p &= c_1 e^x + c_2 e^{-\frac{x}{3}} + c_3 e^{-\frac{3x}{5}} + \left[\frac{15x^2}{64} - \frac{165x}{256} \right] e^x \\
&= \left[c_1 + \frac{15x^2}{64} - \frac{165x}{256} \right] e^x + c_2 e^{-\frac{x}{3}} + c_3 e^{-\frac{3x}{5}}
\end{aligned}$$

- Example 4

Consider the differential equation $y^{(4)} + y'' = 6 \cos(3x) \cos(4x)$. Before we proceed to solve this equation, we wish to rewrite the right-hand side in a form that is recognizable with respect to the general form $b(x) = P(x)e^{qx}$.

Simpson's formulas (see section 5.3) tell us that:

$$\begin{aligned}
6 \cos(3x) \cos(4x) &= 3 [2 \cos(3x) \cos(4x)] \\
&= 3 [\cos(3x - 4x) + \cos(3x + 4x)] \\
&= 3 [\cos(x) + \cos(7x)]
\end{aligned}$$

In a first step, we determine the roots of the characteristic equation corresponding to the homogeneous differential equation:

$$\begin{aligned}
y^{(4)} + y'' = 0 &\Leftrightarrow e^\lambda [\lambda^4 + \lambda^2] = 0 \\
&\Leftrightarrow \lambda^4 + \lambda^2 = 0 \\
&\Leftrightarrow \lambda^2 (\lambda^2 + 1) = 0 \\
&\Leftrightarrow \lambda^2 (\lambda + i) (\lambda - i) = 0
\end{aligned}$$

The first root $\lambda_1 = 0$ is real and has multiplicity $m_1 = 2$, so that its solutions are $y = x^0 e^{\lambda_1 x} = 1$ and $y = x^1 e^{\lambda_1 x} = x$. In contrast, the second and third root $\lambda_2 = \alpha + i\beta = -i$ and $\lambda_3 = \alpha + i\beta = i$ are complex with multiplicity $m_2 = 1$ and $m_3 = 1$, respectively, with corresponding solutions $y = x^0 e^{\alpha x} (\cos x + i \sin x) = \cos x + i \sin x$ and $y = x^0 e^{\alpha x} (\cos x - i \sin x) = \cos x - i \sin x$.

The homogeneous solution y_h is a linear combination of these individual solutions:

$$y_h = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$$

Regarding the particular solution $y_p = x^m P_s(x) e^{q x}$, note that the right-hand side of the differential equation has two parts: $3 \cos x$ and $3 \cos(7x)$. With respect to the first part, given that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, we have that $q_1 = -i$ and $q_2 = i$ with $m = 1$, which correspond to the roots $\lambda_2 = -i$ and $\lambda_3 = i$ of the characteristic equation. The particular solution y_{p1} of the first part has therefore the following form (with A_1 and B_1 two real constants):

$$\begin{aligned} y_{p1} &= x^m P_{s_1}(x) e^{q_1 x} + x^m P_{s_2}(x) e^{q_2 x} = x A_1 e^{-ix} + x B_1 e^{ix} \\ &= x [(A_1 + B_1) \cos x + i (B_1 - A_1) \sin x] \\ &= x (A \cos x + B \sin x) \end{aligned}$$

With respect to the second part $3 \cos(7x)$, the variables $q_1 = 7i$ and $q_2 = -7i$ do not correspond to any of the three roots of the characteristic equation, so that $m = 0$. The particular solution y_{p2} related to the second part has thus the following form (with $A_2, B_2 \in \mathbb{R}$):

$$\begin{aligned} y_{p2} &= x^m P_{s_1}(x) e^{q_1 x} + x^m P_{s_2}(x) e^{q_2 x} = x^0 A_2 e^{7ix} + x^0 B_2 e^{-7ix} \\ &= (A_2 + B_2) \cos(7x) + i (A_2 - B_2) \sin(7x) \\ &= C \cos(7x) + D \sin(7x) \end{aligned}$$

Note furthermore that for both parts the polynomial $P(x)$ is equal to $P(x) = 3$ with a degree $p = 0$, so that the respective polynomial $P_{s_i}(x)$ has been set equal to a constant.

The total particular solution y_p is therefore equal to:

$$y_p = y_{p1} + y_{p2} = x(A \cos x + B \sin x) + C \cos(7x) + D \sin(7x)$$

In a next step, we calculate the second and the fourth derivative of y_p and insert these expressions into the left-hand side of the differential equation. These derivatives are equal to:

$$\begin{aligned} \bullet \quad y_p'' &= (-Ax + 2B) \cos x + (-2A - Bx) \sin x - 49 [C \cos(7x) + D \sin(7x)] \\ \bullet \quad y_p^{(4)} &= (Ax - 4B) \cos x + (4A + Bx) \sin x + 2401 [C \cos(7x) + D \sin(7x)] \end{aligned}$$

If we insert these two expressions into the left-hand side of the differential equation, we obtain the following equation:

$$\begin{aligned}
y^{(4)} + y'' &= 3 [\cos(x) + \cos(7x)] \\
\Leftrightarrow [(Ax - 4B) \cos x + (4A + Bx) \sin x + 2401 [C \cos(7x) + D \sin(7x)]] \\
&+ [(-Ax + 2B) \cos x + (-2A - Bx) \sin x - 49 [C \cos(7x) + D \sin(7x)]] \\
&= 3 [\cos(x) + \cos(7x)] \\
\Leftrightarrow -2B \cos x + 2A \sin x + 2352C \cos(7x) + 2352D \sin(7x) &= 3 [\cos(x) + \cos(7x)]
\end{aligned}$$

This produces the following system of equations :

$$\begin{cases} -2B = 3 \\ 2A = 0 \\ 2352C = 3 \\ 2352D = 0 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = -\frac{3}{2} \\ C = \frac{1}{784} \\ D = 0 \end{cases}$$

The particular solution y_p can then be written in the following way:

$$y_p = y_{p1} + y_{p2} = x(A \cos x + B \sin x) + C \cos(7x) + D \sin(7x) = -\frac{3x}{2} \sin x + \frac{\cos(7x)}{784}$$

For the general solution $y = y_h + y_p$ we find the following result:

$$\begin{aligned}
y = y_h + y_p &= c_1 + c_2x + c_3 \cos x + c_4 \sin x - \frac{3x}{2} \sin x + \frac{\cos(7x)}{784} \\
&= c_1 + c_2x + c_3 \cos x + \frac{\cos(7x)}{784} + \left(c_4 - \frac{3x}{2} \right) \sin x
\end{aligned}$$

- Example 5

Consider the differential equation $y''' - 2y'' - 7y' - 4y = x \cosh x$. In a first step, we determine the roots of the characteristic equation of the corresponding homogeneous differential equation:

$$\begin{aligned}
y''' - 2y'' - 7y' - 4y = 0 &\Leftrightarrow e^\lambda [\lambda^3 - 2\lambda^2 - 7\lambda - 4] = 0 \\
&\Leftrightarrow \lambda^3 - 2\lambda^2 - 7\lambda - 4 = 0 \\
&\Leftrightarrow (\lambda + 1)(\lambda^2 - 3\lambda - 4) = 0 \\
&\Leftrightarrow (\lambda + 1)^2(\lambda - 4) = 0
\end{aligned}$$

The root $\lambda_1 = -1$ has multiplicity $m_1 = 2$ and produces the individual solutions $y = x^0 e^{\lambda_1 x} = e^{-x}$ and $y = x^1 e^{\lambda_1 x} = x e^{-x}$. The solution of the second root $\lambda_2 = 4$ with multiplicity $m_2 = 1$ is equal to $y = x^0 e^{\lambda_2 x} = e^{4x}$.

The homogeneous solution y_h of the differential equation is equal to a linear combination of these three individual solutions:

$$y_h = (c_1 + c_2 x) e^{-x} + c_3 e^{4x}$$

With respect to the particular solution $y_p = x^m P_s(x) e^{qx}$, let us first have a closer look at the right-hand side of the differential equation. Using the definition of the hyperbolic cosine (see section 7.6.9), we can rewrite the right-hand side as follows:

$$b(x) = x \cosh x = x \left[\frac{e^x + e^{-x}}{2} \right] = \frac{x}{2} e^x + \frac{x}{2} e^{-x}$$

With respect to the first part $\frac{x}{2} e^x$, we have that $q = 1$, which does not correspond to any root of the characteristic equation, so that $m = 0$. Given that $P_1(x) = \frac{x}{2}$ with degree $p = 1$, we demand that $P_{s1}(x)$ has the same degree and is thus of the form $P_{s1}(x) = Ax + B$. The particular solution y_{p1} related to the first part is then equal to:

$$y_{p1} = x^m P_{s1}(x) e^{qx} = (Ax + B) e^x$$

Regarding the second part $\frac{x}{2} e^{-x}$, since $q = -1$ corresponds to the first root $\lambda_1 = -1$ with multiplicity $m_1 = m = 2$ and since $P_2(x) = \frac{x}{2}$ with degree $p = 1$, the particular solution y_{p2} related to the second part becomes:

$$y_{p2} = x^m P_{s2}(x) e^{qx} = x^2 (Cx + D) e^{-x}$$

The complete particular solution y_p of the differential equation has therefore the following form:

$$y_p = y_{p1} + y_{p2} = (Ax + B) e^x + x^2 (Cx + D) e^{-x}$$

Next, we calculate the first three derivatives of y_p and insert them into the left-hand side of the differential equation. These derivatives are equal to:

- $y'_p = [Ax + (A + B)] e^x + [-Cx^3 + (3C - D)x^2 + 2Dx] e^{-x}$
- $y''_p = [Ax + (2A + B)] e^x + [Cx^3 + (-6C + D)x^2 + (6C - 4D)x + 2D] e^{-x}$
- $y'''_p = [Ax + (3A + B)] e^x + [-Cx^3 + (9C - D)x^2 + (-18C + 6D)x + (6C - 6D)] e^{-x}$

Inserting these derivatives into the left-hand side of the differential equation gives us the following expression:

$$y_p''' - 2y_p'' - 7y_p' - 4y_p = [-12Ax + (-8A - 12B)]e^x + [-30Cx + 6C - 10D]e^{-x}$$

If we compare this expression with the right-hand side of our differential equation, i.e., $x \cosh x = \frac{x}{2}e^x + \frac{x}{2}e^{-x}$, we obtain the following set of equations:

$$\begin{cases} -12A = \frac{1}{2} \\ -8A - 12B = 0 \\ -30C = \frac{1}{2} \\ 6C - 10D = 0 \end{cases} \Leftrightarrow \begin{cases} A = -\frac{1}{24} \\ B = \frac{1}{36} \\ C = -\frac{1}{60} \\ D = -\frac{1}{100} \end{cases}$$

The particular solution y_p then becomes:

$$y_p = (Ax + B)e^x + x^2(Cx + D)e^{-x} = \left(-\frac{x}{24} + \frac{1}{36}\right)e^x + x^2\left(-\frac{x}{60} - \frac{1}{100}\right)e^{-x}$$

For the general solution $y = y_h + y_p$ we find the following result:

$$\begin{aligned} y = y_h + y_p &= (c_1 + c_2x)e^{-x} + c_3e^{4x} + \left(-\frac{x}{24} + \frac{1}{36}\right)e^x + x^2\left(-\frac{x}{60} - \frac{1}{100}\right)e^{-x} \\ &= \left(c_1 + c_2x - \frac{x^2}{100} - \frac{x^3}{60}\right)e^{-x} + \left(-\frac{x}{24} + \frac{1}{36}\right)e^x + c_3e^{4x} \end{aligned}$$