Physics

Exercises on Kinematics in One, Two, and Three Dimensions

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Summary of Exercises

Exercise 1

Gansukh is driving his car at a velocity of $\vec{v}_{car} = 115 \cdot \vec{i}_x$ km/h along some dusty roads in Tunkh, Mongolia. At a certain moment, the road runs more or less parallel to a railway track, and Gansukh catches up with a train that is following the Trans-Mongolian Railway route and currently travels at $\vec{v}_{train} = 95.0 \cdot \vec{i}_x$ km/h. (1) If you know that the train's length measures $L = 850$ m, how long does it take Gansukh to completely overtake the train? (2) What distance has he covered during this manoeuvre? (3) If Gansukh were moving in the opposite direction to the train, what would your answers be to the questions in part (1) and (2)? (4) Suppose in the scenario of part (1) that at some point Gansukh slows down for about 3.20 s, after which he drives right beside the mid point of the train at the same constant speed. At which position relative to the train did Gansukh start to decelerate?

Exercise 2

In the midst of a police car pursuit close to Parque Kanata in Cochabamba, Bolivia, Blanca, who is the police woman in the front passenger seat with an extremely keen eyesight, suddenly spots a squirrel at a distance of $d = 150$ m and shouts to her colleague Manuela in the driver's seat: "Watch out, a squirrel!" It takes Manuela approximately $t_r = 1.20$ s to understand the situation before slamming the breaks at a deceleration of $\vec{a} = -11.1$ m/s^2. (1) At what distance from the squirrel does Manuela manage to get the car to a complete standstill, knowing that she is driving at $\vec{v}_p = 165 \cdot \vec{i}_x$ km/h? (2) How much time has passed from the moment her colleague Blanca warned her? (3) Suppose that Blanca saw the same squirrel but only this time it is running towards the police car at $\vec{v}_{sq} = -2.30 \cdot \vec{i}_x$ m/s. If you know that Manuela stopped the car at a distance of $d_s = 10.0$ cm from the squirrel, how much harder did Manuela have to hit the breaks?

Exercise 3

Kopano is a physics teacher at the Newton International School in Gaborone, Botswana, and he just bought some new measuring devices for the physics course he is teaching at the students of Grade 12. Kopano is trying out his new equipment whereby he places a tennis ball pitching machine on the ground next to a building and vertically shoots a tennis ball up in the air. Kopano is standing at the window on the second floor and has measured that the ball takes $t_w = 0.25$ s to travel from the window sill to the top edge of the window. (1) If you know that the sill is positioned at a distance of $d_s = 6.5$ m above the ground and that the height of the window is equal to $h = 1.5$ m, with which speed did the tennis ball depart from the pitching machine? (2) How high will the ball go? (3) Assuming that the tennis ball is traveling upwards, how many seconds ago was the ball launched when it is now at eye level with Kopano (suppose that this is half way the height h of the window)? (4) Finally, after how many seconds, starting from the point in part (3), will the tennis ball fall to the ground?

Exercise 4

On a crisp Sunday afternoon, you are cycling at an average velocity of $\vec{v} = 22.0 \cdot \vec{i}_x$ km/h in the middle of the countryside nearby the town Katvari in Latvia. At a certain moment, another cyclist passes by you and after $t_{react} = 2.00$ s you realize it's Ilja, a friend of yours. You then start to accelerate ($\vec{a} = 0.385 \cdot \vec{i}_x$ m/s^2) and overtake Ilja after exactly $t_{sprint} = 12.3$ s. (1) Determine the average velocity $\vec{v}_{friend}$ at which your friend is cycling. (2) Instead of overtaking Ilja, you wish to end up cycling right next to Ilja. Therefore, after $t_1 = 10.0$ s of sprinting, you slow down at a rate of $\vec{a}_s = -a \cdot \vec{i}_x$. What is the distance d_s covered during this period of deceleration? (3) How long does it now take you to catch up with Ilja under the scenario described in part (2)? (4) How much more total distance do you now need compared to the situation in part (2)?

Exercise 5

Bart and Tina are training in their hometown Ninove, Belgium, for the ultramarathon "The Bali Hope Ultra", which takes place in Bali, one of the islands of the Indonesian archipelago, in September 2022 and whereby the participants are crossing the island overnight from north to south over a distance of 84.9 km. According to their training schedule, they are running today a distance of 35.0 km. About $d = 88.0$ m before the end of their run, Bart maintains a velocity of $\vec{v}_B = 5.04 \cdot \vec{i}_x$ m/s, while Tina is running at a higher pace of $\vec{v}_T = 6.22 \cdot \vec{i}_x$ m/s and is a distance of $x_{BT} = 24.5$ m ahead of Bart. Tina feels a cramp coming up in the calf muscles of her right leg which slows her down at a rate of $\vec{a}_T = -0.27 \cdot \vec{i}_x$ m/s^2. (1) If Bart wishes to overtake Tina just $d_f = 5.00$ m before the end of their run, at what rate should Bart then start to accelerate? (2) What is Bart's velocity right at the moment when his GPS marks a total distance covered of 35.0 km? (3) Suppose that Bart maintains his acceleration calculated in part (1), but now after $t_{rec} = 8.50$ s Tina recovers from her cramp and runs the last bit with a constant velocity. Who sees the 35.0 km mark on their GPS first?

Exercise 6

Luca is standing at the edge of the White Cliffs of Dover in the county of Kent, the United Kingdom, and he wants to show his son Michael how to estimate their height h when you only have the elements of nature at your disposal. When Luca throws a cobblestone vertically into the air with an initial velocity of $\vec{v}_0 = -2.50 \cdot \vec{i}_x$ m/s, they hear the characteristic splash of water about $t_{tot} = 5.32$ s later. (1) Given a speed of sound equal to $v_s = 343$ m/s, what is the estimated height h of the White Cliffs of Dover that Luca and his son have calculated? (2) With what velocity does the cobblestone hit the water? (3) Michael also throws a cobblestone at an initial speed of $v_M = 4.35$ m/s under an angle of $\theta = 61.5°$ with the horizontal. When do they hear the splash?

Exercise 7

Fig. 6 shows the position-time graph for two swans swimming in the same direction in the Evros Delta in the north of Greece. Formulate an answer for the following questions: (1) Do the swans at some point move at the same velocity? (2) Which swan has the greatest acceleration? (3) Are the swans overtaking each other at any particular instant? (4) Which swan registers the largest instantaneous velocity? (5) Which swan undergoes the greatest displacement? (6) How do the average velocities of the swans compare to one another?

Exercise 8

Marwah is traveling at $\vec{v}_M = 115 \cdot \vec{i}_x$ km/h on a two-lane expressway east of Imilchil, Morocco, and approaching a $L = 45.0$ m-long double-trailer truck, which is going at $\vec{v}_t = 90.0 \cdot \vec{i}_x$ km/h. To overtake the truck, Marwah needs $d_L = 12.0$ m of leeway at each end of the truck. However, there is an oncoming minivan traveling at a speed of $\vec{v}_m = -85.0 \cdot \vec{i}_x$ km/h approximately $d = 500$ m away. (1) Can Marwah safely overtake the truck without accelerating? (2) If not, by how much should Marwah accelerate if she wants the minivan to pass by her at least $t_{margin} = 2.00$ s after she has overtaken the truck, and what would be her velocity after this manoeuvre? (3) How much distance has Marwah traveled during this manoeuvre?

Exercise 9

Yusef and Hasan are sitting at the back of the classroom in the Jordan National School in Irbid, Jordan, and are interested in anything but the current geography class. They started to make little wads of paper which they subsequently flick away with the tip of their finger from the edge of their desk, giving them a horizontal velocity of $\vec{v}_{0x} = 3.50 \cdot \vec{i}_x$ m/s. (1) At which angle and with what velocity does one little ball of paper hit the ground, measured at $h = 1.50$ m below the edge of the desk? (2) How far (horizontally speaking) from the edge of the desk does the paper land? (3) If Yusef drops a wad of paper vertically from the edge of the desk at the same time that Hasan flicks another, identical wad of paper away from the desk, which one touches the floor first? (4) If a wad of paper is flung away at a greater horizontal velocity of $\vec{v}_{0x2} = 4.15 \cdot \vec{i}_x$ m/s, how does this change the answer to part (3)?

Exercise 10

You feel like throwing a pineapple from your bedroom window into the rectangular pond you had recently installed in the garden. You launch the pineapple at an initial speed of $v_0 = 8.80$ m/s under an angle θ and it hits the water $d_{we} = 1.10$ m from the edge of the pond closest to the back of your house. (1) If your bedroom is located $h = 9.50$ m above ground level and the distance between the house and the closest edge measures $d_{he} = 7.00$ m, under what angle are you throwing the pineapple? (2) If the pond is half a meter deep and the water slows the pineapple down with an acceleration whose magnitude is equal to $a_w = 65.0$ m/s^2, will the pineapple hit the bottom of the pond?

Exercise 11

Ning is a professional archer and she is currently training for the Paris 2024 Olympic Games. During her next training, she aims to hit a target on the ground while leaning out of a helicopter, which is flying horizontally at $\vec{v}_h = 139 \cdot \vec{i}_x$ km/h at an altitude of $h = 40.0$ m near the Dongqian Lake in

Ningbo, China. On the day of her training, there is a forecasted headwind of $\vec{v}_w = -54.0 \cdot \vec{i}_x$ km/h and Ning has brought a compound bow that is able to shoot arrows at a speed of $v_a = 80.0$ m/s. (1) How far will Ning's arrows travel under these conditions? (2) If Ning lowers her bow, making an angle of $\theta = 20.0°$ with the direction of flight, at what horizontal distance must the helicopter be away from the target if she wants to hit the mark? (3) Suppose that in part (1) the headwind is making an angle $\alpha < 90°$ with the vertical, so that the y-component of the arrow's velocity right before hitting the mark is equal to $\vec{v}_y = -29.9 \cdot \vec{i}_y$ m/s. At what distance from the mark did Ning release her arrow?

Exercise 12

Due to the lower gravity on the planet Mars, its atmosphere is thinner and more volatile relative to Earth. One of the consequences is that dust is easily swept up and lingers throughout the atmosphere. Some of the chemical compounds and elements that constitute Martian dust include phosphorus pentoxide (P_4O_{10}), titanium dioxide (TiO_2), zinc (Zn), and manganese(II) oxide (MnO). Suppose now that a TiO_2 compound finds itself above the Gusev Crater and that, at $t = 0$ s, at the position $\vec{r}(t) = 1.0 \cdot \vec{i}_x + 1.0 \cdot \vec{i}_y + 2.0 \cdot \vec{i}_z$ m, it has a velocity of $\vec{v}_0(t) = 6.5 \cdot \vec{i}_y + 2.0 \cdot \vec{i}_z$ m/s. If the TiO_2 compound is being accelerated by an upcoming dust storm at a rate of $\vec{a}(t) = 2.3 \cdot \vec{i}_x - 3.1 \cdot \vec{i}_y + 0.5 \cdot \vec{i}_z$ m/s^2, what is the compound's position and velocity when the y coordinate reaches its maximum?

Exercise 13

Juan is a locally famous stuntman in the region around the city of Bayamo, Cuba, and he is about to try out a new stunt in his home in the outskirts of Bayamo. Juan takes place in his self-made ejector seat, which is installed upon a rotatable and tiltable platform, so that he sits $s = 1.55$ m above ground level. Facing north, he directs his seat in a straight line with the ridge of the barn that is right in front of him, and tilts it in the forward direction until it makes a $\theta = 65.0°$ angle with the ground. When Juan gets ejected from his seat, he manages to travel a horizontal distance of $d_x = 10.5$ m and ends up on the $w = 4.00$ m wide balcony that is attached perfectly symmetrical to the front of the barn. (1) If you know that the balcony hangs $h = 8.65$ m above the ground, what is the launching speed v_0 of his home-made ejector seat? (2) After a couple of runs, one of the bolts in the platform is partially unscrewed, so that by the time of the next ejection round Juan's seat has effectively rotated $\phi = 5.00°$ east of north. Will Juan still make it to the balcony?

Exercise 14

Giulia, an extreme sports fanatic, stands on the edge of the 754 m high sea cliff Cape Enniberg at the Faroe Islands and is looking over the Norwegian Sea ready to take her next parachute jump. She takes a run-up in the northeastern (NE) direction and dives from the cliff head first with a speed of 3.50 m/s under a 25.0° angle with the horizontal. After a free fall of $t_{free} = 8.50$ s, she opens her parachute and about $t_{open} = 2.50$ s later Giulia is descending at a constant velocity with her horizontal speed reduced to 1.00 m/s. At the very moment that her downward velocity becomes constant, a southeastern (SE) wind kicks in with an initial speed of 2.00 m/s and gradually picks up speed with a rate of 0.112 m/s^2. After the onset of the SE wind, Giulia touches the water $t_{descend} = 38.9$ s later.

(1) At what height does Giulia's downward velocity become constant and what is the value of this velocity? (2) What is the total acceleration during the opening of the parachute? (3) What are the coordinates of Giulia's landing spot? (4) With what velocity and under which angle does Giulia land into the water?

Exercise 15

You're practicing your snowboard skills in the indoor ski resort Sayama Indoor Skiing Ground in Tokorozawa, Japan, and when you come to a halt at the bottom of the last slope, you take some time to rest. While you're tossing around a snowball, your mind wanders off to that last slope and suddenly it dawns on you how to solve that particular physics problem you've been thinking about for the past two weeks: If I know the angle ϕ of a slope, under which angle θ with the horizontal should I throw a snowball with a given initial velocity v_0, so that it ends up the farthest as possible on the slope (point d)? Write down the solution you have in your mind.

Exercise 16

Sarki is participating in the Kenyan national competition of acrobatic aircraft racing and during the semifinals, he is required to steer his Zirko Edge 540 plane with a wingspan of 7.42 m right between two buildings that stand 10.0 m apart from each other. The opening through which the aircraft has to pass lies in the south-southwest (SSW) direction and the Zirko Edge 540 has an average air speed of $v_{plane} = 275$ km/h. If Sarki has to deal with a sturdy west wind of $v_w = 65.0$ km/h on the day of his competition, at what angle (west of south) should he better steer his airplane so that it safely whizzes through the opening between the two buildings? What is the magnitude of the resultant (effective) velocity v_R at which Sarki pulls off this manoeuvre?

Exercise 17

Nastya and Keril are taking part in a local sports competition in Nizhny Novgorod, Russia, that involves four main parts: long-distance running, archery, mountain biking, and swimming. They are in the lead and reached the last activity, i.e., swimming. The Volga river is the final leg of the race that stands between them and the finish line, which lies right across the other side of the Volga. Since there is a current of $\vec{v}_{river} = -1.05 \cdot \vec{i}_x$ m/s, they might end up some distance away from the finish line, in which case they have to sprint the last couple of meters.

(1) Given the magnitudes of their swimming and running velocity v_{swim} and v_{run}, respectively, determine a general formula for the fastest route across the Volga. (2) While Nastya is a faster swimmer than Keril ($v_{swim,N} = 1.95$ m/s versus $v_{swim,K} = 1.85$ m/s), she runs at a lower pace ($v_{run,N} = 4.15$ m/s versus $v_{run,K} = 5.20$ m/s). If Keril has an advantage of 19.5 s with respect to Nastya, who wins the competition if both follow their optimal routes? Where do the athletes come ashore? Suppose that the Volga is $d = 850$ m wide at the point where they enter the water.

Exercise 18

Sophia is casually riding her brand-new snowboard on a 32°-blue square slope of the Whistler Mountain in Canada. Being all warmed up after an hour of doing slaloms, Sophia heads towards a first jump, which makes a 13° angle with the horizontal, and pulls off a Chicken Salad grab. She successfully lands her trick 28 m down the hill. (1) What was Sophia's initial velocity? (2) What is her landing velocity? (3) What is the airtime of her jump?

Exercise 19

Tommaso is cruising at sunset at $v_{Cessna} = 232$ km/h in his Cessna 172 Skyhawk above the hilly landscape of Val d'Orcia, Italy. As he is headed north-west towards the town of Siena, he is enjoying the endless vineyards and the picturesque villages, such as Pienza, Monticchiello, and Bagno Vignoni. Due to this mesmerizing scenery, Tommaso forgot to check his instruments during the past 50.0 minutes, and it appears that he already covered 210 km since he last checked and that he is actually flying in the direction of 27.5° west of north. What is the magnitude and direction of the wind velocity $\vec{v}_{wind}$ that is responsible for the shift in his trajectory?

Exercise 20

After spending a day on turbulent waters in the Gulf of Siam, Rangsei is steering her shrimp boat $\theta_i = 30.0°$ north of east towards her docking station at the port of Sihanoukville, Cambodia. When she is 1.50 km away from the port, Rangsei receives a radio call from the local command centre with the message that she must dock 300 m north-west from her usual docking station due to some hindrance caused by local festivities. Given a north-west current of $v_{cur} = 1.20$ m/s, determine the angle θ under which Rangsei must redirect her shrimp boat to safely reach her new docking station, if you know that the boat maintains a velocity of $v_{boat} = 6.52$ kts (1 knot is equal to 1.852 km/h) with respect to still water.

Exercise 1

Problem Statement

Gansukh is driving his car at a velocity of $\vec{v}_{car} = 115 \cdot \vec{i}_x$ km/h along some dusty roads in Tunkh, Mongolia. At a certain moment, the road runs more or less parallel to a railway track, and Gansukh catches up with a train that is following the Trans-Mongolian Railway route and currently travels at $\vec{v}_{train} = 95.0 \cdot \vec{i}_x$ km/h. (1) If you know that the train's length measures $L = 850$ m, how long does it take Gansukh to completely overtake the train? (2) What distance has he covered during this manoeuvre? (3) If Gansukh were moving in the opposite direction to the train, what would your answers be to the questions in part (1) and (2)? (4) Suppose in the scenario of part (1) that at some point Gansukh slows down for about 3.20 s, after which he drives right beside the mid point of the train at the same constant speed. At which position relative to the train did Gansukh start to decelerate?

Solution

(1) We first need to establish the velocity $\vec{v}_{car,r}$ of the car seen from the perspective of someone sitting in the train. If we consider the coordinate system (x',y') that moves along with the train and whereby the positive x-direction points towards the front of the train, then the velocity of the train is equal to the null vector ($\vec{v}_{train} = 0 \cdot \vec{i}_{x'}$ km/h) and the surroundings are passing by in the negative x-direction, from the front towards the back of the train, at a velocity of $\vec{v}_{surroundings} = -95.0 \cdot \vec{i}_{x'}$ km/h.

By simply adding the velocity vectors, a car traveling at $\vec{v}_{car} = 115 \cdot \vec{i}_x$ km/h in the positive x-direction of the coordinate system (x,y) *relative to the ground* is moving at $\vec{v}_{car,r} = (v_{car} - v_{surroundings}) \cdot \vec{i}_{x'} = (115 - 95.0) \cdot \vec{i}_{x'} = 20.0 \cdot \vec{i}_{x'}$ km/h in the positive x'-direction of the coordinate system (x',y') *relative to the train.*

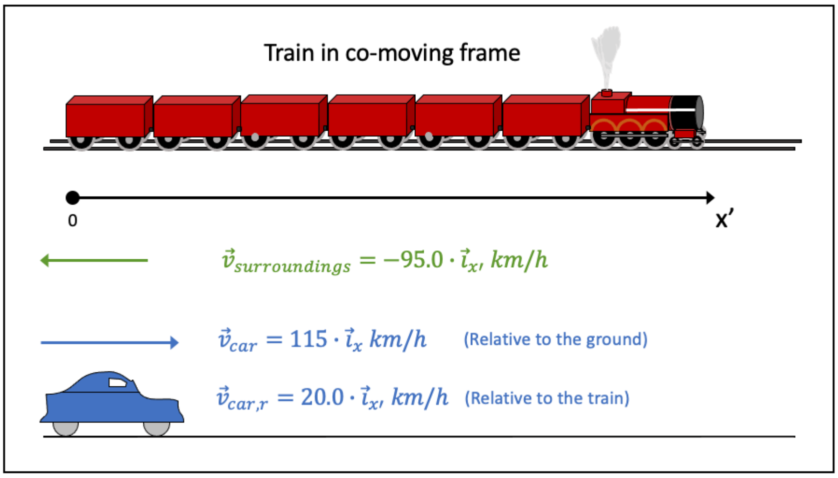

Figure 1

Taking into account the train's length of $L = 850$ m, the time necessary for the car to overtake the train is then calculated as follows (viewed from the reference frame moving along with the train):

$$t = \frac{L}{v_{car,r}} = \frac{850}{5.56} = 153 \text{ s or } 2 \text{ min } 33 \text{ sec}$$

Bear in mind that in the above calculation the speed expressed in km/h is converted into m/s: $v_{car,r} = 20.0$ km/h corresponds to $v_{car,r} = \frac{20.0}{3.60} = 5.56$ m/s.

(2) To find the distance d that Gansukh has driven during this amount of time, we need to switch back to the coordinate system (x,y) *relative to the ground*. That is, we must consider the car's velocity of $\vec{v}_{car} = 115 \cdot \vec{i}_x$ km/h. The distance d becomes:

$$d = v_{car} \cdot t = 31.9 \cdot 153 = 4.89 \times 10^3 \text{ m or } 4.89 \text{ km}$$

(3) In the case where the car is passing the train in the opposite direction, i.e., from the front towards the back of the train at a velocity $\vec{v}_{car,o} = -115 \cdot \vec{i}_x$ km/h, the car's velocity $\vec{v}_{car,ro}$ *relative to the train* is equal to $\vec{v}_{car,ro} = -\left(v_{car,o} + v_{surroundings}\right) \cdot \vec{i}_{x'} = -(115 + 95.0) = -210 \cdot \vec{i}_{x'}$ km/h. The duration t_o during which the car travels alongside the train together with the distance d_o covered are then the following:

$$\begin{cases} t_o = \dfrac{L}{v_{car,ro}} = \dfrac{850}{58.3} = 14.6 \text{ s} \\ \\ d_o = v_{car,o} \cdot t_o = 31.9 \cdot 14.6 = 465 \text{ m} \end{cases}$$

(4) The deceleration $\vec{a}_d$ during the time period of $t_d = 3.20$ s is equal to (whereby $v_f = 0$ m/s is the speed of the train in the co-moving frame of reference):

$$\vec{v}_f = \vec{v}_{car,r} + \vec{a}_d \cdot t_d \quad \Leftrightarrow \quad \vec{a}_d = \left(\frac{v_f - v_{car,r}}{t_d}\right) \cdot \vec{i}_{x'} = \frac{0 - 5.56}{3.20} = -1.74 \cdot \vec{i}_{x'} \text{ m/s}^2$$

Given that the origin of the coordinate system (x',y') is located at the back of the train, the position $\vec{x}_d$ at which Gansukh started to slow down is calculated as follows (whereby $\vec{x}_f = \frac{L}{2} \cdot \vec{i}_{x'}$ is the position of the mid point of the train):

$$\vec{x}_f = \vec{x}_d + \vec{v}_{car,r} \cdot t_d + \frac{\vec{a}_d}{2} \cdot t_d^2 \quad \Leftrightarrow \quad \vec{x}_d = \vec{x}_f - \vec{v}_{car,r} \cdot t_d - \frac{\vec{a}_d}{2} \cdot t_d^2$$

$$= \left[\frac{L}{2} - v_{car,r} \cdot t_d - \frac{(-a_d)}{2} \cdot t_d^2\right] \cdot \vec{i}_{x'}$$

$$= \left[\frac{850}{2} - 5.56 \cdot 3.20 - \frac{(-1.74)}{2} \cdot 3.20^2\right] \cdot \vec{i}_{x'}$$

$$= 416 \cdot \vec{i}_{x'} \text{ m}$$

Exercise 2

Problem Statement

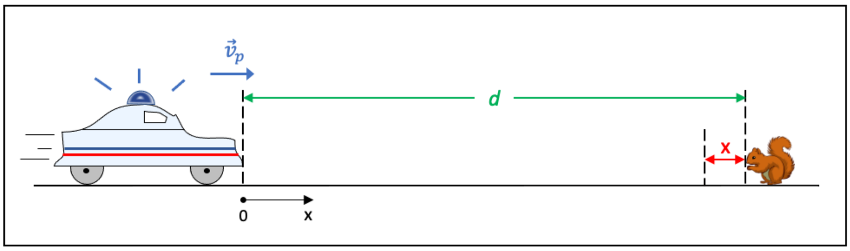

Figure 2

In the midst of a police car pursuit close to Parque Kanata in Cochabamba, Bolivia, Blanca, who is the police woman in the front passenger seat with an extremely keen eyesight, suddenly spots a squirrel at a distance of $d = 150$ m and shouts to her colleague Manuela in the driver's seat: "Watch out, a squirrel!" It takes Manuela approximately $t_r = 1.20$ s to understand the situation before slamming the breaks at a deceleration of $\vec{a} = -11.1$ m/s^2. (1) At what distance from the squirrel does Manuela manage to get the car to a complete standstill, knowing that she is driving at $\vec{v}_p = 165 \cdot \vec{i}_x$ km/h? (2) How much time has passed from the moment her colleague Blanca warned her? (3) Suppose that Blanca saw the same squirrel but only this time it is running towards the police car at $\vec{v}_{sq} = -2.30 \cdot \vec{i}_x$ m/s. If you know that Manuela stopped the car at a distance of $d_s = 10.0$ cm from the squirrel, how much harder did Manuela have to hit the breaks?

Solution

(1) Before hitting the brakes, Manuela needs $t_r = 1.20$ s to react, during which the car travels a distance of:

$$x_r = v_p \cdot t_r = 45.8 \cdot 1.20 = 55.0 \text{ m}$$

This leaves Manuela with a distance of $d_r = d - x_r = 150 - 55.0 = 95.0$ m to avoid hitting the squirrel. The distance d_b covered during breaking is calculated as follows:

$$v_f^2 - v_p^2 = 2 \cdot (-a) \cdot d_b \quad \Leftrightarrow \quad d_b = \frac{v_f^2 - v_p^2}{2 \cdot (-a)} = \frac{0^2 - 45.8^2}{2 \cdot (-11.1)} = 94.6 \text{ m}$$

This means that the distance x (see Fig. 2) between the police car and the squirrel is equal to $x = d_r - d_b = 95.0 - 94.6 = 0.374$ m or 37.4 cm.

(2) The time t_b needed during breaking can be calculated as follows:

$$v_f = v_p - a \cdot t_b \quad \Leftrightarrow \quad t_b = \frac{v_p - v_f}{a} = \frac{45.8 - 0}{11.1} = 4.13 \text{ s}$$

The total time t_{tot} that has passed from the moment Blanca spotted the squirrel to bringing the car to a complete standstill is then equal to $t_{tot} = t_r + t_b = 1.20 + 4.13 = 5.33$ s.

(3) We know from part (1) that the breaking distance d_{b2} is expressed as $d_{b2} = \frac{v_p^2}{2 \cdot a_s}$, with a_s the magnitude of the deceleration $\vec{a}_s$ needed to put the car to a stop at a distance d_s from the squirrel. We also know that the breaking distance is equal to $d_{b2} = d_r - d_s - (v_{sq} \cdot t_{tot2})$, with "$v_{sq} \cdot t_{tot2}$" the distance covered by the squirrel during the total time t_{tot2}, which is the time required for Manuela to stop the car and is equal to $t_{tot2} = t_r + t_{b2}$. The breaking time t_{b2} is expressed as:

$$v_f = 0 = v_p - a_s \cdot t_{b2} \quad \Leftrightarrow \quad t_{b2} = \frac{v_p}{a_s}$$

The total time then becomes $t_{tot2} = t_r + t_{b2} = t_r + \frac{v_p}{a_s}$. If we insert this expression back into our second expression for the breaking distance and subsequently putting the two expressions for the breaking distance equal to each other, we find the magnitude a_s for the deceleration $\vec{a}_s$:

$$\frac{v_p^2}{2 \cdot a_s} = d_r - d_s - \left(v_{sq} \cdot \left[t_r + \frac{v_p}{a_s} \right] \right)$$

$$\Leftrightarrow \quad a_s = \frac{v_p \cdot (v_p + 2 \cdot v_{sq})}{2 \cdot (d_r - d_s - v_{sq} \cdot t_r)}$$

$$= \frac{45.8 \cdot (45.8 + 2 \cdot 2.30)}{2 \cdot (95.0 - 0.10 - 2.30 \cdot 1.20)}$$

$$= 12.5 \text{ m/s}^2$$

With a total time equal to $t_{tot2} = t_r + \frac{v_p}{a_s} = 1.20 + \frac{45.8}{12.5} = 4.85$ s, the squirrel ran a distance of $d_{sq} = v_{sq} \cdot t_{tot2} = 2.30 \cdot 4.85 = 11.2$ m, which explains the greater deceleration of the police car ($a_s = 12.5 \text{ m/s}^2 > a = 11.1 \text{ m/s}^2$) since Manuela now has less distance to stop her car with respect to the situation in part (1).

Exercise 3

Problem Statement

Kopano is a physics teacher at the Newton International School in Gaborone, Botswana, and he just bought some new measuring devices for the physics course he is teaching at the students of Grade 12. Kopano is trying out his new equipment whereby he places a tennis ball pitching machine on the ground next to a building and vertically shoots a tennis ball up in the air. Kopano is standing at the window on the second floor and has measured that the ball takes $t_w = 0.25$ s to travel from the window sill to the top edge of the window. (1) If you know that the sill is positioned at a distance of $d_s = 6.5$ m above the ground and that the height of the window is equal to $h = 1.5$ m, with which speed did the tennis ball depart from the pitching machine? (2) How high will the ball go? (3) Assuming that the tennis ball is traveling upwards, how many seconds ago was the ball launched when it is now at eye level with Kopano (suppose that this is half way the height h of the window)? (4) Finally, after how many seconds, starting from the point in part (3), will the tennis ball fall to the ground?

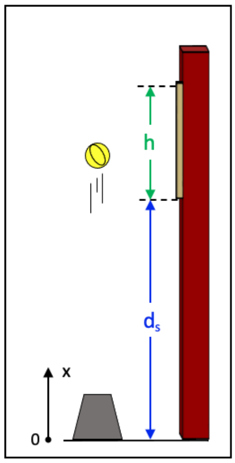

Figure 3

Solution

(1) What we will first calculate is the speed v_w of the tennis ball at the height of the window sill while traveling upwards. Given that the ball needs $t_w = 0.25$ s to cover the height of the window, we find the speed v_w as follows:

$$x_f = x_0 + v_0 \cdot t + \frac{a_x}{2} \cdot t^2 \quad \Leftrightarrow \quad h + d_s = d_s + v_w \cdot t_w - \frac{g}{2} \cdot t_w^2$$

$$\Leftrightarrow \quad v_w = \frac{h}{t_w} + \frac{g \cdot t_w}{2} = \frac{1.5}{0.25} + \frac{9.81 \cdot 0.25}{2} = 7.2 \text{ m/s}$$

In a next step, we determine the initial speed v_0 of the tennis ball when it is launched from the pitching machine:

$$v_w^2 - v_0^2 = 2 \cdot (-g) \cdot h \quad \Leftrightarrow \quad v_0 = \sqrt{v_w^2 + 2 \cdot g \cdot h} = \sqrt{7.2^2 + 2 \cdot 9.81 \cdot 6.5} = 13 \text{ m/s}$$

(2) The maximum height h_{max} the tennis ball reaches during its flight is equal to:

$$v_f^2 - v_0^2 = 2 \cdot (-g) \cdot h_{max} \quad \Leftrightarrow \quad h_{max} = \frac{v_f^2 - v_0^2}{2 \cdot (-g)} = \frac{0^2 - 13.4^2}{2 \cdot (-9.81)} = 9.2 \text{ m}$$

(3) To find the amount of time t_{eye} it takes the ball to arrive at Kopano's eye level, i.e., at a height of $d_{eye} = d_s + \frac{h}{2} = 6.5 + \frac{1.5}{2} = 7.3$ m above the ground, we have to solve the following quadratic equation:

$$x_f = x_0 + v_0 \cdot t + \frac{a_x}{2} \cdot t^2 \quad \Leftrightarrow \quad d_{eye} = 0 + v_0 \cdot t_{eye} - \frac{g}{2} \cdot t_{eye}^2$$

$$\Leftrightarrow \quad 0 = -7.3 + 13 \cdot t_{eye} - \frac{9.81}{2} \cdot t_{eye}^2$$

Since we assumed that the tennis ball is traveling upwards, we take the solution of the above quadratic equation that has the lowest value, i.e., $t_{eye} = 0.74$ s.

(4) Finally, to calculate the remaining time t_{ground} the tennis ball needs to fall to the ground, we must first find the time t_{max} it takes the ball to reach the maximum height of $h_{max} = 9.2$ m:

$$v_f = v_0 - g \cdot t_{max} \quad \Leftrightarrow \quad t_{max} = \frac{v_f - v_0}{-g} = \frac{0 - 13}{-9.81} = 1.4 \text{ s}$$

The remaining time t_{ground} is then equal to:

$$t_{ground} = 2 \cdot t_{max} - t_{eye} = 2 \cdot 1.4 - 0.74 = 2.0 \text{ s}$$

Exercise 4

Problem Statement

On a crisp Sunday afternoon, you are cycling at an average velocity of $\vec{v} = 22.0 \cdot \vec{i}_x$ km/h in the middle of the countryside nearby the town Katvari in Latvia. At a certain moment, another cyclist passes by you and after $t_{react} = 2.00$ s you realize it's Ilja, a friend of yours. You then start to accelerate ($\vec{a} = 0.385 \cdot \vec{i}_x$ m/s^2) and overtake Ilja after exactly $t_{sprint} = 12.3$ s. (1) Determine the average velocity $\vec{v}_{friend}$ at which your friend is cycling. (2) Instead of overtaking Ilja, you wish to end up cycling right next to Ilja. Therefore, after $t_1 = 10.0$ s of sprinting, you slow down at a rate of $\vec{a}_s = -a \cdot \vec{i}_x$. What is the distance d_s covered during this period of deceleration? (3) How long does it now take you to catch up with Ilja under the scenario described in part (2)? (4) How much more total distance do you now need compared to the situation in part (2)?

Solution

(1) In a first instance, let us calculate the total distance x_{tot} needed to overtake your friend. In order to find this distance, we must first figure out your final velocity $\vec{v}_f$ after your sprint of $t_{sprint} = 12.3$ s:

$$\vec{v}_f = \vec{v}_0 + \vec{a} \cdot t_{sprint} = (6.11 + 0.385 \cdot 12.3) \cdot \vec{i}_x = 10.8 \cdot \vec{i}_x \text{ m/s}$$

The total distance cycled is equal to the distance x_{react} covered before you reacted and started to accelerate plus the distance x_{sprint} during which you sprinted:

$$\begin{aligned} x_{tot} = x_{react} + x_{sprint} = v_0 \cdot t_{react} + \left(\frac{v_f^2 - v_0^2}{2 \cdot a}\right) &= 6.11 \cdot 2.00 + \left(\frac{10.8^2 - 6.11^2}{2 \cdot 0.385}\right) \\ &= 12.2 + 104 \\ &= 117 \text{ m} \end{aligned}$$

The average velocity $\vec{v}_{friend}$ of your friend Ilja then becomes:

$$\vec{v}_{friend} = \frac{x_{tot}}{t_{tot}} \cdot \vec{i}_x = \frac{x_{tot}}{t_{react} + t_{sprint}} \cdot \vec{i}_x = \frac{117}{2.00 + 12.3} \cdot \vec{i}_x = 8.15 \cdot \vec{i}_x \text{ m/s or } 29.3 \cdot \vec{i}_x \text{ km/h}$$

(2) The velocity $\vec{v}_s$ you attain during $t_1 = 10.0$ s of sprinting is equal to:

$$\vec{v}_s = (v_0 + a \cdot t_1) \cdot \vec{i}_x = (6.11 + 0.385 \cdot 10.0) \cdot \vec{i}_x = 9.96 \cdot \vec{i}_x \text{ m/s}$$

The distance d_s traveled during your deceleration is than calculated as follows:

$$v_{friend}^2 - v_s^2 = 2 \cdot (-a) \cdot d_s \quad \Leftrightarrow \quad d_s = \frac{v_{friend}^2 - v_s^2}{2 \cdot (-a)} = \frac{8.15^2 - 9.96^2}{2 \cdot (-0.385)} = 42.6 \text{ m}$$

(3) The total time t_{tot} is equal to $t_{tot} = t_{react} + t_1 + t_s$, whereby t_s represents the time spent during your deceleration and is calculated in the following way:

$$v_{friend} = v_s - a \cdot t_s \quad \Leftrightarrow \quad t_s = \frac{v_{friend} - v_s}{-a} = \frac{8.15 - 9.96}{-0.385} = 4.71 \text{ s}$$

The total time t_{tot} it takes you to catch up with Ilja then becomes:

$$t_{tot} = t_{react} + t_1 + t_s = 2.00 + 10.0 + 4.71 = 16.7 \text{ s}$$

(4) The total distance x_{tot2} covered when you decide to ride next to Ilja instead of overtaking him is equal to the sum of the distance cycled during your reaction time, the distance during your acceleration, and the distance when you slowed down:

$$x_{tot2} = (v_0 \cdot t_{react}) + \left(v_0 \cdot t_1 + \frac{a}{2} \cdot t_1^2\right) + (d_s)$$

$$= (6.11 \cdot 2.00) + \left(6.11 \cdot 10.0 + \frac{0.385}{2} \cdot 10.0^2\right) + (42.6)$$

$$= 135 \text{ m}$$

The extra distance needed to end up cycling right next to Ilja is equal to $d_{extra} = x_{tot2} - x_{tot} = 135 - 117 = 18.0$ m.

Exercise 5

Problem Statement

Bart and Tina are training in their hometown Ninove, Belgium, for the ultramarathon "The Bali Hope Ultra", which takes place in Bali, one of the islands of the Indonesian archipelago, in September 2022 and whereby the participants are crossing the island overnight from north to south over a distance of 84.9 km. According to their training schedule, they are running today a distance of 35.0 km. About $d = 88.0$ m before the end of their run, Bart maintains a velocity of $\vec{v}_B = 5.04 \cdot \vec{i}_x$ m/s, while Tina is running at a higher pace of $\vec{v}_T = 6.22 \cdot \vec{i}_x$ m/s and is a distance of $x_{BT} = 24.5$ m ahead of Bart. Tina feels a cramp coming up in the calf muscles of her right leg which slows her down at a rate of $\vec{a}_T = -0.270 \cdot \vec{i}_x$ m/s^2. (1) If Bart wishes to overtake Tina just $d_f = 5.00$ m before the end of their run, at what rate should Bart then start to accelerate? (2) What is Bart's velocity right at the moment when his GPS marks a total distance covered of 35.0 km? (3) Suppose that Bart maintains his acceleration calculated in part (1), but now after $t_{rec} = 8.50$ s Tina recovers from her cramp and runs the last bit with a constant velocity. Who sees the 35.0 km mark on their GPS first?

Solution

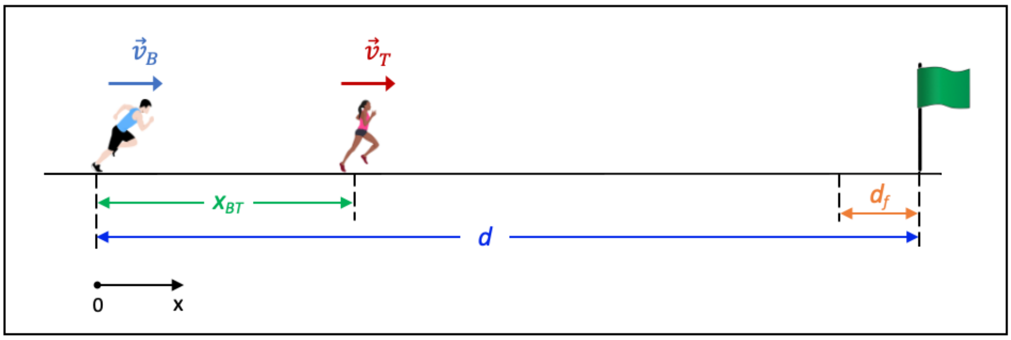

Figure 4

(1) Let us first calculate the time t_T Tina needs to arrive at $d_f = 5.00$ m before the finish line:

$$\Delta x = v_0 \cdot t + \frac{a_x}{2} \cdot t^2 \quad \Leftrightarrow \quad (d - x_{BT} - d_f) = v_T \cdot t_T + \frac{(-a_T)}{2} \cdot t_T^2$$

$$\Leftrightarrow \quad 0 = -(88.0 - 24.5 - 5.00) + 6.22 \cdot t_T - \frac{0.270}{2} \cdot t_T^2$$

The solution to the above quadratic equation that gives a physically sensible solution is equal to $t_T = 13.2$ s (the other solution is equal to $t_T = 32.9$ s, but results later on in a deceleration for Bart instead of an acceleration).

The magnitude of Bart's required acceleration $\vec{a}_B$ if he wants to overtake Tina $d_f = 5.00$ m before the end of the run is calculated as follows:

$$\Delta x = v_0 \cdot t + \frac{a_x}{2} \cdot t^2 \quad \Leftrightarrow \quad (d - d_f) = v_B \cdot t_T + \frac{a_B}{2} \cdot t_T^2$$

$$\Leftrightarrow \quad a_B = [(d - d_f) - v_B \cdot t_T] \cdot \frac{2}{t_T^2}$$

$$= [(88.0 - 5.00) - 5.04 \cdot 13.2] \cdot \frac{2}{13.2^2}$$

$$= 0.192 \text{ m}/s^2$$

(2) At the acceleration rate a_B during the final stretch of $d = 88.0$ m, the velocity $\vec{v}_{Bf}$ at which Bart completes his 35.0 km run is equal to:

$$v_{Bf}^2 - v_B^2 = 2 \cdot a_B \cdot d \quad \Leftrightarrow \quad \vec{v}_{Bf} = \left(\sqrt{v_B^2 + 2 \cdot a_B \cdot d}\right) \cdot \vec{i}_x$$

$$= \left(\sqrt{5.04^2 + 2 \cdot 0.192 \cdot 88.0}\right) \cdot \vec{i}_x$$

$$= 7.69 \cdot \vec{i}_x \text{ m/s or } 27.7 \cdot \vec{i}_x \text{ km/h}$$

(3) Tina's speed v_{recf} after she recovers from her cramp is the constant speed at which she runs the last bit of her training and is equal to:

$$v_{recf} = v_T - a_T \cdot t_{rec} = 6.22 - 0.270 \cdot 8.50 = 3.93 \text{ m/s}$$

The corresponding distance d_{rec} is found to be:

$$d_{rec} = \frac{v_{recf}^2 - v_T^2}{2 \cdot (-a_T)} = \frac{3.93^2 - 6.22^2}{2 \cdot (-0.270)} = 43.1 \text{ m}$$

The remaining time t_{remT} Tina needs to finish her run is calculated as follows (with d_{rem} representing the remaining distance until the end of the run):

$$t_{remT} = \frac{d_{rem}}{v_{recf}} = \frac{(d - x_{BT} - d_{rec})}{v_{recf}} = \frac{(88.0 - 24.5 - 43.1)}{3.93} = 5.19 \text{ s}$$

The total time t_{totT} Tina needs to finish her run is equal to $t_{totT} = t_{rec} + t_{remT} = 8.50 + 5.19 = 13.7$ s. The total time t_{totB} it takes Bart to finish the last bit of the run is equal to $t_{totB} = \frac{v_{Bf} - v_B}{a_B} = \frac{7.69 - 5.04}{0.192} = 13.8$ s. In other words, Tina sees the 35.0 km mark on her GPS first.

Exercise 6

Problem Statement

Luca is standing at the edge of the White Cliffs of Dover in the county of Kent, the United Kingdom, and he wants to show his son Michael how to estimate their height h when you only have the elements of nature at your disposal. When Luca throws a cobblestone vertically into the air with an initial velocity of $\vec{v}_0 = -2.50 \cdot \vec{i}_x$ m/s, they hear the characteristic splash of water about $t_{tot} = 5.32$ s later. (1) Given a speed of sound equal to $v_s = 343$ m/s, what is the estimated height h of the White Cliffs of Dover that Luca and his son have calculated? (2) With what velocity does the cobblestone hit the water? (3) Michael also throws a cobblestone at an initial speed of $v_M = 4.35$ m/s under an angle of $\theta = 61.5°$ with the horizontal. When do they hear the splash?

Solution

(1) The distance that the sound travels from the water all the way to the top of the cliff is the same as the magnitude of the displacement (which is not the distance!) that the cobblestone undergoes. This means we have to equate the following two equations, taking also into account the fact that the total time of $t_{tot} = 5.32$ s consists of the sum of the duration of the cobblestone's trajectory (t_{cob}) and the time it takes the sound to reach Luca's ears (t_s):

$$\begin{cases} h = -v_0 \cdot t_{cob} + \dfrac{g}{2} \cdot t_{cob}^2 \\ h = v_s \cdot t_s \\ \text{whereby } t_{tot} = t_{cob} + t_s \end{cases}$$

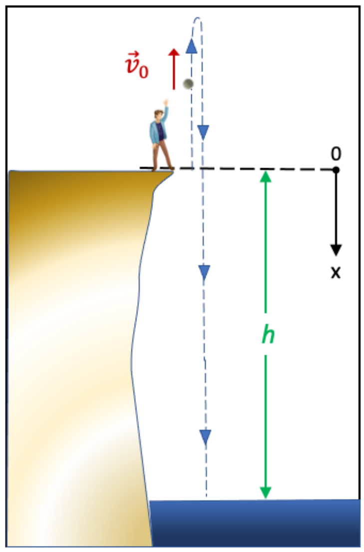

Figure 5

This results in the following equation:

$$- v_0 \cdot t_{cob} + \frac{g}{2} \cdot t_{cob}^2 = v_s \cdot (t_{tot} - t_{cob})$$

$$\Leftrightarrow \frac{g}{2} \cdot t_{cob}^2 + (v_s - v_0) \cdot t_{cob} - v_s \cdot t_{tot} = 0$$

$$\Leftrightarrow \frac{9.81}{2} \cdot t_{cob}^2 + (343 - 2.50) \cdot t_{cob} - 343 \cdot 5.32 = 0$$

For which we find the following physically sensible ($t > 0$) solution:

$$t_{cob} = \frac{1}{9.81}\left[-(343-2.50)+\sqrt{(343-2.50)^2+2\cdot 9.81\cdot 343\cdot 5.32}\right]$$

$$= 5.00 \text{ s}$$

This means that it takes sound $t_s = t_{tot} - t_{cob} = 5.32 - 5.00 = 0.321$ s to travel the height h of the cliff. In other words, the height of the cliff is equal to $h = v_s \cdot t_s = 343 \cdot 0.321 = 110$ m. We can check this result by inserting t_{cob} into the respective equation of motion for the cobblestone:

$$h = -v_0 \cdot t_{cob} + \frac{g}{2} \cdot t_{cob}^2 = -2.50 \cdot 5.00 + \frac{9.81}{2} \cdot 5.00^2 = 110 \text{ m}$$

(2) The velocity $\vec{v}_w$ with which the cobblestone touches the water can be calculated as follows:

$$\vec{v}_w = \vec{v}_0 + \vec{g} \cdot t_{cob} = (-2.50 + 9.81 \cdot 5.00) \cdot \vec{i}_x = 46.5 \cdot \vec{i}_x \text{ m/s}$$

(3) In a first instance, we calculate the time t_{cobM} it takes Michael's cobblestone to reach the water by focusing on the equation of motion in the vertical direction only, i.e., the x-direction, whereby the x-component of $\vec{v}_M$ is equal to $\vec{v}_{M,x} = -(v_M \cdot \sin\theta) \cdot \vec{i}_x$:

$$h = -(v_M \cdot \sin\theta) \cdot t_{cobM} + \frac{g}{2} \cdot t_{cobM}^2 \quad \Leftrightarrow \quad 0 = -110 - [4.35 \cdot \sin(61.5°)] \cdot t_{cobM} + \frac{9.81}{2} \cdot t_{cobM}^2$$

The physically sensible ($t > 0$) solution to the above quadratic equation is equal to $t_{cobM} = 5.14$ s. Luca and Michael will hear the splash of Michael's cobblestone after a total amount of time equal to $t_{totM} = t_{cobM} + t_s = 5.14 + 0.321 = 5.46$ s.

Exercise 7

Problem Statement

Fig. 6 shows the position-time graph for two swans swimming in the same direction in the Evros Delta in the north of Greece. Formulate an answer for the following questions: (1) Do the swans at some point move at the same velocity? (2) Which swan has the greatest acceleration? (3) Are the swans overtaking each other at any particular instant? (4) Which swan registers the largest instantaneous velocity? (5) Which swan undergoes the greatest displacement? (6) How do the average velocities of the swans compare to one another?

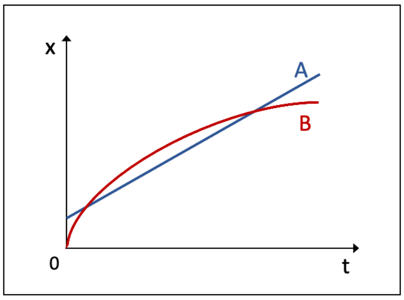

Figure 6

Solution

(1) In a position-time graph, the velocity is indicated by the slope of the curve. Swan A's straight line exhibits the same slope at every moment in time, which means that it swims at a constant velocity. Swan A and B move at the same velocity when two conditions are met: the slope of both curves is equal in magnitude and it has the same sign. This is the case when the tangent of curve B is parallel to the straight line A.

(2) Swan A travels at a constant velocity, so its acceleration is equal to 0 m/s^2. Swan B is slowing down, as its velocity decreases over time. In other words, swan B's acceleration is negative. However, we cannot conclude that swan A's acceleration is larger than that of swan B, because a negative acceleration does not mean that its magnitude is smaller than zero. A negative acceleration means that its magnitude is larger than zero and that it is pointing into the negative direction. At the point where the tangent of swan B's curve becomes horizontal (the slope is zero), the instantaneous velocity becomes zero, but the acceleration does not (the acceleration is negative and constant).

(3) The swans pass by each other when their respective curves intersect. Swan B first overtakes swan A, given swan B's greater velocity, and as swan B is gradually slowing down, swan A eventually catches up with swan B and, in turn, overtakes it.

(4) Before their slopes become equal to each other, swan B's instantaneous velocity is always greater than that of swan A. After that, swan A's instantaneous velocity is larger at every point in time.

(5) Swan A starts out a little farther than swan B and also ends up ahead of it (considering the end of both curves). Comparing the initial and final positions of both swans, the graph seems to indicate that the respective vertical distance remains the same, so that their displacement is also equal.

(6) Given an equal time window as well as the same displacement for both swan A and B, it follows that the average velocity of each swan is also the same.

Exercise 8

Problem Statement

Marwah is traveling at $\vec{v}_M = 115 \cdot \vec{i}_x$ km/h on a two-lane expressway east of Imilchil, Morocco, and approaching a $L = 45.0$ m-long double-trailer truck, which is going at $\vec{v}_t = 90.0 \cdot \vec{i}_x$ km/h. To overtake the truck, Marwah needs $d_L = 12.0$ m of leeway at each end of the truck. However, there is an oncoming minivan traveling at a speed of $\vec{v}_m = -85.0 \cdot \vec{i}_x$ km/h approximately $d = 500$ m away. (1) Can Marwah safely overtake the truck without accelerating? (2) If not, by how much should Marwah accelerate if she wants the minivan to pass by her at least $t_{margin} = 2.00$ s after she has overtaken the truck, and what would be her velocity after this manoeuvre? (3) How much distance has Marwah traveled during this manoeuvre?

Solution

(1) In order to make a decision, we need to switch from a coordinate system fixed to the ground to a reference frame co-moving with the truck and compare the time it takes Marwah to overtake the double-trailer truck to the time required by the minivan to arrive at the front of the truck (at opposite lanes, of course).

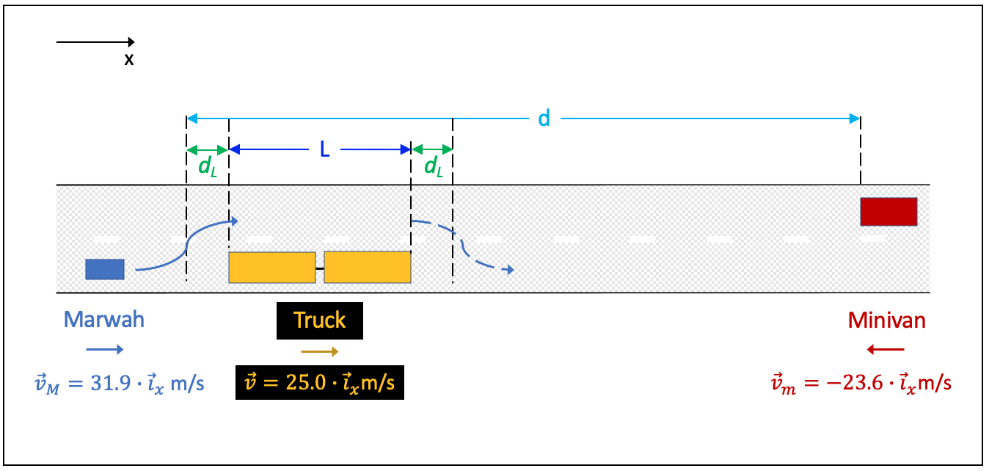

Figure 7

Let us first consider Marwah's car. Bearing in mind the $d_L = 12.0$ m clear room at the rear and the front of the truck, Marwah must cover a distance of $x_{truck} = 2 \cdot d_L + L = 2 \cdot 12.0 + 45.0 = 69.0$ m to overtake the truck, whereby her speed relative to the truck is equal to $v_{Mr} = v_M - v_t = 31.9 - 25.0 = 6.94$ m/s. For this manoeuvre, Marwah needs the following amount of time:

$$t_{pass} = \frac{x_{truck}}{v_{Mr}} = \frac{69.0}{6.94} = 9.94 \text{ s}$$

Next, let us determine when the oncoming minivan reaches the front of the truck, thereby respecting the $d_L = 12.0$ m leeway Marwah needs to pass the truck. The minivan's distance to the front of the truck is equal to $x_m = d - x_{truck} = 500 - 69.0 = 431$ m, and the minivan is traveling at a relative speed of $v_{mr} = v_m + v_t = 23.6 + 25.0 = 48.6$ m/s in the negative x-direction. The time t_m the minivan requires to arrive at the front of the truck equals:

$$t_m = \frac{x_m}{v_{mr}} = \frac{431}{48.6} = 8.87 \text{ s}$$

As Marwah will not be able to overtake the truck without hitting the minivan—it takes her $t_{pass} = 9.94$ s to perform that manoeuvre whereas the minivan reaches the truck in $t_m = 8.87$ s—she must accelerate to avoid any accidents.

(2) The time margin of $t_{margin} = 2.00$ s implies that Marwah must have completed her manoeuvre of overtaking the truck two seconds prior to the moment when the minivan passes by her. In other words, her manoeuvre can at the most last for a time period of $t_{safe} = 8.87 - 2.00 = 6.87$ s.

The magnitude of Marwah's required acceleration $\vec{a}$ is then calculated as follows:

$$x_{truck} = v_{Mr} \cdot t_{safe} + \frac{a}{2} \cdot t_{safe}^2 \quad \Leftrightarrow \quad a = \frac{(x_{truck} - v_{Mr} \cdot t_{safe}) \cdot 2}{t_{safe}^2}$$

$$= \frac{(69.0 - 6.94 \cdot 6.87) \cdot 2}{6.87^2}$$

$$= 0.904 \text{ m/s}^2$$

After safely passing the truck, Marwah's velocity with respect to the ground is equal to:

$$\vec{v}_{Mf} = \vec{v}_M + \vec{a} \cdot t_{safe} = (31.9 + 0.904 \cdot 6.87) \cdot \vec{i}_x = 38.2 \cdot \vec{i}_x \text{ m/s or } 137 \cdot \vec{i}_x \text{ km/h}$$

(3) The total amount of distance covered by Marwah during this manoeuvre is calculated as follows:

$$d_{overtake} = \frac{v_{Mf}^2 - v_M^2}{2 \cdot a} = \frac{38.2^2 - 31.9^2}{2 \cdot 0.904} = 241 \text{ m}$$

This distance should be equal to the sum of the distance driven by the truck and the distance x_{truck}, which Marwah needed to overtake the truck:

$$v_t \cdot t_{safe} + x_{truck} = 25.0 \cdot 6.87 + 69.0 = 241 \text{ m} = d_{overtake}$$

Exercise 9

Problem Statement

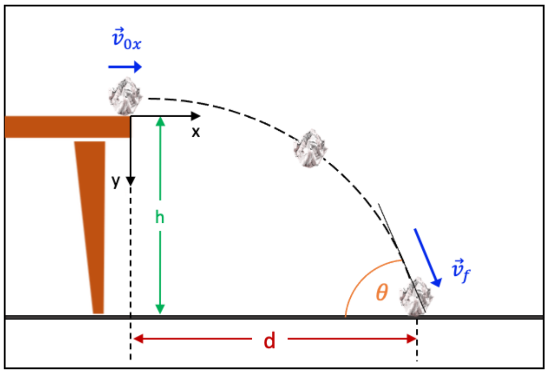

Figure 8

Yusef and Hasan are sitting at the back of the classroom in the Jordan National School in Irbid, Jordan, and are interested in anything but the current geography class. They started to make little wads of paper which they subsequently flick away with the tip of their finger from the edge of their desk, giving them a horizontal velocity of $\vec{v}_{0x} = 3.50 \cdot \vec{i}_x$ m/s. (1) At which angle and with what velocity does one little ball of paper hit the ground, measured at $h = 1.50$ m below the edge of the desk? (2) How far (horizontally speaking) from the edge of the desk does the paper land? (3) If Yusef drops a wad of paper vertically from the edge of the desk at the same time that Hasan flicks another, identical wad of paper away from the desk, which one touches the floor first? (4) If a wad of paper is flung away at a greater horizontal velocity of $\vec{v}_{0x2} = 4.15 \cdot \vec{i}_x$ m/s, how does this change the answer to part (3)?

Solution

(1) Given that there is no horizontal acceleration ($a_x = 0.00$ m/s^2), the x-component v_x of the final velocity vector $\vec{v}_f$ is equal to the magnitude of the initial velocity $\vec{v}_{0x} = 3.50 \cdot \vec{i}_x$ m/s. Let us now calculate the y-component v_y of the final velocity vector:

$$v_y^2 - v_{0y}^2 = 2 \cdot a_y \cdot h \quad \Leftrightarrow \quad v_y = \sqrt{v_{0y}^2 + 2 \cdot a_y \cdot h}$$

$$= \sqrt{0 + 2 \cdot 9.81 \cdot 1.50}$$

$$= 5.42 \text{ m/s}$$

The magnitude of the final velocity $\vec{v}_f$ and the angle θ can then be found as follows:

$$\begin{cases} v_f & = \sqrt{v_x^2 + v_y^2} \\ & = \sqrt{3.50^2 + 5.42^2} \\ & = 6.46 \text{ m/s} \\ & \\ \tan\theta & = \dfrac{v_y}{v_x} = \dfrac{5.42}{3.50} \quad \Leftrightarrow \quad \theta = 57.2^\circ \end{cases}$$

(2) To find the horizontal distance d between the edge of the table and the final position of the wad of paper, we first must establish how long the paper traveled through the air. Considering the y-component of the final velocity, we calculate the time t_{air} as follows:

$$v_y = v_{0y} + a_y \cdot t_{air} \quad \Leftrightarrow \quad t_{air} = \frac{v_y - v_{0y}}{a_y} = \frac{5.42 - 0}{9.81} = 0.553 \text{ s}$$

As a result, the distance d is equal to:

$$d = v_x \cdot t_{air} = 3.50 \cdot 0.553 = 1.94 \text{ m}$$

(3) In order to figure out whether Hasan's wad of paper (flung horizontally) or Yusef's paper (dropped vertically) hits the ground first, we calculate the amount of time t_{wadY} Yusef's ball of paper spends in the air:

$$y = y_0 + v_{0y} \cdot t_{wadY} + \frac{a_y}{2} \cdot t_{wadY}^2 \quad \Leftrightarrow \quad 1.50 = 0 + 0 \cdot t_{wadY} + \frac{9.81}{2} \cdot t_{wadY}^2$$

$$\Leftrightarrow \quad t_{wadY} = \sqrt{\frac{1.50 \cdot 2}{9.81}} = 0.553 \text{ s}$$

Since $t_{wadY} = t_{air}$, it follows that both wads of paper hit the ground simultaneously. This is not surprising, given that both pieces of paper are identical in shape which makes them experience the same air resistance.

(4) A greater horizontal velocity does not change the answer to part (3) because the wad of paper is still being flicked away from the same height. In other words, it is the vertical component of the velocity vector that determines the time the paper dwells in the air (which in turn is determined by the amount of air resistance). The only difference now is that the paper lands farther away from the edge of the desk:

$$d = v_{ox2} \cdot t_{air} = 4.15 \cdot 0.553 = 2.29 \text{ m}$$

Exercise 10

Problem Statement

You feel like throwing a pineapple from your bedroom window into the rectangular pond you had recently installed in the garden. You launch the pineapple at an initial speed of $v_0 = 8.80$ m/s under an angle θ and it hits the water $d_{we} = 1.10$ m from the edge of the pond closest to the back of your house. (1) If your bedroom is located $h = 9.50$ m above ground level and the distance between the house and the closest edge measures $d_{he} = 7.00$ m, under what angle are you throwing the pineapple? (2) If the pond is half a meter deep and the water slows the pineapple down with an acceleration whose magnitude is equal to $a_w = 65.0$ m/s^2, will the pineapple hit the bottom of the pond?

Solution

(1) Measured from the back of the house, the pineapple reaches the water at a horizontal distance of $x = d_{he} + d_{we} = 7.00 + 1.10 = 8.10$ m. We can now write the two following equations of motion:

$$\begin{cases} \quad x = x_0 + v_{0x} \cdot t \\ \Leftrightarrow \quad 8.10 = 0 + (8.80 \cdot \cos\theta) \cdot t \\ \\ \quad y = y_0 + v_{0y} \cdot t + \dfrac{a_y}{2} \cdot t^2 \\ \Leftrightarrow \quad 0 = 9.50 + (8.80 \cdot \sin\theta) \cdot t + \dfrac{(-9.81)}{2} \cdot t^2 \end{cases}$$

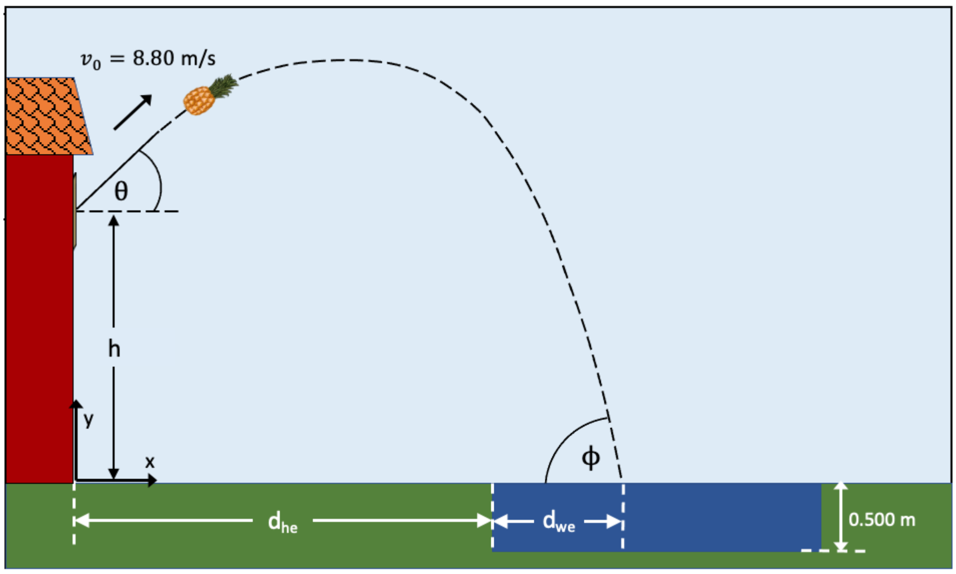

Figure 9

Let us now replace the variable t in the second equation (the vertical dimension) with the expression for t based on the first equation $\left(t = \frac{8.10}{(8.80 \cdot \cos\theta)}\right)$. This gives the following equation for the y-direction:

$$0 = 9.50 + (8.80 \cdot \sin\theta) \cdot \left[\frac{8.10}{(8.80 \cdot \cos\theta)}\right] + \frac{(-9.81)}{2} \cdot \left[\frac{8.10}{(8.80 \cdot \cos\theta)}\right]^2$$

$$\Leftrightarrow \;\; 0 = 9.50 + 8.10 \cdot \tan\theta + \left[\frac{(-9.81)}{2} \cdot \left(\frac{8.10}{8.80}\right)^2\right] \cdot \left[\frac{1}{\cos\theta}\right]^2$$

Dividing the trigonometric identity "$\cos^2\theta + \sin^2\theta = 1$" by "$\cos^2\theta$", we can express "$\frac{1}{\cos^2\theta}$" as "$1 + \tan^2\theta$". In addition, if we introduce the substitution $\tan\theta = s$, we can rewrite the above equation as follows:

$$0 = 9.50 + 8.10 \cdot s + \left[\frac{(-9.81)}{2} \cdot \left(\frac{8.10}{8.80}\right)^2\right] \cdot \left(1 + s^2\right)$$

$$\Leftrightarrow \;\; 0 = \left[9.50 - \frac{9.81}{2} \cdot \left(\frac{8.10}{8.80}\right)^2\right] + 8.10 \cdot s - \left[\frac{9.81}{2} \cdot \left(\frac{8.10}{8.80}\right)^2\right] \cdot s^2$$

$$\Leftrightarrow \;\; 0 = 5.34 + 8.10 \cdot s - 4.16 \cdot s^2$$

Solving for s and putting the solution into our previous substitution of $\tan\theta = s$, we obtain the following two angles under which the pineapple is allowed to leave your bedroom window:

$$\begin{cases} s_- = \dfrac{-8.10 - \sqrt{8.10^2 - 4 \cdot (-4.16) \cdot 5.34}}{2 \cdot (-4.16)} = 2.47 \;\; \Leftrightarrow \;\; \theta_- = 68.0^\circ \\ \\ s_+ = \dfrac{-8.10 + \sqrt{8.10^2 - 4 \cdot (-4.16) \cdot 5.34}}{2 \cdot (-4.16)} = -0.521 \;\; \Leftrightarrow \;\; \theta_+ = -27.5^\circ \end{cases}$$

In the rest of this exercise, we will work out the first case only ($\theta_- = 68.0^\circ$) and leave the second case as an exercise for the reader (the solutions are: $\vec{v}_y = -14.2 \cdot \vec{i}_y$ m/s, $\vec{v}_x = 7.81 \cdot \vec{i}_x$ m/s, $\phi = 61.3^\circ$, and $\Delta y = -2.15$ m).

(2) To answer the question whether the pineapple touches the bottom of the pond, we first need to find the y-component of the final velocity (v_y) together with the angle ϕ under which the pineapple lands in the water. Given that the pineapples spends $t = \frac{x}{v_x} = \frac{8.10}{8.80 \cdot \cos(68.0^\circ)} = 2.45$ s in the air, we can calculate $\vec{v}_y$ and ϕ as follows:

$$\begin{cases} \vec{v}_y & = \vec{v}_{0y} + \vec{a}_y \cdot t \\ & = [8.80 \cdot \sin(68.0°) + (-9.81) \cdot 2.45] \cdot \vec{i}_y \\ & = -15.9 \cdot \vec{i}_y \text{ m/s} \\ \\ \tan\phi & = \dfrac{v_y}{v_x} = \dfrac{15.9}{8.80 \cdot \cos(68.0°)} = 4.83 \quad \Leftrightarrow \quad \phi = 78.3° \end{cases}$$

If the water slows down the sinking pineapple with an acceleration of $a_w = 65.0$ m/s^2, we find the vertical depth at which the pineapple comes to a halt through the following calculation:

$$v_{yf}^2 - v_y^2 = 2 \cdot a_w \cdot \Delta y$$

$$\Leftrightarrow \; 0 - 15.9^2 = 2 \cdot [-9.81 + 65.0 \cdot \sin(78.3°)] \cdot \Delta y$$

$$\Leftrightarrow \; \Delta y = \frac{-15.9^2}{2 \cdot [-9.81 + 65.0 \cdot \sin(78.3°)]} = -2.35 \text{ m}$$

Given the pond's depth of 0.500 m, we can conclude that the pineapple does not have enough distance to naturally come to a halt and will therefore hit the bottom.

Exercise 11

Problem Statement

Ning is a professional archer and she is currently training for the Paris 2024 Olympic Games. During her next training, she aims to hit a target on the ground while leaning out of a helicopter, which is flying horizontally at $\vec{v}_h = 139 \cdot \vec{i}_x$ km/h at an altitude of $h = 40.0$ m near the Dongqian Lake in Ningbo, China. On the day of her training, there is a forecasted headwind of $\vec{v}_w = -54.0 \cdot \vec{i}_x$ km/h and Ning has brought a compound bow that is able to shoot arrows at a speed of $v_a = 80.0$ m/s. (1) How far will Ning's arrows travel under these conditions? (2) If Ning lowers her bow, making an angle of $\theta = 20.0°$ with the direction of flight, at what horizontal distance must the helicopter be away from the target if she wants to hit the mark? (3) Suppose that in part (1) the headwind is making an angle $\alpha < 90°$ with the vertical, so that the y-component of the arrow's velocity right before hitting the mark is equal to $\vec{v}_y = -29.9 \cdot \vec{i}_y$ m/s. At what distance from the mark did Ning release her arrow?

Solution

(1) First, we have to calculate how long the arrows travel through the air by considering the vertical dimension of motion:

$$y = h + v_{0y} \cdot t + \frac{a_y}{2} \cdot t^2$$

$$\Leftrightarrow \quad 0 = 40.0 + 0 \cdot t + \frac{(-9.81)}{2} \cdot t^2$$

$$\Leftrightarrow \quad t = \sqrt{\frac{80.0}{9.81}} = 2.86 \text{ s}$$

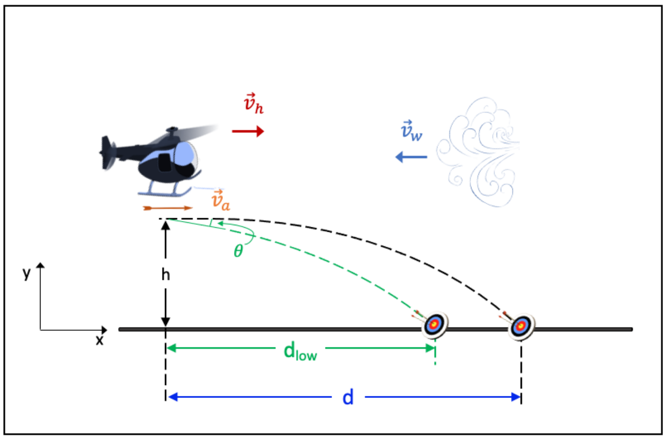

Figure 10

With this information, we can find the horizontal distance that the arrows spend in the air under the given weather conditions as follows:

$$d = v_x \cdot t = (v_h + v_a - v_w) \cdot t = (38.6 + 80.0 - 15.0) \cdot 2.86 = 296 \text{ m}$$

(2) If Ning lowers her bow by $\theta = 20.0°$, the time t_{low} the arrow whizzes through the air is equal to:

$$y = y_0 + v_{0y} \cdot t_{low} + \frac{a_y}{2} \cdot t_{low}^2 \quad \Leftrightarrow \quad 0 = 40.0 - 80.0 \cdot \sin(20.0°) \cdot t_{low} + \frac{(-9.81)}{2} \cdot t_{low}^2$$

$$\Leftrightarrow \quad t_{low} = \frac{\left(80.0 \cdot \sin(20.0°) - \sqrt{[80.0 \cdot \sin(20.0°)]^2 - 4 \cdot \frac{(-9.81)}{2} \cdot 40.0}\right)}{-9.81} = 1.20 \text{ s}$$

By lowering her bow, the horizontal distance that the arrow travels reduces from 296 m to:

$$d_{low} = v_x \cdot t_{low} = (v_h + v_a \cdot \cos\theta - v_w) \cdot t_{low} = (38.6 + 80.0 \cdot \cos(20.0°) - 15.0) \cdot 1.20 = 119 \text{ m}$$

(3) Given the y-component $\vec{v}_y = -29.9 \cdot \vec{i}_y$ m/s of the arrow's final velocity and the y-component $\vec{v}_{0y} = -v_w \cdot \cos\alpha \cdot \vec{i}_y$ m/s of its initial velocity, we first calculate the angle α as follows:

$$v_y^2 - (v_w \cdot \cos\alpha)^2 = 2 \cdot (-g) \cdot (-h) \quad \Leftrightarrow \quad \alpha = \cos^{-1}\left(\frac{\sqrt{(v_y^2 - 2 \cdot g \cdot h)}}{v_w}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{(29.9^2 - 2 \cdot 9.81 \cdot 40.0)}}{15.0}\right)$$

$$= 45.8°$$

Next, we calculate the time t_{air} the arrow spends in the air:

$$-v_y = -v_w \cdot \cos\alpha - g \cdot t_{air} \quad \Leftrightarrow \quad t_{air} = \frac{v_y - v_w \cdot \cos\alpha}{g} = \frac{29.9 - 15.0 \cdot \cos(45.8°)}{9.81} = 1.98 \text{ s}$$

In a final step, we determine the horizontal distance Δx between the mark and the helicopter at the moment when Ning shot her arrow:

$$\Delta x = (v_h + v_a - v_w \cdot \sin\alpha) \cdot t_{air} = [38.6 + 80.0 - 15.0 \cdot \sin(45.8°)] \cdot 1.98 = 214 \text{ m}$$

Exercise 12

Problem Statement

Due to the lower gravity on the planet Mars, its atmosphere is thinner and more volatile relative to Earth. One of the consequences is that dust is easily swept up and lingers throughout the atmosphere. Some of the chemical compounds and elements that constitute Martian dust include phosphorus pentoxide (P_4O_{10}), titanium dioxide (TiO_2), zinc (Zn), and manganese(II) oxide (MnO). Suppose now that a TiO_2 compound finds itself above the Gusev Crater and that, at $t = 0$ s, at the position $\vec{r}(t) = 1.0 \cdot \vec{i}_x + 1.0 \cdot \vec{i}_y + 2.0 \cdot \vec{i}_z$ m, it has a velocity of $\vec{v}_0(t) = 6.5 \cdot \vec{i}_y + 2.0 \cdot \vec{i}_z$ m/s. If the TiO_2 compound is being accelerated by an upcoming dust storm at a rate of $\vec{a}(t) = 2.3 \cdot \vec{i}_x - 3.1 \cdot \vec{i}_y + 0.5 \cdot \vec{i}_z$ m/s^2, what is the compound's position and velocity when the y coordinate reaches its maximum?

Solution

First, we would like to know when the y-coordinate reaches its maximum so that we find an expression for the time variable t_{max}. Therefore, we calculate the derivative of the equation of motion of the y coordinate with respect to the time variable t and equate it to zero:

$$\begin{cases} y(t) = y_0 + v_{0y} \cdot t + \dfrac{a_y}{2} \cdot t^2 \\ \\ \dfrac{dy(t)}{dt} = 0 \;\Leftrightarrow\; v_{0y} + a_y \cdot t = 0 \;\Leftrightarrow\; t_{max} = \dfrac{(-v_{0y})}{a_y} \end{cases}$$

We now write for all three spatial directions the most general form of the equation of motion that reflects the position $\vec{r}(t)$ of the compound:

$$\vec{r}(t) = \left[x_0 + v_{0x} \cdot t + \frac{a_x}{2} \cdot t^2\right] \cdot \vec{i}_x + \left[y_0 + v_{0y} \cdot t + \frac{a_y}{2} \cdot t^2\right] \cdot \vec{i}_y + \left[z_0 + v_{0z} \cdot t + \frac{a_z}{2} \cdot t^2\right] \cdot \vec{i}_z$$

Replacing t by the expression found for t_{max}, the position vector $\vec{r}(t)$ of the dust particle becomes the following:

$$\begin{aligned} \vec{r}(t_{max}) \;=\; & \left[x_0 + v_{0x} \cdot \frac{(-v_{0y})}{a_y} + \frac{a_x}{2} \cdot \frac{(-v_{0y})^2}{a_y^2}\right] \cdot \vec{i}_x + \left[y_0 + v_{0y} \cdot \frac{(-v_{0y})}{a_y} + \frac{a_y}{2} \cdot \frac{(-v_{0y})^2}{a_y^2}\right] \cdot \vec{i}_y + \\ & \left[z_0 + v_{0z} \cdot \frac{(-v_{0y})}{a_y} + \frac{a_z}{2} \cdot \frac{(-v_{0y})^2}{a_y^2}\right] \cdot \vec{i}_z \end{aligned}$$

$$= \left[x_0 - \frac{v_{0x} \cdot v_{0y}}{a_y} + \frac{a_x \cdot v_{0y}^2}{2 \cdot a_y^2}\right] \cdot \vec{i}_x + \left[y_0 - \frac{v_{0y}^2}{2 \cdot a_y}\right] \cdot \vec{i}_y + \left[z_0 - \frac{v_{0y} \cdot v_{0z}}{a_y} + \frac{a_z \cdot v_{0y}^2}{2 \cdot a_y^2}\right] \cdot \vec{i}_z$$

$$= \left[1.0 - \frac{0 \cdot 6.5}{-3.1} + \frac{2.3 \cdot 6.5^2}{2 \cdot (-3.1)^2}\right] \cdot \vec{i}_x + \left[1.0 - \frac{6.5^2}{2 \cdot (-3.1)}\right] \cdot \vec{i}_y +$$

$$\left[2.0 - \frac{6.5 \cdot 2.0}{-3.1} + \frac{0.5 \cdot 6.5^2}{2 \cdot (-3.1)^2}\right] \cdot \vec{i}_z$$

$$= 6.1 \cdot \vec{i}_x + 7.8 \cdot \vec{i}_y + 7.3 \cdot \vec{i}_z \text{ m}$$

With respect to the velocity vector $\vec{v}(t)$, it has the following general form:

$$\vec{v}(t) = [v_{0x} + a_x \cdot t] \cdot \vec{i}_x + [v_{0y} + a_y \cdot t] \cdot \vec{i}_y + [v_{0z} + a_z \cdot t] \cdot \vec{i}_z$$

At t_{max}, the TiO_2 compound has the following velocity:

$$\vec{v}(t_{max}) = \left[v_{0x} + a_x \cdot \frac{(-v_{0y})}{a_y}\right] \cdot \vec{i}_x + \left[v_{0y} + a_y \cdot \frac{(-v_{0y})}{a_y}\right] \cdot \vec{i}_y + \left[v_{0z} + a_z \cdot \frac{(-v_{0y})}{a_y}\right] \cdot \vec{i}_z$$

$$= \left[v_{0x} - \frac{v_{0y} \cdot a_x}{a_y}\right] \cdot \vec{i}_x + 0 \cdot \vec{i}_y + \left[v_{0z} - \frac{v_{0y} \cdot a_z}{a_y}\right] \cdot \vec{i}_z$$

$$= \left[0 - \frac{6.5 \cdot 2.3}{-3.1}\right] \cdot \vec{i}_x + \left[2.0 - \frac{6.5 \cdot 0.5}{-3.1}\right] \cdot \vec{i}_z$$

$$= 4.8 \cdot \vec{i}_x + 3.0 \cdot \vec{i}_z \text{ m/s}$$

Exercise 13

Problem Statement

Juan is a locally famous stuntman in the region around the city of Bayamo, Cuba, and he is about to try out a new stunt in his home in the outskirts of Bayamo. Juan takes place in his self-made ejector seat, which is installed upon a rotatable and tiltable platform, so that he sits $s = 1.55$ m above ground level. Facing north, he directs his seat in a straight line with the ridge of the barn that is right in front of him, and tilts it in the forward direction until it makes a $\theta = 65.0°$ angle with the ground. When Juan gets ejected from his seat, he manages to travel a horizontal distance of $d_x = 10.5$ m and ends up on the $w = 4.00$ m wide balcony that is attached perfectly symmetrical to the front of the barn. (1) If you know that the balcony hangs $h = 8.65$ m above the ground, what is the launching speed v_0 of his home-made ejector seat? (2) After a couple of runs, one of the bolts in the platform is partially unscrewed, so that by the time of the next ejection round Juan's seat has effectively rotated $\phi = 5.00°$ east of north. Will Juan still make it to the balcony?

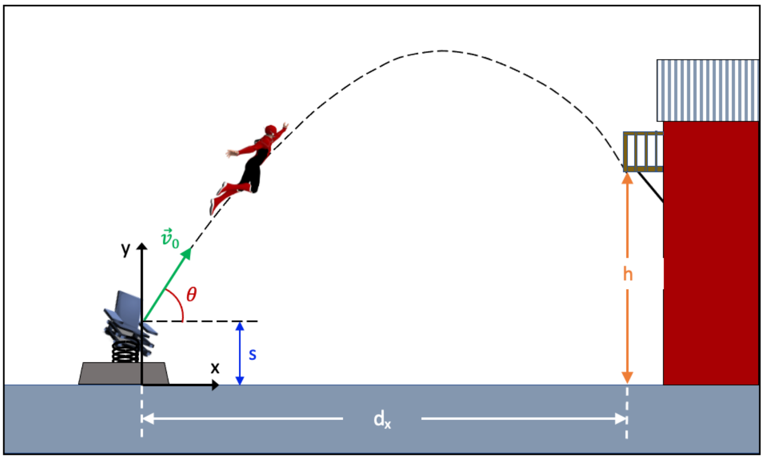

Figure 11

Solution

(1) We start with writing the two equations of motion for the x- and y-direction, respectively:

$$x(t) = x_0 + v_{0x} \cdot t \qquad y(t) = y_0 + v_{0y} \cdot t + \frac{a_y}{2} \cdot t^2$$

$$\Leftrightarrow \; d_x = 0 + v_0 \cdot \cos\theta \cdot t \qquad \Leftrightarrow \; h = s + v_0 \cdot \sin\theta \cdot t + \frac{(-g)}{2} \cdot t^2$$

$$\Leftrightarrow \; 10.5 = v_0 \cdot \cos(65.0°) \cdot t \qquad \Leftrightarrow \; 8.65 = 1.55 + v_0 \cdot \sin(65.0°) \cdot t - \frac{9.81}{2} \cdot t^2$$

If we replace t in the y-equation with the expression for t obtained from the x-equation $\left(t = \frac{10.5}{v_0 \cdot \cos(65.0°)}\right)$, we can calculate the initial velocity with which Juan is being ejected from his seat:

$$8.65 = 1.55 + v_0 \cdot \sin(65.0°) \cdot \left[\frac{10.5}{v_0 \cdot \cos(65.0°)}\right] - \frac{9.81}{2} \cdot \left[\frac{10.5}{v_0 \cdot \cos(65.0°)}\right]^2$$

$$\Leftrightarrow \quad 0 = -7.10 + 10.5 \cdot \tan(65.0°) - \frac{9.81 \cdot 10.5^2}{2 \cdot v_0^2 \cdot \cos^2(65.0°)}$$

$$\Leftrightarrow \quad v_0 = \sqrt{\frac{9.81 \cdot 10.5^2}{2 \cdot \cos^2(65.0°) \cdot (10.5 \cdot \tan(65.0°) - 7.10)}} = 14.0 \text{ m/s}$$

(2) For the second part of the problem, we have to deal with motion in three dimensions, as the loose bolt has caused the platform to rotate by $\phi = 5.00°$ east of north.

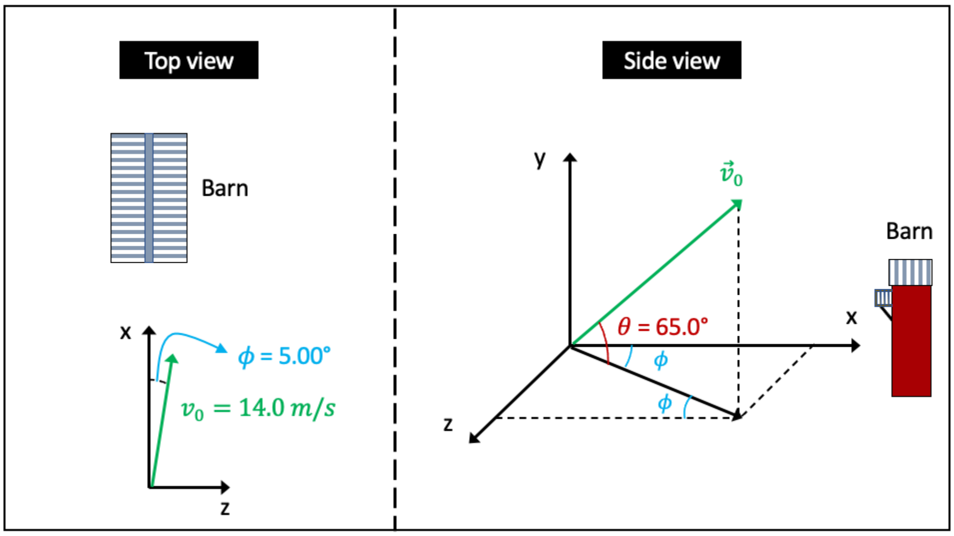

Figure 12

The equations of motion for the x-, y-, and z-direction, respectively, are the following:

$$\begin{cases} x(t) &= (v_0 \cdot \cos\theta) \cdot \cos\phi \cdot t \\ &= [14.0 \cdot \cos(65.0°)] \cdot \cos(5.00°) \cdot t \\ y(t) &= s + (v_0 \cdot \sin\theta) \cdot t - \frac{g}{2} \cdot t^2 \\ &= 1.55 + [14.0 \cdot \sin(65.0°)] \cdot t - \frac{9.81}{2} \cdot t^2 \\ z(t) &= (v_0 \cdot \cos\theta) \cdot \sin\phi \cdot t \\ &= [14.0 \cdot \cos(65.0°)] \cdot \sin(5.00°) \cdot t \end{cases}$$

If we wish to find out whether Juan is able to reach the balcony given his altered trajectory, we solve the second equation (y-dimension) for the time variable, so that we subsequently can determine what his x- and z-coordinates are when he finds himself at an altitude of $y(t) = h = 8.65$ m, i.e., the height of the balcony. We write:

$$8.65 = 1.55 + [14.0 \cdot \sin(65.0°)] \cdot t - \frac{9.81}{2} \cdot t^2$$

$$\Leftrightarrow \quad 0 = -7.10 + [14.0 \cdot \sin(65.0°)] \cdot t - \frac{9.81}{2} \cdot t^2$$

For which we find the following two solutions:

$$\begin{cases} t_+ = \dfrac{\left(-14.0 \cdot \sin(65.0°) + \sqrt{(14.0 \cdot \sin(65.0°))^2 - 4 \cdot \frac{(-9.81)}{2} \cdot (-7.10)}\right)}{(-9.81)} = 0.816 \text{ s} \\ \\ t_- = \dfrac{\left(-14.0 \cdot \sin(65.0°) - \sqrt{(14.0 \cdot \sin(65.0°))^2 - 4 \cdot \frac{(-9.81)}{2} \cdot (-7.10)}\right)}{(-9.81)} = 1.77 \text{ s} \end{cases}$$

Filling out these two values in the above equations of motion, we then obtain two position vectors for the two moments during which when Juan finds himself at $h = 8.65$ m above the ground (one for when he is ascending in the air (t_+) and one for on his way down (t_-)):

$$\begin{cases} \vec{r}(t_+) = (4.82, 8.65, 0.421) \text{ m} \\ \\ \vec{r}(t_-) = (10.5, 8.65, 0.915) \text{ m} \end{cases}$$

The x-coordinate of the position vector $\vec{r}(t_-)$ seems to indicate that Juan indeed makes it to the balcony, despite some technical malfunctioning. However, the value $x(t_-) = 10.5$ m is the result of rounding off the number 10.46004434, which we obtained from the equation $x(t_-) = (v_0 \cdot \cos\theta) \cdot \cos\phi \cdot t_- = [14.01380483 \cdot \cos(65.0°)] \cdot \cos(5.00°) \cdot 1.772903 = 10.46004434$. So, it would be more accurate to say that Juan *almost* makes it to the balcony, just short of 4.00 cm in the x-direction. In order to avoid an unfortunate fall to the ground, he might get hold of the balustrade $z(t_-) = 0.915$ m to the right of the center of the balcony (remember that the balcony is $w = 4.00$ m wide) by stretching his arms.

Exercise 14

Problem Statement

Giulia, an extreme sports fanatic, stands on the edge of the 754 m high sea cliff Cape Enniberg at the Faroe Islands and is looking over the Norwegian Sea ready to take her next parachute jump. She takes a run-up in the northeastern (NE) direction and dives from the cliff head first with a speed of 3.50 m/s under a 25.0° angle with the horizontal. After a free fall of $t_{free} = 8.50$ s, she opens her parachute and about $t_{open} = 2.50$ s later Giulia is descending at a constant velocity with her horizontal speed reduced to 1.00 m/s. At the very moment that her downward velocity becomes constant, a southeastern (SE) wind kicks in with an initial speed of 2.00 m/s and gradually picks up speed with a rate of 0.112 m/s^2. After the onset of the SE wind, Giulia touches the water $t_{descend} = 38.9$ s later.

(1) At what height does Giulia's downward velocity become constant and what is the value of this velocity? (2) What is the total acceleration during the opening of the parachute? (3) What are the coordinates of Giulia's landing spot? (4) With what velocity and under which angle does Giulia land into the water?

Solution

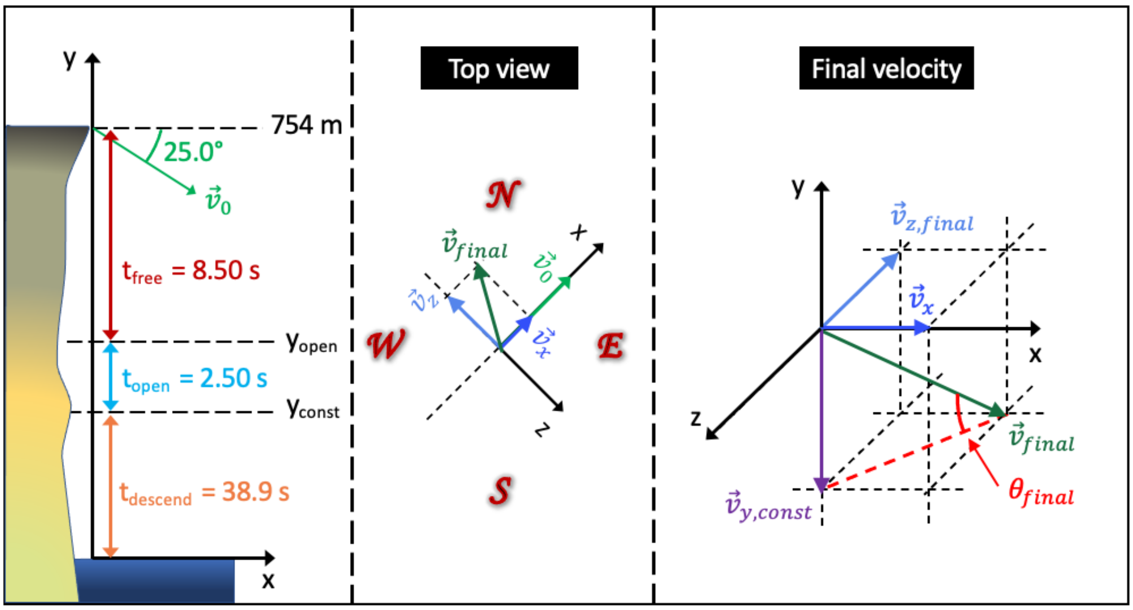

Figure 13

(1) First, we determine the height at which Giulia opens her parachute as well as her velocity right before that moment:

$$\begin{cases} y_{open} &= y_0 + v_{0y} \cdot t_{free} + \dfrac{a_y}{2} \cdot t_{free}^2 \\ &= 754 - 3.50 \cdot \sin(25.0°) \cdot 8.50 - \dfrac{9.81}{2} \cdot 8.50^2 \\ &= 387 \text{ m} \\ \\ v_{y,open} &= v_{0y} + a_y \cdot t_{free} \\ &= -3.50 \cdot \sin(25.0°) - 9.81 \cdot 8.50 \\ &= -84.9 \text{ m/s} \end{cases}$$

At this point, we have three unknown variables, i.e., the height at which Giulia's downward velocity becomes constant (y_{const}), the constant downward velocity after the parachute is fully deployed ($v_{y,const}$), and the upwards acceleration during the opening of the parachute (a_y). We now write the following three equations of motion:

$$\begin{cases} v_{y,const} = v_{y,open} + a_y \cdot t_{open} \\ \qquad = -84.9 + a_y \cdot 2.50 \\ \\ \quad v_{y,const}^2 - v_{y,open}^2 = 2 \cdot a_y \cdot (y_{const} - y_{open}) \\ \Leftrightarrow v_{y,const}^2 - 84.9^2 = 2 \cdot a_y \cdot (y_{const} - 387) \\ \\ \quad y_{final} = y_{const} + v_{y,const} \cdot t_{descend} \\ \Leftrightarrow 0 = y_{const} + v_{y,const} \cdot 38.9 \end{cases}$$

If we replace a_y in the second equation with the expression for a_y obtained from the first equation ($a_y = \frac{v_{y,const}+84.9}{2.50}$) and y_{const} with the one obtained from the third equation ($y_{const} = -v_{y,const} \cdot 38.9$), we can write the second equation in the following way:

$$v_{y,const}^2 - 84.9^2 = 2 \cdot \left(\frac{v_{y,const} + 84.9}{2.50} \right) \cdot (-v_{y,const} \cdot 38.9 - 387)$$

$$\Leftrightarrow \quad (2.50 + 2 \cdot 38.9) \cdot v_{y,const}^2 + 2 \cdot (387 + 84.9 \cdot 38.9) \cdot v_{y,const} + 84.9 \cdot (2 \cdot 387 - 2.5 \cdot 84.9) = 0$$

The physically relevant solution for this quadratic equation is $v_{y,const} = -7.00$ m/s. As a result, the height at which Giulia starts descending with this constant velocity is equal to:

$$y_{const} = -v_{y,const} \cdot 38.9 = -(-7.00) \cdot 38.9 = 272 \text{ m}$$

(2) The x- and y-component of the total acceleration $\vec{a}$ that Giulia undergoes during the opening of her parachute are calculated as follows:

$$\begin{cases} & v_x &= v_{0x} + a_x \cdot t_{open} \\ \Leftrightarrow & 1.00 &= 3.50 \cdot \cos(25.0°) + a_x \cdot 2.50 \\ \Leftrightarrow & a_x &= \dfrac{1.00 - 3.50 \cdot \cos(25.0°)}{2.50} &= -0.869 \text{ m/s}^2 \\ \\ & v_{y,const} &= v_{y,open} + a_y \cdot t_{open} \\ \Leftrightarrow & -7.00 &= -84.9 + a_y \cdot 2.50 \\ \Leftrightarrow & a_y &= \dfrac{84.9 - 7.00}{2.50} &= 31.1 \text{ m/s}^2 \end{cases}$$

Therefore, the magnitude of the total acceleration $\vec{a}$ and the corresponding angle (with respect to the y-axis) become:

$$\begin{cases} a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.869)^2 + 31.1^2} = 31.2 \text{ m/s}^2 \\ \\ \theta_{acc} = \tan^{-1}\left(\dfrac{a_x}{a_y}\right) = \tan^{-1}\left(\dfrac{0.869}{31.1}\right) = 1.60° \end{cases}$$

(3) The coordinates of Giulia's final position vector can be found in the following way:

$$\begin{cases} x_{final} &= [v_{0x} \cdot t_{free}] + \left[\dfrac{(v_x^2 - v_{0x}^2)}{2 \cdot a_x}\right] + [v_x \cdot t_{descend}] \\ \\ &= [3.50 \cdot \cos(25.0°) \cdot 8.50] + \left[\dfrac{1.00^2 - [3.50 \cdot \cos(25.0°)]^2}{2 \cdot (-0.869)}\right] + [1.00 \cdot 38.9] \\ \\ &= 71.1 \text{ m} \\ \\ y_{final} &= \left[y_0 + v_{0y} \cdot t_{free} + \dfrac{a_y}{2} \cdot t_{free}^2\right] + [y_{const} - y_{open}] + [v_{y,const} \cdot t_{descend}] \\ \\ &= \left[754 - 3.50 \cdot \sin(25.0°) \cdot 8.50 + \dfrac{(-9.81)}{2} \cdot 8.50^2\right] + [272 - 387] + [(-7.00) \cdot 38.9] \\ \\ &= 0.00 \text{ m} \\ \\ z_{final} &= v_{0z} \cdot t_{descend} + \dfrac{a_z}{2} \cdot t_{descend}^2 \\ \\ &= -2.00 \cdot 38.9 + \dfrac{(-0.112)}{2} \cdot 38.9^2 \\ \\ &= -163 \text{ m} \end{cases}$$

(4) Finally, the velocity and the angle (with respect to the water surface) with which Giulia lands in the water are equal to (with $v_{z,final} = v_{0z} + a_z \cdot t_{descend}$):

$$\begin{cases} v_{final} &= \sqrt{v_x^2 + v_{y,const}^2 + (v_{0z} + a_z \cdot t_{descend})^2} \\ &= \sqrt{1.00^2 + (-7.00)^2 + [-2.00 + (-0.112) \cdot 38.9]^2} \\ &= 9.51 \text{ m/s} \\ \\ \theta_{final} &= \sin^{-1}\left(\frac{v_{y,const}}{v_{final}}\right) = \sin^{-1}\left(\frac{7.00}{9.51}\right) = 47.4^\circ \end{cases}$$

Exercise 15

Problem Statement

You're practicing your snowboard skills in the indoor ski resort Sayama Indoor Skiing Ground in Tokorozawa, Japan, and when you come to a halt at the bottom of the last slope, you take some time to rest. While you're tossing around a snowball, your mind wanders off to that last slope and suddenly it dawns on you how to solve that particular physics problem you've been thinking about for the past two weeks: If I know the angle ϕ of a slope, under which angle θ with the horizontal should I throw a snowball with a given initial velocity v_0, so that it ends up the farthest as possible on the slope (point d)? Write down the solution you have in your mind.

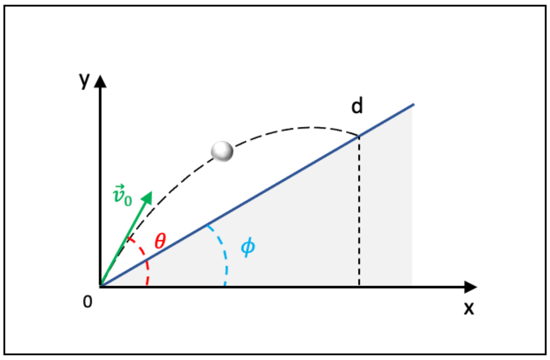

Figure 14

Solution

In a first step, we need to write an equation for when the parabolic trajectory of the snowball intersects with the straight incline, which has the general form of $y(t) = \tan\phi \cdot x(t)$. The equations of motion for the x- and y-direction take the following form:

$$\begin{cases} x(t) &= v_0 \cdot \cos\theta \cdot t \\ y(t) &= v_0 \cdot \sin\theta \cdot t - \dfrac{g}{2} \cdot t^2 \end{cases}$$

Replacing t in the second equation with the expression for t obtained from the first equation $\left(t = \frac{x(t)}{v_0 \cdot \cos\theta}\right)$ and equating y(t) to the general form of a straight slope ($y(t) = \tan\phi \cdot x(t)$), we can write the second equation as follows:

$$\tan\phi \cdot x(t) = v_0 \cdot \sin\theta \cdot \left(\frac{x(t)}{v_0 \cdot \cos\theta}\right) - \frac{g}{2} \cdot \left(\frac{x(t)}{v_0 \cdot \cos\theta}\right)^2$$

$$\Leftrightarrow \quad x(t) = \frac{2 \cdot v_0^2}{g} \cdot (\tan\theta - \tan\phi) \cdot \cos^2\theta$$

What we want is to find an expression for the distance d in terms of the angle θ. From Fig. 14 it is clear that we can write x(t) as $x(t) = d \cdot \cos\phi$, so that the above equation becomes:

$$d = \frac{2 \cdot v_0^2}{g \cdot \cos\phi} \cdot (\tan\theta - \tan\phi) \cdot \cos^2\theta$$

Let us now rewrite that equation:

$$\begin{aligned} d &= \frac{2 \cdot v_0^2}{g \cdot \cos\phi} \cdot \left[\frac{\sin\theta}{\cos\theta} - \frac{\sin\phi}{\cos\phi}\right] \cdot \cos^2\theta \\ &= \frac{2 \cdot v_0^2}{g \cdot \cos\phi} \cdot \left[\frac{\sin\theta \cdot \cos\theta}{\cos\theta} - \frac{\sin\phi \cdot \cos\theta}{\cos\phi}\right] \cdot \cos\theta \\ &= \frac{2 \cdot v_0^2}{g \cdot \cos\phi} \cdot \left[\sin\theta - \frac{\sin\phi \cdot \cos\theta}{\cos\phi}\right] \cdot \cos\theta \\ &= \frac{2 \cdot v_0^2}{g \cdot \cos^2\phi} \cdot (\sin\theta \cdot \cos\phi - \sin\phi \cdot \cos\theta) \cdot \cos\theta \end{aligned}$$

Given the angle subtraction theorem "$\sin(A - B) = \sin A \cos B - \cos A \sin B$", the above last line can be reformulated in the following way:

$$d = \frac{2 \cdot v_0^2}{g \cdot \cos^2\phi} \cdot \sin(\theta - \phi) \cdot \cos\theta$$

To find the condition for the maximum distance, we first take the derivative of the above expression for d with respect to the angle θ:

$$\frac{d}{d\theta}\left[\frac{2 \cdot v_0^2}{g \cdot \cos^2\phi} \cdot \sin(\theta - \phi) \cdot \cos\theta\right] = \frac{2 \cdot v_0^2}{g \cdot \cos^2\phi} \cdot (\cos(\theta - \phi) \cdot \cos\theta - \sin(\theta - \phi) \cdot \sin\theta)$$

With the assistance of the angle addition identity "$\cos(A + B) = \cos A \cos B - \sin A \sin B$", the right-hand side of the above equation can be written as follows:

$$\frac{d}{d\theta} \cdot d = \frac{2 \cdot v_0^2}{g \cdot \cos^2\phi} \cdot \cos(2\theta - \phi)$$

Finally, the angle θ that gives the farthest possible distance d on the slope where my snowball will land is found when equating the above derivative to zero. In other words:

$$\frac{d}{d\theta} \cdot d = 0 \quad \Leftrightarrow \quad \frac{2 \cdot v_0^2}{g \cdot \cos^2 \phi} \cdot \cos(2\theta - \phi) = 0$$

$$\Leftrightarrow \quad \cos(2\theta - \phi) = 0$$

$$\Leftrightarrow \quad (2\theta - \phi) = \frac{\pi}{2}$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{4} + \frac{\phi}{2}$$

Exercise 16

Problem Statement

Sarki is participating in the Kenyan national competition of acrobatic aircraft racing and during the semifinals, he is required to steer his Zirko Edge 540 plane with a wingspan of 7.42 m right between two buildings that stand 10.0 m apart from each other. The opening through which the aircraft has to pass lies in the south-southwest (SSW) direction and the Zirko Edge 540 has an average air speed of $v_{plane} = 275$ km/h. If Sarki has to deal with a sturdy west wind of $v_w = 65.0$ km/h on the day of his competition, at what angle (west of south) should he better steer his airplane so that it safely whizzes through the opening between the two buildings? What is the magnitude of the resultant (effective) velocity v_R at which Sarki pulls off this manoeuvre?

Solution

Based on the Pythagorean theorem and given that the SSW direction entails an angle of 22.5° west of south, Fig. 15 lets us write the following four equations:

$$\begin{cases} 275 \cdot \sin\alpha = 65.0 + a \\ 275 \cdot \cos\alpha = b \\ d \cdot \sin(22.5°) = a \\ d \cdot \cos(22.5°) = b \end{cases}$$

Combining the second and the fourth equation, we find the following expression for d:

$$d = \frac{275 \cdot \cos\alpha}{\cos(22.5°)}$$

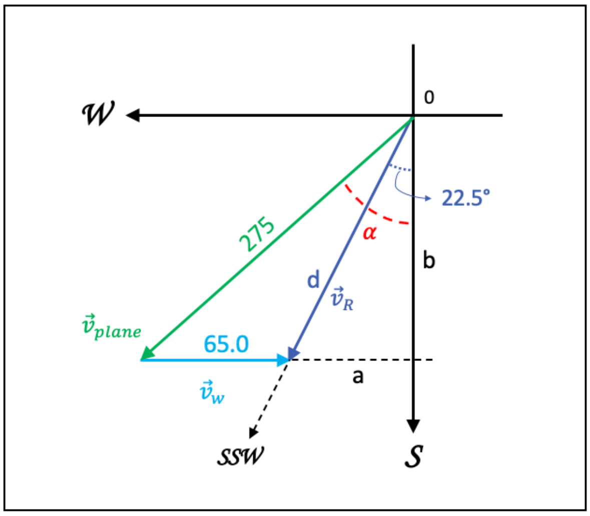

Figure 15

If we plug this expression into the third equation and insert it subsequently into the first equation (to replace the variable a), we can reformulate the first equation in the following way:

$$275 \cdot \sin\alpha = 65.0 + 275 \cdot \tan(22.5°) \cdot \cos\alpha$$

Given the goniometric identity "$\cos^2\alpha + \sin^2\alpha = 1$", we replace $\sin\alpha$ with $\sqrt{1-\cos^2\alpha}$ in the above equation:

$$275 \cdot \sqrt{1-\cos^2\alpha} \;=\; 65.0 + 275 \cdot \tan(22.5°) \cdot \cos\alpha$$

$$\Leftrightarrow \quad 275^2 \cdot (1-\cos^2\alpha) \;=\; 65.0^2 + 275^2 \cdot \tan^2(22.5°) \cdot \cos^2\alpha + 2 \cdot 65.0 \cdot 275 \cdot \tan(22.5°)\cos\alpha$$

$$\Leftrightarrow \quad 275^2 \cdot (1 + \cdot \tan^2(22.5°)) \cdot \cos^2\alpha + 2 \cdot 65.0 \cdot 275 \cdot \tan(22.5°)\cos\alpha + (65.0^2 - 275^2) = 0$$

If we introduce the new variable s whereby $s = \cos\alpha$, the equation becomes:

$$275^2 \cdot (1 + \cdot \tan^2(22.5°)) \cdot s^2 + 2 \cdot 65.0 \cdot 275 \cdot \tan(22.5°) \cdot s + (65.0^2 - 275^2) = 0$$

For this quadratic equation, we find the following two solutions:

$$\begin{cases} s_+ = 0.818 & \Leftrightarrow \quad \alpha_+ = 35.1° \\ s_- = -0.985 & \Leftrightarrow \quad \alpha_- = 170° \end{cases}$$

As we are considering the angle west of south, the physically relevant solution is $\alpha_+ = 35.1°$. The resultant velocity vector $\vec{v}_R$ in the SSW direction has therefore a magnitude of:

$$d = \frac{275 \cdot \cos\alpha}{\cos(22.5°)} = \frac{275 \cdot \cos(35.1°)}{\cos(22.5°)} = 243 \text{ km/h}$$

Perhaps a more efficient and shorter method to find the angle α is to work with the sine rule. If we consider the triangle formed by the vectors $\vec{v}_{plane}$, $\vec{v}_w$, and $\vec{v}_R$, we can write:

$$\frac{\sin(90.0° + 22.5°)}{v_{plane}} = \frac{\sin(90.0° - \alpha)}{v_R} = \frac{\sin(\alpha - 22.5°)}{v_w}$$

Equalling the first and the last term, we can calculate the angle α:

$$\frac{\sin(90.0° + 22.5°)}{v_{plane}} = \frac{\sin(\alpha - 22.5°)}{v_w} \quad \Leftrightarrow \quad \alpha - 22.5° = \sin^{-1}\left(\frac{v_w}{v_{plane}} \cdot \sin(90.0° + 22.5°)\right) = 12.6°$$

$$\Leftrightarrow \quad \alpha = 35.1°$$

The magnitude of the resultant velocity $\vec{v}_R$ is then found as follows:

$$\frac{\sin(90.0° + 22.5°)}{v_{plane}} = \frac{\sin(90.0° - \alpha)}{v_R} \quad \Leftrightarrow \quad v_R = v_{plane} \cdot \frac{\sin(90.0° - \alpha)}{\sin(90.0° + 22.5°)} = 275 \cdot \frac{\sin(90.0° - 35.1°)}{\sin(90.0° + 22.5°)}$$

$$= 243 \text{ km/h}$$

Exercise 17

Problem Statement

Nastya and Keril are taking part in a local sports competition in Nizhny Novgorod, Russia, that involves four main parts: long-distance running, archery, mountain biking, and swimming. They are in the lead and reached the last activity, i.e., swimming. The Volga river is the final leg of the race that stands between them and the finish line, which lies right across the other side of the Volga. Since there is a current of $\vec{v}_{river} = -1.05 \cdot \vec{i}_x$ m/s, they might end up some distance away from the finish line, in which case they have to sprint the last couple of meters.

(1) Given the magnitudes of their swimming and running velocity v_{swim} and v_{run}, respectively, determine a general formula for the fastest route across the Volga. (2) While Nastya is a faster swimmer than Keril ($v_{swim,N} = 1.95$ m/s versus $v_{swim,K} = 1.85$ m/s), she runs at a lower pace ($v_{run,N} = 4.15$ m/s versus $v_{run,K} = 5.20$ m/s). If Keril has an advantage of 19.5 s with respect to Nastya, who wins the competition if both follow their optimal routes? Where do the athletes come ashore? Suppose that the Volga is $d = 850$ m wide at the point where they enter the water.

Solution

(1) The total time (t_{tot}) consists of the time it takes to cross the river (t_{cross}) plus the time needed to run to the finish line (t_{run}) in case they end up at some distance greater than zero from the finish line. To establish a general formula, we will assume that the athletes start swimming upstream, i.e., an angle θ east of north. The time to swim across the Volga (y-direction) is then found as follows:

$$t_{cross} = \frac{d}{v_{swim} \cdot \cos\theta}$$

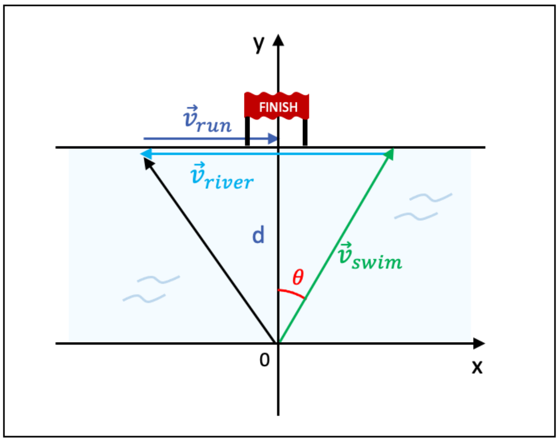

Figure 16

In order to calculate the time t_{run}, we first establish the point of arrival on the opposite bank of the Volga. The velocity in the x-direction is equal to:

$$v_{swim} \cdot \sin\theta - v_{river}$$

If we multiply that with the time the athletes needed to cross the Volga, we know the displacement along the other shore:

$$(v_{swim} \cdot \sin\theta - v_{river}) \cdot t_{cross} = (v_{swim} \cdot \sin\theta - v_{river}) \cdot \frac{d}{v_{swim} \cdot \cos\theta}$$

Now we want to determine the time it takes to run to the finish line, which is equal to the above expression for the displacement divided by their running speed:

$$t_{run} = \frac{(v_{swim} \cdot \sin\theta - v_{river})}{v_{run}} \cdot \frac{d}{v_{swim} \cdot \cos\theta}$$

I invite you to reflect for a moment on the equation we've found for t_{run}. Regardless of whether Nastya and Keril arrive to the right or to the left of the finish line, we need the time to be a positive number. However, bearing in mind our coordinate system, if they arrive to the right of the finish line, the displacement in the x-direction will be positive but the direction of v_{run} will be negative, as they must run to the left towards the finish. This would give a negative amount of time. Similarly, for a point of arrival to the left of the finish, time, again, becomes negative, since the displacement will be negative but with a positive v_{run} (they must move to the right to get to the finish). In order to obtain a positive amount of time, we introduce a minus sign in the above equation:

$$t_{run} = \frac{(v_{river} - v_{swim} \cdot \sin\theta)}{v_{run}} \cdot \frac{d}{v_{swim} \cdot \cos\theta}$$

The total amount of time then becomes:

$$\begin{aligned} t_{tot} &= t_{cross} + t_{run} \\ &= \frac{d}{v_{swim} \cdot \cos\theta} + \frac{(v_{river} - v_{swim} \cdot \sin\theta)}{v_{run}} \cdot \frac{d}{v_{swim} \cdot \cos\theta} \\ &= \frac{d}{v_{swim} \cdot \cos\theta} \cdot \left(\frac{v_{run} + v_{river} - v_{swim} \cdot \sin\theta}{v_{run}}\right) \\ &= \frac{d}{v_{swim} \cdot v_{run}} \cdot \left(\frac{v_{run} + v_{river} - v_{swim} \cdot \sin\theta}{\cos\theta}\right) \end{aligned}$$

To find the fastest time, we take the derivative of t_{tot} with respect to the angle θ:

$$\frac{d(t_{tot})}{d\theta} = \frac{d}{v_{swim} \cdot v_{run}} \cdot \left[\frac{-v_{swim} \cdot \cos^2\theta + (v_{run} + v_{river} - v_{swim} \cdot \sin\theta) \cdot \sin\theta}{\cos^2\theta}\right]$$

$$= \frac{d}{v_{swim} \cdot v_{run}} \cdot \left[\frac{-v_{swim} \cdot (\cos^2\theta + \sin^2\theta) + (v_{run} + v_{river}) \cdot \sin\theta}{\cos^2\theta} \right]$$

$$= \frac{d}{v_{swim} \cdot v_{run}} \cdot \left[\frac{-v_{swim} + (v_{run} + v_{river}) \cdot \sin\theta}{\cos^2\theta} \right]$$

The find the angle θ under which t_{tot} becomes minimal, we equate the above derivative to zero:

$$\frac{d(t_{tot})}{d\theta} = 0 \quad \Leftrightarrow \quad \frac{d}{v_{swim} \cdot v_{run}} \cdot \left[\frac{-v_{swim} + (v_{run} + v_{river}) \cdot \sin\theta}{\cos^2\theta} \right] = 0$$

$$\Leftrightarrow \quad \frac{-v_{swim} + (v_{run} + v_{river}) \cdot \sin\theta}{\cos^2\theta} = 0$$

$$\Leftrightarrow \quad -v_{swim} + (v_{run} + v_{river}) \cdot \sin\theta = 0$$

$$\Leftrightarrow \quad \sin\theta = \frac{v_{swim}}{v_{run} + v_{river}}$$

$$\Leftrightarrow \quad \theta = \sin^{-1}\left(\frac{v_{swim}}{v_{run} + v_{river}} \right)$$

(2) The optimal angle for Nastya (θ_N) and Keril (θ_K), respectively, is equal to:

$$\begin{cases} \theta_N = \sin^{-1}\left(\dfrac{1.95}{4.15 + 1.05} \right) = 22.0° \\ \theta_K = \sin^{-1}\left(\dfrac{1.85}{5.20 + 1.05} \right) = 17.2° \end{cases}$$

The best choice for the athletes is thus to start swimming upstream. The total time it takes Nastya ($t_{tot,N}$) and Keril ($t_{tot,K}$), respectively, to arrive at the finish line becomes:

$$\begin{cases} t_{tot,N} = \dfrac{850}{1.95 \cdot 4.15} \cdot \left[\dfrac{4.15 + 1.05 - 1.95 \cdot \sin(22.0°)}{\cos(22.0°)} \right] \\ \quad = 506 \text{ s} \text{ or } 8 \text{ min } 26.3 \text{ sec} \\ t_{tot,K} = \dfrac{850}{1.85 \cdot 5.20} \cdot \left[\dfrac{5.20 + 1.05 - 1.85 \cdot \sin(17.2°)}{\cos(17.2°)} \right] \\ \quad = 527 \text{ s} \text{ or } 8 \text{ min } 47.5 \text{ sec} \end{cases}$$

Since Nastya lags 19.5 s behind Keril, we add this amount to her total time, so that she finishes in 8 min 45.8 s and wins the competition by a margin of 1.66 s.

To determine at what exact location the two athletes come ashore, we work out the following equations of motion in the x-direction:

$$
\begin{cases}
x_N(t_{cross}) &= [v_{swim,N} \cdot \sin(\theta_N) - v_{river}] \cdot t_{cross} \\
&= [v_{swim,N} \cdot \sin(\theta_N) - v_{river}] \cdot \dfrac{d}{v_{swim,N} \cdot \cos(\theta_N)} \\
&= [1.95 \cdot \sin(22.0°) - 1.05] \cdot \dfrac{850}{1.95 \cdot \cos(22.0°)} \\
&= -150 \text{ m} \\
\\
x_K(t_{cross}) &= [v_{swim,K} \cdot \sin(\theta_K) - v_{river}] \cdot t_{cross} \\
&= [v_{swim,K} \cdot \sin(\theta_K) - v_{river}] \cdot \dfrac{d}{v_{swim,K} \cdot \cos(\theta_K)} \\
&= [1.85 \cdot \sin(17.2°) - 1.05] \cdot \dfrac{850}{1.85 \cdot \cos(17.2°)} \\
&= -242 \text{ m}
\end{cases}
$$

Exercise 18

Problem Statement

Sophia is casually riding her brand-new snowboard on a 32°-blue square slope of the Whistler Mountain in Canada. Being all warmed up after an hour of doing slaloms, Sophia heads towards a first jump, which makes a 13° angle with the horizontal, and pulls off a Chicken Salad grab. She successfully lands her trick 28 m down the hill. (1) What was Sophia's initial velocity? (2) What is her landing velocity? (3) What is the airtime of her jump?

Solution

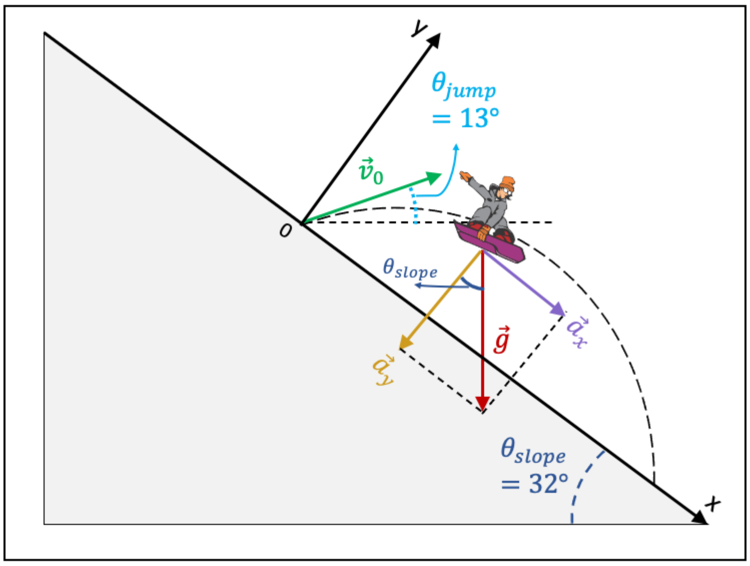

Figure 17

(1) The angle of the jump of $\theta_{jump} = 13°$ with respect to the horizontal is equal to $\theta_{tot} = \theta_{jump} + \theta_{slope} = 13° + 32° = 45°$ from the perspective of someone standing on the ski slope. With that in mind, starting from the equation of motion in the y-direction we can obtain an expression for the airtime of Sophia's jump in terms of the initial velocity:

$$y(t_{air}) = y_0 + v_{0y} \cdot t_{air} + \frac{a_y}{2} \cdot t_{air}^2 \Leftrightarrow y(t_{air}) = y_0 + v_0 \cdot \sin(\theta_{tot}) \cdot t_{air} - \frac{g \cdot \cos(\theta_{slope})}{2} \cdot t_{air}^2$$

$$\Leftrightarrow 0.0 = 0.0 + v_0 \cdot \sin(45°) \cdot t_{air} - \frac{9.81 \cdot \cos(32°)}{2} \cdot t_{air}^2$$

$$\Leftrightarrow t_{air} = \frac{2 \cdot v_0 \cdot \sin(45°)}{9.81 \cdot \cos(32°)}$$

If we replace the above expression for t_{air} in the equation of motion in the x-direction, we find the initial velocity (whereby we make use of the angle subtraction theorem "$\cos(A - B) = \cos A \cos B + \sin A \sin B$"):

$$x(t_{air}) = x_0 + v_{0x} \cdot t_{air} + \frac{a_x}{2} \cdot t_{air}^2$$

$$\Leftrightarrow \quad x(t_{air}) = x_0 + v_0 \cdot \cos(\theta_{tot}) \cdot t_{air} + \frac{g \cdot \sin(\theta_{slope})}{2} \cdot t_{air}^2$$

$$\Leftrightarrow \quad 28 = 0.0 + v_0 \cdot \cos(45°) \cdot \left[\frac{2 \cdot v_0 \cdot \sin(45°)}{9.81 \cdot \cos(32°)}\right] + \frac{9.81 \cdot \sin(32°)}{2} \cdot \left[\frac{2 \cdot v_0 \cdot \sin(45°)}{9.81 \cdot \cos(32°)}\right]^2$$

$$\Leftrightarrow \quad 28 = \frac{2 \cdot v_0^2}{9.81} \cdot \sin(45°) \cdot \left[\frac{\cos(45°)}{\cos(32°)} + \frac{\sin(32°) \cdot \sin(45°)}{\cos^2(32°)}\right]$$

$$\Leftrightarrow \quad 28 = \frac{2 \cdot v_0^2}{9.81} \cdot \frac{\sin(45°)}{\cos^2(32°)} \cdot [\cos(45°) \cdot \cos(32°) + \sin(32°) \cdot \sin(45°)]$$

$$\Leftrightarrow \quad 28 = \frac{2 \cdot v_0^2}{9.81} \cdot \frac{\sin(45°)}{\cos^2(32°)} \cdot \cos(45° - 32°)$$

$$\Leftrightarrow \quad v_0 = \sqrt{\frac{28 \cdot 9.81 \cdot \cos^2(32°)}{2 \cdot \sin(45°) \cdot \cos(13°)}}$$

$$= 12 \text{ m/s}$$

(2) We can retrieve the landing velocity v_x from the following equation of motion in the x-direction:

$$\begin{aligned} v_x^2 \quad - \quad & v_{0x}^2 = 2 \cdot a_x \cdot \Delta x \\ \Leftrightarrow \quad v_x \quad = \quad & \sqrt{v_{0x}^2 + 2 \cdot a_x \cdot \Delta x} \\ = \quad & \sqrt{[v_0 \cdot \cos(\theta_{tot})]^2 + 2 \cdot [g \cdot \sin(\theta_{slope})] \cdot \Delta x} \\ = \quad & \sqrt{[12 \cdot \cos(45°)]^2 + 2 \cdot [9.81 \cdot \sin(32°)] \cdot 28} = 19 \text{ m/s} \end{aligned}$$

(3) The airtime t_{air} can now be calculated with the help of the expression found in part (1):

$$t_{air} = \frac{2 \cdot v_0 \cdot \sin(45°)}{9.81 \cdot \cos(32°)} = \frac{2 \cdot 12 \cdot \sin(45°)}{9.81 \cdot \cos(32°)} = 2.0 \text{ s}$$

Exercise 19

Problem Statement

Tommaso is cruising at sunset at $v_{Cessna} = 232$ km/h in his Cessna 172 Skyhawk above the hilly landscape of Val d'Orcia, Italy. As he is headed north-west towards the town of Siena, he is enjoying the endless vineyards and the picturesque villages, such as Pienza, Monticchiello, and Bagno Vignoni. Due to this mesmerizing scenery, Tommaso forgot to check his instruments during the past 50.0 minutes, and it appears that he already covered 210 km since he last checked and that he is actually flying in the direction of 27.5° west of north. What is the magnitude and direction of the wind velocity $\vec{v}_{wind}$ that is responsible for the shift in his trajectory?

Solution

In a first instance, let us determine the speed at which his Cessna 172 is actually traveling.

Given that the plane covered 210 km in 50.0 min, the speed is equal to $v_{tot} = 210 \cdot \frac{60.0}{50.0} = 252$ km/h, rather than the assumed 232 km/h.

Since north-west corresponds with the direction of 45.0° west of north, the angle by which the actual trajectory deviates from the north-west direction is measured as $\theta_{dev} = 45.0° - 27.5° = 17.5°$.

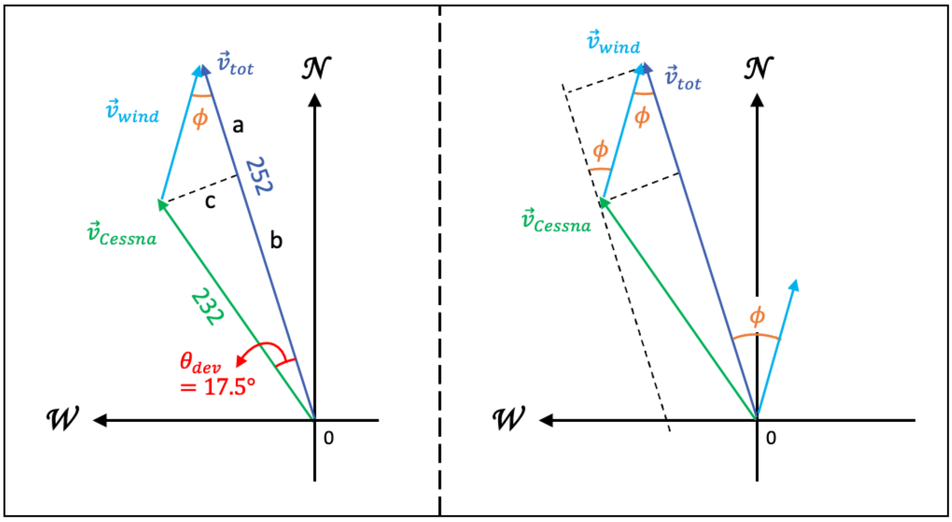

Figure 18

With the assistance of the Pythagorean theorem, we find the following values for the parameters b, c, and a, respectively:

$$\begin{cases} b = v_{Cessna} \cdot \cos(\theta_{dev}) \qquad c = v_{Cessna} \cdot \sin(\theta_{dev}) \qquad a = v_{tot} - b \\ = 232 \cdot \cos(17.5°) \qquad = 232 \cdot \sin(17.5°) \qquad = 252 - 221 \\ = 221 \text{ km/h} \qquad = 69.8 \text{ km/h} \qquad = 30.7 \text{ km/h} \end{cases}$$

The magnitude of the wind vector $\vec{v}_{wind}$ is then calculated as follows:

$$v_{wind} = \sqrt{a^2 + c^2} = \sqrt{30.7^2 + 69.8^2} = 76.2 \text{ km/h}$$

In order to identify the direction of the wind within our given coordinate system, we first need to find the angle ϕ:

$$\phi = \tan^{-1}\left(\frac{c}{a}\right) = \tan^{-1}\left(\frac{69.8}{30.7}\right) = 66.2^\circ$$

Put another way, if a vector identical to $\vec{v}_{tot}$ is rotated clockwise for a number of 66.2 degrees, it would be parallel to the vector $\vec{v}_{wind}$. Given that the angle between $\vec{v}_{tot}$ and north measures 27.5°, the direction of the wind vector $\vec{v}_{wind}$ is equal to $66.2^\circ - 27.5^\circ = 38.7^\circ$ east of north.

Exercise 20

Problem Statement

After spending a day on turbulent waters in the Gulf of Siam, Rangsei is steering her shrimp boat $\theta_i = 30.0°$ north of east towards her docking station at the port of Sihanoukville, Cambodia. When she is 1.50 km away from the port, Rangsei receives a radio call from the local command centre with the message that she must dock 300 m north-west from her usual docking station due to some hindrance caused by local festivities. Given a north-west current of $v_{cur} = 1.20$ m/s, determine the angle θ under which Rangsei must redirect her shrimp boat to safely reach her new docking station, if you know that the boat maintains a velocity of $v_{boat} = 6.52$ kts (1 knot is equal to 1.852 km/h) with respect to still water.

Solution

In order to write the equations of motion in the x- and y-direction, we first identify the required displacement in both directions:

$$\begin{cases} \Delta x = b - a & \Delta y = c + e \\ \quad = d \cdot \cos\theta_i - f \cdot \cos\theta_{NW} & \quad = d \cdot \sin\theta_i + f \cdot \sin\theta_{NW} \\ \quad = 1500 \cdot \cos(30.0°) - 300 \cdot \cos(45.0°) & \quad = 1500 \cdot \sin(30.0°) + 300 \cdot \sin(45.0°) \\ \quad = 1087 \text{ m} & \quad = 962 \text{ m} \end{cases}$$

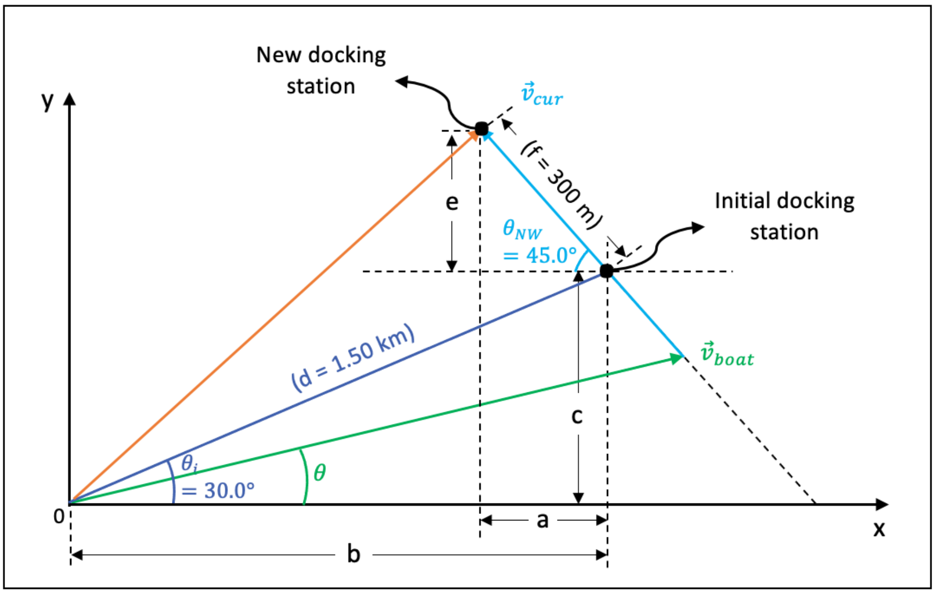

Figure 19

Keeping in mind that the speed of the boat is equal to $v_{boat} = \frac{6.52 \cdot 1.852}{3.6} = 3.35$ m/s and that $\vec{v}_{cur}$ makes an angle of $\theta_{NW} = 45.0°$ north of west, the two equations of motion are the following:

$$\begin{cases} \Delta x = (v_{boat} \cdot \cos\theta - v_{cur} \cdot \cos\theta_{NW}) \cdot t \\ \Leftrightarrow\ 1087 = (3.35 \cdot \cos\theta - 1.20 \cdot \cos(45.0°)) \cdot t \\ \\ \Delta y = (v_{boat} \cdot \sin\theta + v_{cur} \cdot \sin\theta_{NW}) \cdot t \\ \Leftrightarrow\ 962 = (3.35 \cdot \sin\theta + 1.20 \cdot \sin(45.0°)) \cdot t \end{cases}$$

If we replace the variable t in the second equation by the expression for t obtained from the first equation, the second equation then becomes:

$$962 = (3.35 \cdot \sin\theta + 1.20 \cdot \sin(45.0°)) \cdot \left[\frac{1087}{(3.35 \cdot \cos\theta - 1.20 \cdot \cos(45.0°))}\right]$$

$$\Leftrightarrow\ 3.35 \cdot (962 \cdot \cos\theta - 1087 \cdot \sin\theta) = 1.20 \cdot [1087 \cdot \sin(45.0°) + 962 \cdot \cos(45.0°)]$$

$$\Leftrightarrow\ 3.35 \cdot (962 \cdot \cos\theta - 1087 \cdot \sin\theta) = 1.20 \cdot \frac{\sqrt{2}}{2} \cdot (1087 + 962)$$

It is useful to know that the linear combination of a cosine and a sine function, i.e., "$a \cdot \cos\theta + b \cdot \sin\theta$", can be replaced by a single cosine function "$c \cdot \cos(\theta + \phi)$", whereby $c = sgn(a)\sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(-\frac{b}{a})$.

Therefore, in the case of our above equation, we can rewrite the linear combination "$962 \cdot \cos\theta - 1087 \cdot \sin\theta$" as "$1452 \cdot \cos(\theta + 48.5°)$", whereby $c = +\sqrt{962^2 + (-1087)^2} = 1452$ and $\phi = \tan^{-1}\left[-\frac{(-1087)}{962}\right] = 48.5°$. If we implement this new expression, we can calculate the value of the angle θ:

$$3.35 \cdot [1452 \cdot \cos(\theta + 48.5°)] = 1.20 \cdot \frac{\sqrt{2}}{2} \cdot (1087 + 962)$$

$$\Leftrightarrow\ \cos(\theta + 48.5°) = \frac{1.20 \cdot \frac{\sqrt{2}}{2} \cdot 2049}{3.35 \cdot 1452}$$

$$\Leftrightarrow\ \theta + 48.5° = \cos^{-1}\left(\frac{1.20 \cdot \frac{\sqrt{2}}{2} \cdot 2049}{3.35 \cdot 1452}\right) = 69.1°$$

$$\Leftrightarrow\ \theta = 20.6°$$

So, if Rangsei readjusts her direction to 20.6° north of east, she will safely arrive at her new docking station, as requested.