Physics

Exercises on Applications of Newton's Laws in One, Two, and Three Dimensions

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Summary of Exercises

Exercise 1

Suppose you live at the geographical coordinates 41°21′ 46.4″N 15°18′36.5″E, whereby your street makes an angle of $\gamma = 9.50°$ with the horizontal. Today, the outdoor temperature indicates 27.3° and so you planned a lunch outside with the neighbours in front of your house. You have just put a glass of water ($m_w = 0.450$ kg) on the table but forgot that the tablecloth is a smooth (i.e., frictionless) surface. As a result, the glass starts sliding towards the edge of the table. (1) What is the effective acceleration $\vec{a}_{ef}$ of the glass if you take into account the fact that you're living on a rotating planet, i.e., you are subject to a centrifugal force? (2) How does $\vec{a}_{ef}$ compare to the situation whereby you ignore the Earth's centripetal acceleration $\vec{a}_{cp}$? Remember that the Earth's radius is equal to $r_E = 6.38 \times 10^6$ m and assume that the center of the Earth is stable enough to be the origin of an inertial coordinate system (x,y).

Exercise 2

In Nagano, Japan, Yuuto's fourteen-year-old daughter Koharu (m_K) is coming home from school and as usual, before going inside to do her homework, she runs towards the marble incline, which makes an angle θ with the horizontal, right beside the entry porch and slides down on her feet to see how far she gets without falling. Today, it has been freezing and while the upper part of the incline (with a length of d_1) is slightly wet, the lower part of the incline is covered with a thin layer of ice for a distance of d_2.

(1) If the coefficient of kinetic friction for rubber on wet marble and ice is represented by μ_{k1} and μ_{k2}, respectively, formulate an expression for the final velocity v_{f2} of Koharu at the bottom of the incline in terms of the initial velocity v_0, the angle θ, the gravitational constant g, the coefficients of friction μ_{k1} and μ_{k2}, and the distances d_1 and d_2. (2) Formulate an expression for the length L of the incline in terms of the velocities v_0, v_{f1}, and v_{f2}, the angle θ, and the coefficients for kinetic friction μ_{k1} and μ_{k2}, whereby v_{f1} is the velocity at the end of distance d_1. (3) If $d_1 = 3.50$ m, $\mu_{k1} = 0.622$, $\theta = 28.5°$, $v_0 = 3.50$ m/s, $v_{f2} = 6.70$ m/s, and the time Koharu spends on the ice $t_2 = 0.933$ s, what is the value of the coefficient of kinetic friction for rubber on ice μ_{k2}? (4) Using the formula derived in part (2) and the provided data in part (3), what is the length L of the incline?

Exercise 3

Harry just got back home from buying a gift ($m_b = 4.85$ kg) for his husband Leo in the Meadowhall Shopping Centre in Sheffield, United Kingdom. When Harry is in the midst of gift wrapping his present on his knees in the living room, Leo unexpectedly arrives home. Harry rushes to hide the gift between his back and the wall, and he casually gets up on his feet in $t = 1.90$ s while pushing on the gift with his back, in the hope that it stays in place so that Leo is not on to him. If the force $\vec{F}$ with which Harry pushes against his gift makes an angle of $\theta = 64.1°$ with the xz-plane in the positive y-direction and $\phi = 69.4°$ with the xy-plane in the negative z-direction (whereby the y-axis is directed upwards and the z-axis points out of the wall), (1) what is the distance d that the gift has traveled across the wall when Harry is standing upright? (2) What is the magnitude of the force $\vec{F}$? Assume that the origin of the coordinate system sits at the centre of the gift (at its initial position on the wall), that the coefficient of kinetic friction of wrapping paper on the wall equals $\mu_k = 0.730$, and that the net acceleration $\vec{a}$ and the gift's trajectory run parallel with the projection of $\vec{F}$ onto the xy-plane.

Exercise 4

Near an amusement park in the centre of al-Hilla in Iraq, Bibi ($m_B = 58$ kg) is practicing her kick-flip and 360 spin backside with her brand new Core C2 skateboard ($m_s = 3.1$ kg). Bibi notices a semi-circle steel rail with a height of $h = 0.45$ m and a radius of $r = 7.5$ m and starts building up speed so that she can grind the rail as far as possible. By the time she jumps onto the rail, Bibi has reached a speed of $v_0 = 6.5$ m/s. (1) Under what angle with the horizontal should Bibi hit the rail with her skateboard deck? (2) As Bibi slides along the rail, she manages to keep her balance until she comes to a halt. At what angle is she now positioning her skateboard to avoid falling from the rail? (3) How far did Bibi slide (work with average values)? Assume that the coefficient of kinetic and static friction for wood on steel is equal to $\mu_k = 0.29$ and $\mu_s = 0.42$, respectively.

Exercise 5

A rock with a mass of $m_r = 20.0$ ton is hurtling through space in the vertical direction and when it is $r = 1,250$ km away from the surface of the Earth at an angle of $\theta = 59.5°$ with the vertical, the rock possesses a speed of $v_0 = 2,500$ m/s and is accelerating at $a_0 = 275$ m/s^2. Will it hit the Earth? As an approximate criterion for the condition of "hitting the Earth", consider the critical distance x_c. Given that the gravitational force $\vec{F}_G$ dynamically changes as the rock approaches Earth, use the average between $\vec{F}_G$'s magnitude at the rock's current position and that at the surface of the Earth. Assume furthermore that the Earth and the rock are moving in the xy-plane. Finally, remember that the radius of the Earth is equal to $r_E = 6.38 \times 10^6$ m, the Earth's mass to $M = 5.98 \times 10^{24}$ kg, and the universal gravitational constant to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2).

Exercise 6

Emilio ($m_E = 72.9$ kg) just drove an hour from his home in Viseu, Portugal, to go water skiing in the Atlantic Ocean along the coast of Aveiro. His friend Isabela agreed to take him onto the water with her boat. At one point, Isabela takes a turn with a radius of $r = 125$ m at a constant speed. Meanwhile, Emilio is firmly holding the tow rope, which makes an angle of $\theta = 13.7°$ with the horizontal, and in the curve, he is following a path (without skidding) that lies radially $d = 5.10$ m more outwards compared to Isabela's position, producing an angle ϕ between the tow rope and his velocity vector $\vec{v} = 18.4 \cdot \vec{i}_y$ m/s. The water that Emilio pushes away sideways in the curve exerts a force of $\vec{F}_w = 150 \cdot \vec{i}_x$ N upon him, and he is also experiencing a kinetic friction force $\vec{F}_k$ opposite to his direction of motion—assume a kinetic friction coefficient of skis on water of $\mu_k = 0.175$. What is the value of the angle ϕ?

Exercise 7

Feeling re-energized after a weekend of hiking close to the Bavarian Sea, Grgur is analyzing astronomical data with a pair of fresh eyes on Monday morning in the ESO's headquarters in Garching, Germany. Apparently, the mental and physical recharging over the weekend have payed off as Grgur identifies a pattern between two newly discovered objects after one hour of work. The first object, which he named MS-X52R, is orbiting the planet Mars (in a circular orbit), while the second object, called SN-Y22T, circles the planet Saturn. By the time SN-Y22T has completed 1 revolution around Saturn, MS-X52R has already orbited Mars 7.43 times. Moreover, Grgur also found that the orbital height of SN-Y22T above the planet's surface relative to that of MS-X52R is ten times their respective relative velocity. Grgur is interested in calculating these orbital heights. What values does he find? Remember that the radius and mass of Mars and Saturn are equal to $r_M = 3.39 \times 10^6$ m and $r_S = 60.3 \times 10^6$ m and $M_M = 6.42 \times 10^{23}$ kg and $M_S = 5.69 \times 10^{26}$ kg, respectively.

Exercise 8

At this time of the year, the volcano Maat Mons on the planet Venus ($M_V = 4.87 \times 10^{24}$ kg) is highly active. Within the atmospheric region close to the planet's surface, where the 8 km-high volcano resides, the air density is immense at a value of approximately $\rho = 67.0$ kg/m^3. At a certain point, a large basaltic rock ($m_r = 1,250$ kg) is being ejected from Maat Mons. When it reaches the highest point in its trajectory it collides with another rock, thereby effectively eliminating any horizontal motion, so that the rock now starts falling vertically. If you know that the drag force $\vec{F}_D$ has the form of $\vec{F}_D = -b \cdot v^2 \cdot \vec{i}_y$ N (with $b = \frac{1}{2} \cdot c_D \cdot \rho \cdot \pi \cdot r^2$, whereby the drag coefficient equals $c_D = 0.635$), that the radius of Venus measures $r_V = 6.05 \times 10^6$ m, and that the basaltic rock has a diameter of about $d = 92.6$ cm, (1) what is the magnitude of the terminal velocity $\vec{v}_T$? (2) Write an expression for the magnitude of the rock's velocity in terms of the time variable t. Assume that $t = 0$ s when the rock is at the highest point of its trajectory.

Exercise 9

Willow is visiting her grandmother Evie, who lives in Launceston, Tasmania, to spend some quality time with her. During some afternoon tea with traditional Anzac biscuits, Evie tells Willow to go and get an old painting from the attic that she made during her childhood. Once up there, Willow spots the painting on top of a large storage cupboard. While standing on the tips of her toes, she grabs the painting and tilts it away from her by an angle of $\phi = 21.3°$ with the vertical. At that moment, an old golden medallion ($m_m = 0.350$ kg), which was hanging just over the left side of the painting and attached to a chain, which is fixed to the middle of the top edge of the painting, slides from the left side in an arc-like motion towards the middle—initially, the chain, which has a length of $L = 34.4$ cm, was making an angle of $\theta = 63.6°$ with the left side. If the average blink of an eye lasts $t_b = 0.120$ s, how many times can Willow blink before the medallion reaches the middle of the painting? Assume that the kinetic friction coefficient between the metal of the medallion and the canvas is equal to $\mu_k = 0.784$.

Exercise 10

In the western Pacific Ocean, close to the coast of Tobi Island, Palau, two blacktail damselfish are feeling playful. The heavier of the two (m_1) is swimming right behind the other one ($m_2 = 3.92$ kg), who is moving at a constant speed of $\vec{v}_0 = 1.51 \cdot \vec{i}_x$ m/s under an angle of $\theta = 34.8°$ with the horizontal. At a certain moment, the heavier damselfish is pushing his friend in the same direction of her motion with a constant force of $\vec{F}_{21} = 24.4 \cdot \vec{i}_x$ N. The drag force $\vec{F}_D$ in a viscous medium for lower velocities has the general form of $\vec{F}_D = -(K \cdot \eta) \cdot \vec{v}$, with η the viscosity coefficient with a value of $\eta = 1.787 \times 10^{-3}$ kg/(m·s) for water at 0°C. The parameter K depends on the shape of the object, and if we approximate the fish by a sphere, we obtain $K = 6 \cdot \pi \cdot r$ (in m), with $r = 12.0$ cm. If we ignore the buoyancy force in our problem, how fast is the first blacktail damselfish going after being pushed for $t = 6.25$ s?

Exercise 11

The thirteen-year-old Bahadur is visiting the new science fair with his dad Husani in the Planetarium Science Center in Alexandria, Egypt. In one of the activities, Bahadur has to pull a large block ($M = 8.50$ kg), which is moving on a frictionless rail. On top of the large block, a small block ($m = 4.50$ kg) is positioned precisely 12.0 cm to the right of a marked area. Bahadur is asked to pull the lower block for just the right amount of time t_*, so that the upper block moves to the left and comes to rest precisely within the marked area. Since the top surface of the large block is slightly roughened, the small block needs a minimum amount of force $\vec{F}_s$ before it can start moving (the static friction coefficient is equal to $\mu_s = 0.115$). Once the small block is set in motion, it experiences a slightly lower amount of (kinetic) friction, i.e., $\mu_k = 0.102$. If Bahadur pulls the large block M with a force of $\vec{F}_{pull} = 15.0 \cdot \vec{i}_x$ N, for how long (t_*) should he sustain this force? Assume that the time t_s corresponds to the time needed for block m to overcome the static friction—during this time, block m is not yet moving—and is equal to $t_s = \frac{t_*}{10}$ s.

Exercise 12

Lagrange points are relatively stable orbits of objects of little mass in the presence of two heavier masses (with one mass (M_1) larger than the other (M_2) for a minimum ratio of $\frac{M_1}{M_2} = 24.96$), which are all orbiting around a common center of mass, i.e., the barycenter. In our Solar System, examples of such massive bodies include the Sun-Jupiter and the Sun-Earth duo as well as the Earth-Moon system. The gravitational interplay within these systems allows for the existence of five Lagrange points, i.e., L_1 up to L_5. From the perspective of a *rotating* reference frame, the relatively stable circular orbit of the object of little mass (say, m_1) is the result of the combined gravitational impact on m_1, due to the large masses M_1 and M_2, being balanced by a pseudo-force, i.e., the centrifugal force, experienced by m_1 from the center of mass. The final effect is such that the period T of the object m_1 is equal to that of both mass M_1 and M_2—the period T is the amount of time during which an object completes one revolution around another object.

With regard to point L_3 of the Earth-Moon system, the orbit of mass m_1 lies a little bit farther from the barycenter with respect to the Moon (M_2) and there is a small distance d between the position of m_1 and M_2, if both objects would be located at the same side of mass M_1 (with m_1 being closer to M_1). If you know that the mass of the Earth and the Moon are equal to $M_1 = 5.972 \times 10^{24}$ kg and $M_2 = 7.342 \times 10^{22}$ kg, respectively, and that the Earth-Moon distance measures $R = 3.844 \times 10^5$ km, how far lies the Lagrange point L_3 from the center of the Earth?

Exercise 13

On a 5.00 m-wide gravel road outside of Balkanabat, Turkmenistan, Melek is going $\vec{v}_0 = 47.2 \cdot \vec{i}_y$ m/s, when she suddenly notices that a 90° left curve is ahead. Melek hits the breaks over a distance of $\Delta x_0 = 156$ m and $t_0 = 4.30$ s later she enters the curve, which has a radius of $r = 85.0$ m, at a velocity $\vec{v}_{in}$ at 1.00 m from the left guardrail. (1) Will Melek skid in the curve? (2) If yes, will she hit the guardrail on the right-hand side? (3) If so, when? If not, at what distance from the right rail does Melek exit the curve? Given that Melek may change her distance from the left rail while going through the curve, apply average values over the width of the road between her point of entry and the right rail when dealing with circular motion. Assume furthermore that Melek maintains her speed v_{in} throughout the curve and that the kinetic friction coefficient between gravel and rubber tires is equal to $\mu_k = 0.718$.

Exercise 14

Halima is doing research at the Copperbelt University in Zambia on superclusters, which are aggregate systems of various galaxy groups and smaller clusters, whereby one of them, the Ophiuchus Supercluster, which is located at a distance of roughly 370 million light-years away from us (1 light-year is equal to 9.46×10^{15} m), particularly interests her. Halima suspects to have found a black hole at the edge of the Ophiuchus Supercluster around which three other objects are orbiting in a circular fashion. So far, Halima has managed to retrieve the following information from the orbiting objects: object 1 has a period of $T_1 = 163$ Earth days, the distance from object 2 to the center of the black hole is equal to $r_2 = 5.93 \times 10^7$ km, the distance from object 1 to the black hole is twice as large relative to that of object 3, and the distance from object 3 to the black hole is 1.26 times greater with respect to object 2. (1) Halima wants to calculate the mass of the black hole in terms of the mass of our Sun, which is equal to $M_s = 1.99 \times 10^{30}$ kg. What value does she find? (2) What is the period (in Earth days) for object 2 and 3? (3) What are the orbital velocities of the three objects? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$s^2) and assume that, due to the overwhelmingly strong gravitational influence of the black hole, the gravitational interactions between the three objects are minimal and can therefore be ignored, and that the mass M_{BH} of the black hole remains constant.

Exercise 15

On 23 January 1960, Jacques Piccard and Don Walsh descended in their 18 m-long small submarine, called a bathyscaphe, to a depth of $d = 10{,}911$ m in the Mariana Trench in the Pacific Ocean. During the descent, both men spent nearly five hours in a 2.16 m-wide pressure sphere. Suppose that at one moment, Jacques was holding a magazine of length $L = 25.00$ cm horizontally with both hands, and on top of it, a set of keys ($m_{sk} = 0.3850$ kg) was resting. The keys were connected to one end of an elastic rubber spring, while the other end was attached to a metal ring through which

Jacques had put his index finger of his right hand. Due to a sudden disturbance in the bathyscaphe's balance, Jacques removed his left hand from the magazine to hold on to the side of the pressure sphere. As a result, Jacques tilted the magazine by an angle of $\theta = 46.80°$ with the vertical and the set of keys slid downwards (from the top side of the magazine cover), stretching thereby the rubber spring (the metal ring was still around Jacques' index finger of his right hand). If the restoring force in a spring has the general form of $\vec{F}_r = -k \cdot x \cdot \vec{i}_x$, with k the spring constant, which for this particular rubber spring is equal to $k = 9.450$ N/m, (1) did the set of keys slide off of the bottom of the magazine? (2) If they did, at what distance did the keys dangle from the bottom of the magazine? (3) If Jacques would have held the magazine in the same way when sitting in his living room at home, what would the results have been then? Remember that the universal gravitational constant G is equal to $G = 6.673 \times 10^{-11}$ m^3/(kg$\cdot$ s^2), and the mass and the radius of the Earth to $M_E = 5.9722 \times 10^{24}$ kg and $r_E = 6.3781 \times 10^6$ m, respectively.

Exercise 16

The four largest moons—called the Galilean moons—orbiting (anti-clockwise) around the planet Jupiter ($M_j = 1.898 \times 10^{27}$ kg) are among the largest within our Solar System. Of this quartet, the two innermost moons orbiting Jupiter are Io ($M_{io} = 8.93 \times 10^{22}$ kg) and Europa ($M_{eur} = 4.80 \times 10^{22}$ kg), whereby Io travels at a height of $h_{io} = 350,500$ km above Jupiter's surface. Suppose that about 6.5 years ago the China National Space Administration (CNSA) launched a space probe ($m_{sp} = 2,850$ kg), which just now successfully settled into Europa's orbit at a distance of roughly $s = 526,800$ km behind the moon with an orbital speed of $v_{sp} = 13,739$ m/s. (1) What is the net gravitational force $\vec{F}_G$ experienced by the probe when at the moment of arrival Io is located right above Jupiter whereas Europa makes an angle of $\theta = 45.0°$ with the horizontal? (2) Suppose that, after being in orbit for 21.3 hours, the CNSA decides to bring the probe into Io's orbit. It takes the probe 22.4 hours to reach Io's orbit at an angle of $\phi = 25.0°$ south of west. When the probe arrives at its new location, what angle does Io's position make with the vertical and at what distance is the probe ahead of or behind the moon Io? (3) At that moment, where is Europa located in its orbit with respect to both the vertical and the probe's position? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2) and that Jupiter's radius measures about $r_j = 7.15 \times 10^7$ m, and assume furthermore circular orbits.

Exercise 17

On a sunny Sunday afternoon, Micaela is practicing one of her favourite sport activities, i.e., clay target shooting, at the Club de Cazadores in Tucumán, Argentina. If the target ($m_t = 105$ g) leaves the shooting station, which is installed at $d = 45.5$ cm above the ground, with an initial speed of $v_0 = 23.6$ m/s under an angle of $\theta = 35.2°$ with the horizontal, while undergoing a drag force $\vec{F}_D = -b \cdot \vec{v}$ (with a drag coefficient of $b = 0.0068$ kg/s), at what distance h from the ground does the target find itself when it's at its highest point?

Exercise 18

Trans-Neptunian Objects (TNOs) are dwarf planets (or minor planets) in the outer Solar System whereby their average orbiting distance to the Sun ($M_s = 1.99 \times 10^{30}$ kg) is larger than that of Neptune, i.e., the outermost planet within our Solar System. Eris and Sedna are two TNOs following elliptical trajectories around the Sun, whereby the orbit of the dwarf planet Eris, which has an elliptical eccentricity equal to $e_E = 0.436$ and a semi-minor axis of length $b_E = 9.14 \times 10^{12}$ m, is the consequence of historically significant gravitational interactions with Neptune—it is therefore assigned to the sub-classification of "scattered-disk objects". In contrast, due to the much larger orbit of the dwarf planet Sedna, which is most likely the result of a collision with some planet-sized object or star, Sedna is only marginally experiencing Neptune's gravitational influence and therefore belongs, arguably, to the sub-classification of "detached objects". (1) If the eccentricity of Sedna's orbit is 1.95 times greater with respect to Eris and if the distance between one of the foci and the center of Sedna's orbit is 14.54 times larger compared to Eris, how do the orbital velocities of these two dwarf planets compare at their perihelion, i.e., the point on their elliptical orbit closest to the massive body around which they orbit? Use the vis-viva equation, i.e., $v^2 = G \cdot M_s \cdot \left(\frac{2}{r} - \frac{1}{a}\right)$, with G the universal gravitational constant ($G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2)) and a the semi-major axis of the ellipse, to calculate the velocities. (2) What are the periods of the dwarf planets Eris and Sedna? For this problem, put the origin of the respective coordinate system in the focus point to the right of the center of the ellipse and use polar coordinates.

Exercise 19

During this cold and snowy month of December in Erzurum, Turkey, Mehmet ($m_M = 72.5$ kg) has dressed up as Noel Baba to bring his little brother Omer some long-desired gifts. Mehmet wants to do it in style, so he takes his sled ($m_s = 5.50$ kg) and slides down the incline—which makes an angle of $\theta = 16.4°$ with the horizontal—behind their house while holding three gifts ($m_1 = 3.50$ kg, $m_2 = 2.50$ kg, and $m_3 = 1.50$ kg), all stacked on top of each other. (1) If the kinetic friction coefficient for the sled on snow is equal to $\mu_{k,s} = 0.0455$ and the static friction coefficient for paper on paper to $\mu_s = 0.545$, how will gift 2 and 3 behave relative to gift 1? (2) What is the minimum value that μ_s should have if the gifts have to remain steady? (3) Suppose that μ_s has a value of 95% of the minimum value established in part (2) and that the kinetic friction coefficient $\mu_{k,2}$ between gift 1 and 2 is equal to 75% of this minimum value and the coefficient $\mu_{k,3}$ between gift 2 and 3 to $\mu_{k,3} = \frac{\mu_{k,2}}{2}$. How do the gifts behave now?

Exercise 20

Amina is doing postdoctoral research at the Sultan Qaboos University, in Muscat, Oman, whereby she specializes in binary star systems, i.e., gravitationally bound systems in which two stars orbit around their common center of mass called the barycenter (x_{bc}). Amina is currently studying data from the Lepus constellation, which lies at a declination of 20° south of the celestial equator, and has identified a new binary star system of circular orbits. Star 1 has a mass of $m_1 = 1.45 \cdot M_z$, with M_z the mass of the star Zeta Leporis and equal to $M_z = 1.46 \cdot M_s$ (whereby the mass of the Sun measures $M_s = 1.99 \times 10^{30}$ kg), whereas the mass of star 2 is equal to $m_2 = 3.20 \cdot M_z$. Amina has furthermore calculated that the stars complete one orbit in exactly 166 days. (1) What distance did Amina measure between both stars? Express your answer in terms of the Earth-Sun distance $r_{es} = 1.496 \times 10^8$ km. (2) What value does Amina find for the orbital velocity of each star? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2).

Exercise 1

Problem Statement

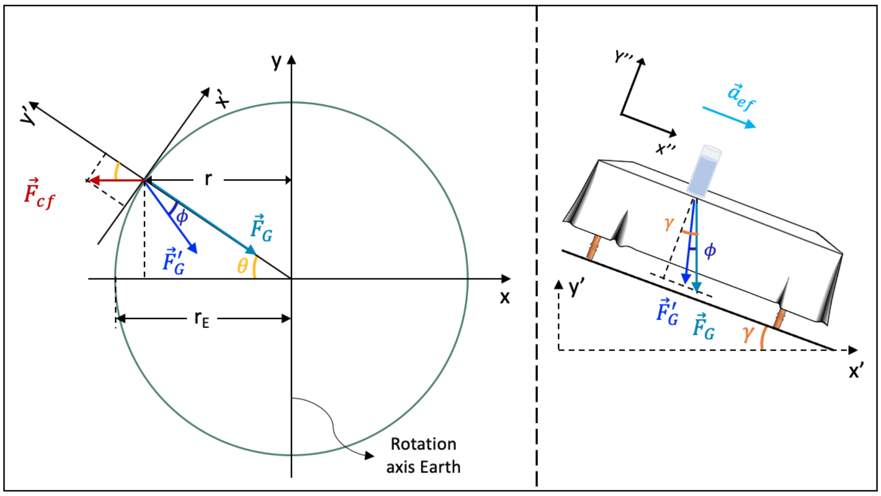

Figure 1

Suppose you live at the geographical coordinates 41°21′ 46.4″N 15°18′36.5″E, whereby your street makes an angle of $\gamma = 9.50°$ with the horizontal. Today, the outdoor temperature indicates 27.3° and so you planned a lunch outside with the neighbours in front of your house. You have just put a glass of water ($m_w = 0.450$ kg) on the table but forgot that the tablecloth is a smooth (i.e., frictionless) surface. As a result, the glass starts sliding towards the edge of the table. (1) What is the effective acceleration $\vec{a}_{ef}$ of the glass if you take into account the fact that you're living on a rotating planet, i.e., you are subject to a centrifugal force? (2) How does $\vec{a}_{ef}$ compare to the situation whereby you ignore the Earth's centripetal acceleration $\vec{a}_{cp}$? Remember that the Earth's radius is equal to $r_E = 6.38 \times 10^6$ m and assume that the center of the Earth is stable enough to be the origin of an inertial coordinate system (x,y).

Solution

(1) Since we view this problem from the perspective of an object, i.e., the glass of water, situated within a coordinate system (x',y') that itself undergoes rotational—and thus accelerating—motion with respect to an inertial framework (x,y), the reference frame (x',y') is a non-inertial framework. This means that in the coordinate system (x',y') Newton's laws do not hold. However, if we introduce pseudo-forces, we can nevertheless apply his laws. The rotational motion implies that a centripetal acceleration $\vec{a}_{cp}$ is directed from the position of the glass of water perpendicular to the Earth's axis of rotation. Therefore, a pseudo-force $\vec{F}_{cf}$, i.e., the centrifugal force, can be introduced that points in the opposite direction of $\vec{a}_{cp}$:

$$\vec{F}_{cf} = m_w \cdot \vec{a}_{cf} = m_w \cdot (-\vec{a}_{cp}) = m_w \cdot (-\frac{v^2}{r}) \cdot \vec{i}_x$$

whereby v represents the orbital speed of the Earth and r the perpendicular (horizontal) distance between the position of the glass of water and the Earth's rotation axis.

Because of the centripetal acceleration $\vec{a}_{cp}$, the actual or effective gravitational force $\vec{F}'_G$, which is also called the plumb line, does not exactly match the gravitational force $\vec{F}_G$, which is directed radially towards the Earth's center, and deflects from it by a very small angle ϕ.

Based on these three forces, we can now write the x'- and y'-component of Newton's second law for the fixed position at the specific latitude of the glass of water with respect to the coordinate system (x',y'):

x'-component

$$-m_w \cdot \frac{v^2}{r} \cdot \sin\theta = m_w \cdot g'_{x'}$$

y'-component

$$-m_w \cdot g + m_w \cdot \frac{v^2}{r} \cdot \cos\theta = m_w \cdot g'_{y'}$$

Using the above equations of Newton's second law and given that the latitude 41°21'46.4"N corresponds to the angle $\theta = 41 + \frac{21}{60} + \frac{46.4}{3600} = 41.4°$, that $r = r_E \cdot \cos\theta$, and that the Earth's orbital speed v at that latitude equals $v = \frac{2\cdot\pi}{T} \cdot r = \frac{2\cdot\pi}{T} \cdot (r_E \cdot \cos\theta) = \frac{2\cdot\pi}{86{,}400} \cdot [6.38 \times 10^6 \cdot \cos(41.4°)] = 348$ m/s, we can calculate the magnitude of the effective gravitational acceleration $\vec{g}\,'$ as follows:

$$\begin{aligned}
g' &= \sqrt{(g'_{x'})^2 + (g'_{y'})^2} \\
&= \sqrt{\left(-\frac{v^2}{r} \cdot \sin\theta\right)^2 + \left(\frac{v^2}{r} \cdot \cos\theta - g\right)^2} \\
&= \sqrt{\left(\frac{v^2}{r}\right)^2 + g^2 - 2 \cdot g \cdot \frac{v^2}{r} \cdot \cos\theta} \\
&= \sqrt{\left(\frac{v^2}{r_E \cdot \cos\theta}\right)^2 + g^2 - 2 \cdot g \cdot \frac{v^2}{r_E}} \\
&= \sqrt{\left(\frac{348^2}{6.38 \times 10^6 \cdot \cos(41.4°)}\right)^2 + 9.81^2 - 2 \cdot 9.81 \cdot \frac{348^2}{6.38 \times 10^6}} \\
&= 9.79 \text{ m/s}^2
\end{aligned}$$

The angle ϕ between the vectors $\vec{g}\,'$ and $\vec{g}$ can then be found in the following manner:

$$\begin{aligned}
\phi &= \tan^{-1}\left(\frac{g'_{x'}}{g'_{y'}}\right) \\
&= \tan^{-1}\left[\frac{(-\frac{v^2}{r}) \cdot \sin\theta}{-g + \frac{v^2}{r} \cdot \cos\theta}\right]
\end{aligned}$$

$$= \tan^{-1}\left[\frac{-(\frac{v^2}{r_E})\cdot\tan\theta}{-(g-\frac{v^2}{r_E})}\right]$$

$$= \tan^{-1}\left[\frac{(\frac{348^2}{6.38\times10^6})\cdot\tan(41.4°)}{(9.81-\frac{348^2}{6.38\times10^6})}\right]$$

$$= 0.0979°$$

We can now address your glass of water on the table. Applying Newton's second law to the glass of water in the coordinate system (x",y") gives us the following magnitude of the effective acceleration $\vec{a}_{ef}$:

$$m_w\cdot a_{ef} = m_w\cdot g'\cdot\sin(\gamma-\phi) \quad\Leftrightarrow\quad a_{ef} = g'\cdot\sin(\gamma-\phi)$$

$$= 9.79\cdot\sin(9.50°-0.0979°)$$

$$= 1.60\ \text{m/s}^2$$

(2) If we approach the problem in (1) using the standard gravitational force $\vec{F}_G$ and ignoring the Earth's rotational motion, then Newton's second law gives us the following magnitude of the acceleration $\vec{a}_w$ of the glass of water:

$$m_w\cdot a_w = m_w\cdot g\cdot\sin\gamma \quad\Leftrightarrow\quad a_w = g\cdot\sin\gamma$$

$$= 9.81\cdot\sin(9.50°)$$

$$= 1.62\ \text{m/s}^2$$

Note that there is another pseudo-force acting on your glass of water due to the fact that the glass is *moving* in a rotating frame of reference. This pseudo-force is called the Coriolis force, which we have ignored in the above problem and is dealt with in another exercise package called "Rotational Motion".

Exercise 2

Problem Statement

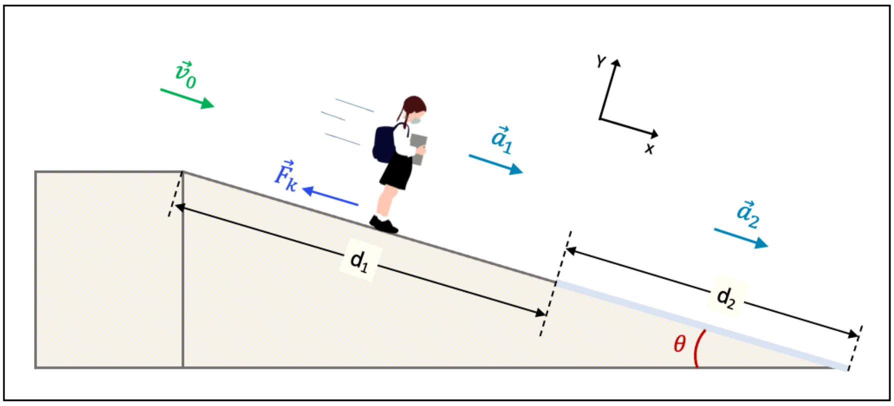

Figure 2

In Nagano, Japan, Yuuto's fourteen-year-old daughter Koharu (m_K) is coming home from school and as usual, before going inside to do her homework, she runs towards the marble incline, which makes an angle θ with the horizontal, right beside the entry porch and slides down on her feet to see how far she gets without falling. Today, it has been freezing and while the upper part of the incline (with a length of d_1) is slightly wet, the lower part of the incline is covered with a thin layer of ice for a distance of d_2.

(1) If the coefficient of kinetic friction for rubber on wet marble and ice is represented by μ_{k1} and μ_{k2}, respectively, formulate an expression for the final velocity v_{f2} of Koharu at the bottom of the incline in terms of the initial velocity v_0, the angle θ, the gravitational constant g, the coefficients of friction μ_{k1} and μ_{k2}, and the distances d_1 and d_2. (2) Formulate an expression for the length L of the incline in terms of the velocities v_0, v_{f1}, and v_{f2}, the angle θ, and the coefficients for kinetic friction μ_{k1} and μ_{k2}, whereby v_{f1} is the velocity at the end of distance d_1. (3) If $d_1 = 3.50$ m, $\mu_{k1} = 0.622$, $\theta = 28.5°$, $v_0 = 3.50$ m/s, $v_{f2} = 6.70$ m/s, and the time Koharu spends on the ice $t_2 = 0.933$ s, what is the value of the coefficient of kinetic friction for rubber on ice μ_{k2}? (4) Using the formula derived in part (2) and the provided data in part (3), what is the length L of the incline?

Solution

(1) Since Koharu's acceleration depends upon the type of surface on which she slides, we apply Newton's second law for both cases of marble and ice, whereby the magnitude of the kinetic friction force equals $F_k = \mu_k \cdot F_N$ (with $\vec{F}_N$ representing the normal force):

	Marble		Ice
$x:$	$m_K \cdot a_1 = -\mu_{k1} \cdot F_N + m_k \cdot g \cdot \sin\theta$	$x:$	$m_K \cdot a_2 = -\mu_{k2} \cdot F_N + m_k \cdot g \cdot \sin\theta$
$y:$	$0 = F_N - m_k \cdot g \cdot \cos\theta$	$y:$	$0 = F_N - m_k \cdot g \cdot \cos\theta$

Inserting the expression for F_N obtained from the y-direction into the equation of the x-direction gives the following two expressions for the acceleration:

Marble | Ice

$$a_1 = g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta) \qquad a_2 = g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)$$

In a next step, we write the equation of motion for each surface type:

Marble | Ice

$$v_{f1}^2 - v_0^2 = 2 \cdot a_1 \cdot d_1 \qquad v_{f2}^2 - v_{f1}^2 = 2 \cdot a_2 \cdot d_2$$

Replacing v_{f1}^2 in the equation for ice by the expression v_{f1}^2 obtained from the equation for marble, we find the following expression for Koharu's final velocity v_{f2}:

$$v_{f2}^2 - \left[v_0^2 + 2 \cdot a_1 \cdot d_1\right] = 2 \cdot a_2 \cdot d_2$$

$$\Leftrightarrow \; v_{f2} = \sqrt{v_0^2 + 2 \cdot a_1 \cdot d_1 + 2 \cdot a_2 \cdot d_2}$$

$$= \sqrt{v_0^2 + 2 \cdot [g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta)] \cdot d_1 + 2 \cdot [g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)] \cdot d_2}$$

$$= \sqrt{v_0^2 + 2 \cdot g \cdot [(d_1 + d_2) \cdot \sin\theta - (\mu_{k1} \cdot d_1 + \mu_{k2} \cdot d_2) \cdot \cos\theta]}$$

(2) Making use of the equations for the acceleration a_1 and a_2 obtained in section (1), we can write the following expression for the length L:

$$L = d_1 + d_2$$

$$= \frac{v_{f1}^2 - v_0^2}{2 \cdot a_1} + \frac{v_{f2}^2 - v_{f1}^2}{2 \cdot a_2}$$

$$= \frac{v_{f1}^2}{2} \cdot \left(\frac{1}{a_1} - \frac{1}{a_2}\right) - \frac{v_0^2}{2 \cdot a_1} + \frac{v_{f2}^2}{2 \cdot a_2}$$

$$= \frac{v_{f1}^2}{2} \cdot \left(\frac{a_2 - a_1}{a_1 \cdot a_2}\right) - \frac{v_0^2 \cdot a_2}{2 \cdot a_1 \cdot a_2} + \frac{v_{f2}^2 \cdot a_1}{2 \cdot a_1 \cdot a_2}$$

$$= \frac{v_{f1}^2 \cdot ([g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)] - [g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta)]) - v_0^2 \cdot [g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)] + v_{f2}^2 \cdot [g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta)]}{2 \cdot [g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta)] \cdot [g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)]}$$

$$= \frac{v_{f1}^2 \cdot (\mu_{k1} - \mu_{k2}) \cdot \cos\theta + (v_{f2}^2 - v_0^2) \cdot \sin\theta + (v_0^2 \cdot \mu_{k2} - v_{f2}^2 \cdot \mu_{k1}) \cos\theta}{2 \cdot g \cdot [\sin^2\theta + \mu_{k1} \cdot \mu_{k2} \cdot \cos^2\theta - (\mu_{k1} + \mu_{k2}) \cdot \sin\theta \cdot \cos\theta]}$$

$$= \frac{v_{f1}^2 \cdot (\mu_{k1} - \mu_{k2}) + (v_{f2}^2 - v_0^2) \cdot \tan\theta + (v_0^2 \cdot \mu_{k2} - v_{f2}^2 \cdot \mu_{k1})}{2 \cdot g \cdot \cos\theta \cdot [\tan^2\theta + \mu_{k1} \cdot \mu_{k2} - (\mu_{k1} + \mu_{k2}) \cdot \tan\theta]}$$

$$= \frac{(\mu_{k2} - \tan\theta) \cdot v_0^2 + (\mu_{k1} - \mu_{k2}) \cdot v_{f1}^2 + (\tan\theta - \mu_{k1}) \cdot v_{f2}^2}{2 \cdot g \cdot \cos\theta \cdot [\tan^2\theta + \mu_{k1} \cdot \mu_{k2} - (\mu_{k1} + \mu_{k2}) \cdot \tan\theta]}$$

(3) First, let us calculate the acceleration a_1 as well as the velocity v_{f1}:

$$a_1 = g \cdot (\sin\theta - \mu_{k1} \cdot \cos\theta)$$

$$= 9.81 \cdot [\sin(28.5°) - 0.622 \cdot \cos(28.5°)]$$

$$= -0.681 \text{ m/s}^2$$

$$v_{f1} = \sqrt{v_0^2 + 2 \cdot a_1 \cdot d_1}$$

$$= \sqrt{3.50^2 + 2 \cdot (-0.681) \cdot 3.50}$$

$$= 2.73 \text{ m/s}$$

To find the value of the coefficient of kinetic friction μ_{k2} for rubber on ice, we consider the following equation of motion:

$$v_{f2} = v_{f1} + a_2 \cdot t_2 = v_{f1} + [g \cdot (\sin\theta - \mu_{k2} \cdot \cos\theta)] \cdot t_2$$

$$\Leftrightarrow \; \mu_{k2} = \tan\theta - \frac{v_{f2} - v_{f1}}{g \cdot t_2 \cdot \cos\theta} = \tan(28.5°) - \frac{(6.70 - 2.73)}{9.81 \cdot 0.933 \cdot \cos(28.5°)} = 0.0500$$

(4) The length L of the incline is calculated as follows:

$$L = \frac{(\mu_{k2} - \tan\theta) \cdot v_0^2 + (\mu_{k1} - \mu_{k2}) \cdot v_{f1}^2 + (\tan\theta - \mu_{k1}) \cdot v_{f2}^2}{2 \cdot g \cdot \cos\theta \cdot [\tan^2\theta + \mu_{k1} \cdot \mu_{k2} - (\mu_{k1} + \mu_{k2}) \cdot \tan\theta]}$$

$$= \frac{[0.0500 - \tan(28.5°)] \cdot 3.50^2 + [0.622 - 0.0500] \cdot 2.73^2 + [\tan(28.5°) - 0.622] \cdot 6.70^2}{2 \cdot 9.81 \cdot \cos(28.5°) \cdot [\tan^2(28.5°) + 0.622 \cdot 0.0500 - (0.622 + 0.0500) \cdot \tan(28.5°)]}$$

$$= 7.90 \text{ m}$$

Exercise 3

Problem Statement

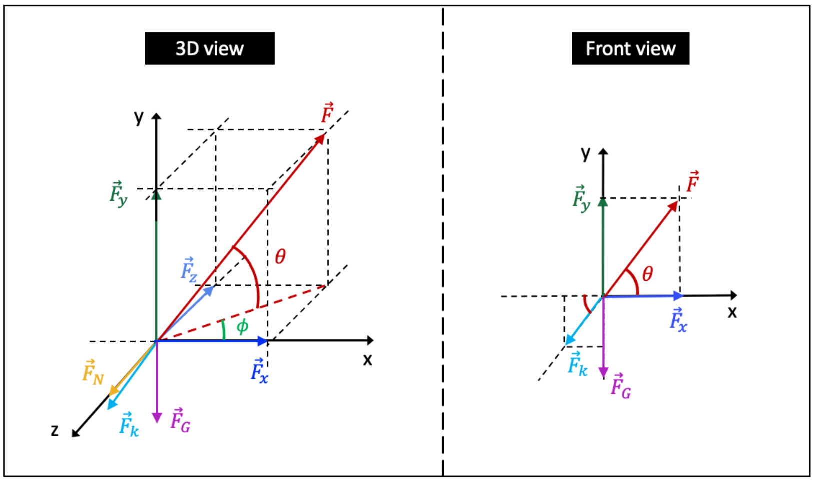

Figure 3

Harry just got back home from buying a gift (m_b = 4.85 kg) for his husband Leo in the Meadowhall Shopping Centre in Sheffield, United Kingdom. When Harry is in the midst of gift wrapping his present on his knees in the living room, Leo unexpectedly arrives home. Harry rushes to hide the gift between his back and the wall, and he casually gets up on his feet in $t = 1.90$ s while pushing on the gift with his back, in the hope that it stays in place so that Leo is not on to him. If the force $\vec{F}$ with which Harry pushes against his gift makes an angle of $\theta = 64.1°$ with the xz-plane in the positive y-direction and $\phi = 69.4°$ with the xy-plane in the negative z-direction (whereby the y-axis is directed upwards and the z-axis points out of the wall), (1) what is the distance d that the gift has traveled across the wall when Harry is standing upright? (2) What is the magnitude of the force $\vec{F}$? Assume that the origin of the coordinate system sits at the centre of the gift (at its initial position on the wall), that the coefficient of kinetic friction of wrapping paper on the wall equals $\mu_k = 0.730$, and that the net acceleration $\vec{a}$ and the gift's trajectory run parallel with the projection of $\vec{F}$ onto the xy-plane.

Solution

(1) Since we assume that the net acceleration $\vec{a}$ as well as the gift's trajectory run parallel with the projection of $\vec{F}$ onto the xy-plane, we know that both the vector $\vec{a}$ and the kinetic friction force $\vec{F}_k$ make an angle θ with the x-axis. Let us now in a first instance write in detail the different components of the force $\vec{F}$ and the friction force $\vec{F}_k$:

Force $\vec{F}$	Friction force $\vec{F}_k$
$\vec{F}_x = (F \cdot \cos\theta \cdot \cos\phi) \cdot \vec{i}_x$	$\vec{F}_{kx} = (-F_k \cdot \cos\theta) \cdot \vec{i}_x$
$\vec{F}_y = (F \cdot \sin\theta) \cdot \vec{i}_y$	$\vec{F}_{ky} = (-F_k \cdot \sin\theta) \cdot \vec{i}_y$
$\vec{F}_z = (-F \cdot \cos\theta \cdot \sin\phi) \cdot \vec{i}_z$	$\vec{F}_{kz} = 0 \cdot \vec{i}_z$

Applying Newton's second law to the gift for each of the three dimensions x, y, and z and given that $F_k = \mu_k \cdot F_N$, we obtain the following three equations:

x-dimension	y-dimension	z-dimension
$F_x - F_{kx} = m_b \cdot a_x$	$F_y - F_{ky} - m_b \cdot g = m_b \cdot a_y$	$-F_z + F_N = 0$

The z-dimension tells us that $F_N = F_z = F \cdot \cos\theta \cdot \sin\phi$, from which follows that $F_k = \mu_k \cdot F_N = \mu_k \cdot F \cdot \cos\theta \cdot \sin\phi$. As a result, the equations related to the x- and y-dimension become, respectively:

$$\begin{cases} F_x - F_{kx} = m_b \cdot a_x \\ \Leftrightarrow (F \cdot \cos\theta \cdot \cos\phi) - [(\mu_k \cdot F \cdot \cos\theta \cdot \sin\phi) \cdot \cos\theta] = m_b \cdot (a \cdot \cos\theta) \\ \Leftrightarrow F \cdot (\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi) = m_b \cdot a \\ \\ F_y - F_{ky} - m_b \cdot g = m_b \cdot a_y \\ \Leftrightarrow (F \cdot \sin\theta) - [(\mu_k \cdot F \cdot \cos\theta \cdot \sin\phi) \cdot \sin\theta] - m_b \cdot g = m_b \cdot (a \cdot \sin\theta) \\ \Leftrightarrow F \cdot (\sin\theta - \mu_k \cdot \cos\theta \cdot \sin\phi \cdot \sin\theta) - m_b \cdot g = m_b \cdot (a \cdot \sin\theta) \end{cases}$$

Replacing F in the second equation (y-dimension) by the expression for F obtained from the first equation (x-dimension), we find an expression for the acceleration of the gift:

$$\left[\frac{m_b \cdot a}{(\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi)}\right] \cdot (\sin\theta - \mu_k \cdot \cos\theta \cdot \sin\phi \cdot \sin\theta) - m_b \cdot g = m_b \cdot (a \cdot \sin\theta)$$

$$\Leftrightarrow \frac{a \cdot (\sin\theta - \mu_k \cdot \cos\theta \cdot \sin\phi \cdot \sin\theta)}{(\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi)} - a \cdot \sin\theta = g$$

$$\Leftrightarrow a \cdot \left[\frac{\sin\theta - \mu_k \cdot \cos\theta \cdot \sin\phi \cdot \sin\theta - \sin\theta \cdot \cos\phi + \mu_k \cdot \cos\theta \cdot \sin\phi \cdot \sin\theta}{\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi}\right] = g$$

$$\Leftrightarrow a \cdot \left[\frac{\sin\theta \cdot (1 - \cos\phi)}{\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi}\right] = g$$

$$\Leftrightarrow a = \frac{g \cdot (\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi)}{\sin\theta \cdot (1 - \cos\phi)}$$

$$= \frac{9.81 \cdot [\cos(69.4°) - 0.730 \cdot \cos(64.1°) \cdot \sin(69.4°)]}{\sin(64.1°) \cdot [1 - \cos(69.4°)]}$$

$$= 0.898 \text{ m/s}^2$$

The distance that the gift has traveled in the x- and y-direction, respectively, is equal to:

$$\begin{cases} x = \dfrac{a_x}{2} \cdot t^2 = \dfrac{a \cdot \cos\theta}{2} \cdot t^2 = \dfrac{0.898 \cdot \cos(64.1°)}{2} \cdot 1.90^2 = 0.708 \text{ m} \\ y = \dfrac{a_y}{2} \cdot t^2 = \dfrac{a \cdot \sin\theta}{2} \cdot t^2 = \dfrac{0.898 \cdot \sin(64.1°)}{2} \cdot 1.90^2 = 1.46 \text{ m} \end{cases}$$

Therefore, the total distance d covered by the gift is found to be:

$$d = \sqrt{x^2 + y^2} = \sqrt{0.708^2 + 1.46^2} = 1.62 \text{ m}$$

(2) The magnitude of the force $\vec{F}$ can be calculated by considering the equation of Newton's second law with respect to, for instance, the x-dimension as determined in part (1):

$$F \cdot (\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi) = m_b \cdot a$$

$$\Leftrightarrow F = \frac{m_b \cdot a}{(\cos\phi - \mu_k \cdot \cos\theta \cdot \sin\phi)}$$

$$= \frac{4.85 \cdot 0.898}{[\cos(69.4°) - 0.730 \cdot \cos(64.1°) \cdot \sin(69.4°)]}$$

$$= 81.6 \text{ N}$$

Exercise 4

Problem Statement

Near an amusement park in the centre of al-Hilla in Iraq, Bibi ($m_B = 58$ kg) is practicing her kickflip and 360 spin backside with her brand new Core C2 skateboard ($m_s = 3.1$ kg). Bibi notices a semi-circle steel rail with a height of $h = 0.45$ m and a radius of $r = 7.5$ m and starts building up speed so that she can grind the rail as far as possible. By the time she jumps onto the rail, Bibi has reached a speed of $v_0 = 6.5$ m/s. (1) Under what angle with the horizontal should Bibi hit the rail with her skateboard deck? (2) As Bibi slides along the rail, she manages to keep her balance until she comes to a halt. At what angle is she now positioning her skateboard to avoid falling from the rail? (3) How far did Bibi slide (work with average values)? Assume that the coefficient of kinetic and static friction for wood on steel is equal to $\mu_k = 0.29$ and $\mu_s = 0.42$, respectively.

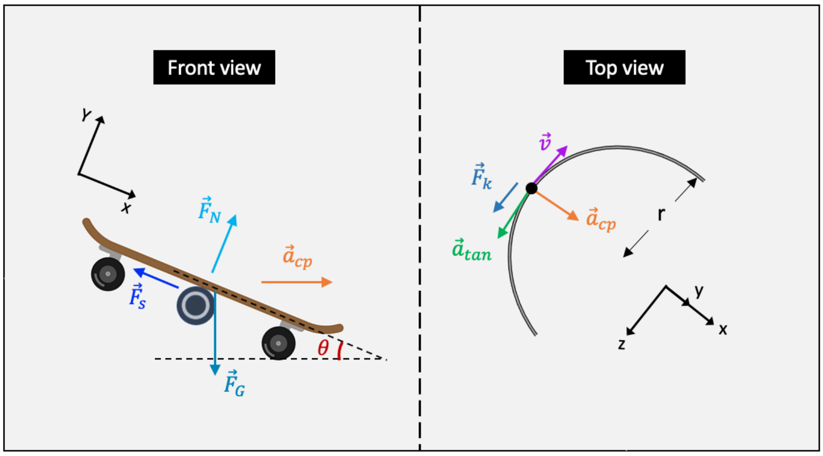

Figure 4

Solution

(1) Let us start with applying Newton's second law to the system "Bibi plus skateboard deck" for the x-and y-direction, whereby $m_{tot} = m_B + m_s$, $\vec{F}_k$ ($\vec{F}_s$) the force related to kinetic (static) friction, $\vec{F}_N$ the normal force, and $\vec{a}_{cp}$ the centripetal acceleration:

x-direction	y-direction
$-F_s + m_{tot} \cdot g \cdot \sin\theta = m_{tot} \cdot a_{cp} \cdot \cos\theta$	$F_N - m_{tot} \cdot g \cdot \cos\theta = m_{tot} \cdot a_{cp} \cdot \sin\theta$

Given that $F_s = \mu_s \cdot F_N$ and inserting the expression for F_N obtained from the y-direction into the equation of the x-direction, we can calculate the appropriate angle θ when hitting the rail:

$$-(\mu_s \cdot F_N) + m_{tot} \cdot g \cdot \sin\theta = m_{tot} \cdot a_{cp} \cdot \cos\theta$$

$$\Leftrightarrow \; -(\mu_s \cdot [m_{tot} \cdot g \cdot \cos\theta + m_{tot} \cdot a_{cp} \cdot \sin\theta]) + m_{tot} \cdot g \cdot \sin\theta = m_{tot} \cdot a_{cp} \cdot \cos\theta$$

$$\Leftrightarrow \; -(\mu_s \cdot [g + a_{cp} \cdot \tan\theta]) + g \cdot \tan\theta = a_{cp}$$

$$\Leftrightarrow \tan\theta = \frac{(a_{cp} + \mu_s \cdot g)}{(g - \mu_s \cdot a_{cp})}$$

$$\Leftrightarrow \theta = \tan^{-1}\left[\frac{(a_{cp} + \mu_s \cdot g)}{(g - \mu_s \cdot a_{cp})}\right]$$

$$= \tan^{-1}\left[\frac{(\frac{v_0^2}{r} + \mu_s \cdot g)}{(g - \mu_s \cdot \frac{v_0^2}{r})}\right] = \tan^{-1}\left[\frac{(\frac{6.5^2}{7.5} + 0.42 \cdot 9.81)}{(9.81 - 0.42 \cdot \frac{6.5^2}{7.5})}\right] = 53°$$

(2) When Bibi eventually comes to a stop after skillfully handling her skateboard deck, her speed v will be equal to zero along with the centripetal acceleration a_{cp}. Under these conditions, the angle θ_s becomes:

$$\theta_s = \tan^{-1}\left[\frac{(a_{cp} + \mu_s \cdot g)}{(g - \mu_s \cdot a_{cp})}\right] = \tan^{-1}\left[\frac{(0 + \mu_s \cdot g)}{(g - \mu_s \cdot 0)}\right] = \tan^{-1}(\ \mu_s\ ) = \tan^{-1}(\ 0.42\ ) = 23°$$

(3) As she slides along the rail, Bibi experiences kinetic friction $\vec{F}_k$ (directed towards the positive z-direction) so that the net acceleration, i.e., the tangential acceleration $\vec{a}_{tan}$, is pointing in the opposite direction of Bibi's motion. This causes the magnitude of Bibi's velocity $\vec{v}$, which points in the negative z-direction, to decrease continuously.

This means, in turn, that the magnitude of the centripetal acceleration $\vec{a}_{cp}$ as well as the angle θ and, consequently, the magnitude of the normal force $\vec{F}_N$ constantly change, i.e., their value declines. Therefore, to find the distance that Bibi slid on the rail, we consider average values. For an average speed of $v_{av} = \frac{6.5+0.0}{2} = 3.3$ m/s, the corresponding average angle is equal to $\theta_{av} = 31°$ (by using the expression obtained in part (1)). We can now find the average tangential acceleration a_{tan} by applying Newton's second law in the z-direction:

$$F_k = m_{tot} \cdot a_{tan}$$

$$\Leftrightarrow \mu_k \cdot \left[m_{tot} \cdot g \cdot \cos\theta_{av} + m_{tot} \cdot \frac{v_{av}^2}{r} \cdot \sin\theta_{av}\right] = m_{tot} \cdot a_{tan}$$

$$\Leftrightarrow a_{tan} = \mu_k \cdot \left[g \cdot \cos\theta_{av} + \frac{v_{av}^2}{r} \cdot \sin\theta_{av}\right] = 0.29 \cdot \left[9.81 \cdot \cos(31°) + \frac{3.3^2}{7.5} \cdot \sin(31°)\right] = 2.6\text{m/s}^2$$

The corresponding displacement Δz by Bibi on the rail is then equal to:

$$v^2 - v_0^2 = 2 \cdot a_{tan} \cdot \Delta z$$

$$\Leftrightarrow \Delta z = \frac{v^2 - v_0^2}{2 \cdot a_{tan}} = \frac{0 - 6.5^2}{2 \cdot 2.6} = -8.0 \text{ m}$$

Exercise 5

Problem Statement

A rock with a mass of $m_r = 20.0$ ton is hurtling through space in the vertical direction and when it is $r = 1,250$ km away from the surface of the Earth at an angle of $\theta = 59.5°$ with the vertical, the rock possesses a speed of $v_0 = 2,500$ m/s and is accelerating at $a_0 = 275$ m/s^2. Will it hit the Earth? As an approximate criterion for the condition of "hitting the Earth", consider the critical distance x_c. Given that the gravitational force $\vec{F}_G$ dynamically changes as the rock approaches Earth, use the average between $\vec{F}_G$'s magnitude at the rock's current position and that at the surface of the Earth. Assume furthermore that the Earth and the rock are moving in the xy-plane. Finally, remember that the radius of the Earth is equal to $r_E = 6.38 \times 10^6$ m, the Earth's mass to $M = 5.98 \times 10^{24}$ kg, and the universal gravitational constant to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2).

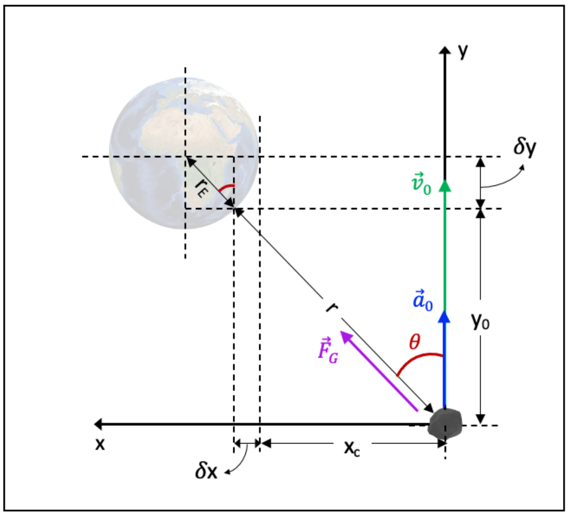

Figure 5

Solution

According to classical mechanics, as the rock approaches Earth, it is subjected to a growing gravitational force $\vec{F}_G = G \cdot \frac{m_r \cdot M}{(r+r_E)^2} \cdot \vec{i}_r = m_r \cdot \vec{a}_g$, since the distance between both objects becomes smaller. The result is that the rock is being pulled closer towards the Earth. Before applying Newton's second law to the rock, let us calculate the average magnitude a_{av} of the gravitational acceleration $\vec{a}_g$ between the current position and the surface of the Earth:

$$
\begin{aligned}
a_{av} &= \frac{1}{2} \cdot (a_{g,current} + a_{g,surface}) \\
&= \frac{1}{2} \cdot \left[\frac{G \cdot M}{(r + r_E)^2} + \frac{G \cdot M}{r_E^2} \right] \\
&= \frac{G \cdot M}{2} \cdot \left[\frac{1}{(r + r_E)^2} + \frac{1}{r_E^2} \right] \\
&= \frac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24}}{2} \cdot \left[\frac{1}{(1.25 \times 10^6 + 6.38 \times 10^6)^2} + \frac{1}{(6.38 \times 10^6)^2} \right] \\
&= 8.33 \text{ m/s}^2
\end{aligned}
$$

As per Newton's second law, we write the following equations for the rock:

x-direction	y-direction
$m_r \cdot a_x = F_G \cdot \sin\theta$	$m_r \cdot a_y = F_G \cdot \cos\theta + m_r \cdot a_0$
$= m_r \cdot a_{av} \cdot \sin\theta$	$= m_r \cdot a_{av} \cdot \cos\theta + m_r \cdot a_0$
$\Leftrightarrow \quad a_x = a_{av} \cdot \sin\theta$	$\Leftrightarrow \quad a_y = a_{av} \cdot \cos\theta + a_0$
$= 8.33 \cdot \sin(59.5°)$	$= 8.33 \cdot \cos(59.5°) + 275$
$= 7.17 \text{ m/s}^2$	$= 279 \text{ m/s}^2$

In a next step, we wish to find out how long it takes the rock to travel the distance $y_0 + \delta y$. This distance is equal to:

$$y_0 + \delta y = (r + r_E) \cdot \cos\theta = (1.25 \times 10^6 + 6.38 \times 10^6) \cdot \cos(59.5°) = 3.87 \times 10^6 \text{ m}$$

The time it takes to travel the distance $y_0 + \delta y$ is calculated by solving the below equation of motion for the y-direction:

$$y_0 + \delta y = v_0 \cdot t + \frac{a_y}{2} \cdot t^2 \quad \Leftrightarrow \quad 3.87 \times 10^6 = 2.50 \times 10^3 \cdot t + \frac{279}{2} \cdot t^2$$

The physically sensible ($t \geq 0$) solution is equal to $t = 158$ s. This allows us to determine the horizontal distance that the rock has covered during this time:

$$x = \frac{a \cdot \sin\phi}{2} \cdot t^2 = \frac{279 \cdot \sin(1.47°)}{2} \cdot 158^2 = 89,348 \text{ m}$$

In order to know whether the rock has hit the Earth, we first must calculate the critical distance x_c:

$$x_c = r \cdot \sin\theta - \delta x = r \cdot \sin\theta - r_E \cdot (1 - \sin\theta)$$

$$= 1.25 \times 10^6 \cdot \sin(59.5°) - 6.38 \times 10^6 \cdot [1 - \sin(59.5°)]$$

$$= 194,230 \text{ m}$$

In other words, at the height of $y_0 + \delta y$, the rock passes by the Earth at a distance of $d = x_c - x = 194,230 - 89,348 = 104,883$ m or 105 km from the surface. The rock does not hit the Earth, but given that the Earth's atmosphere begins around the same altitude, there is a possibility that the rock will be slowed down and burnt by colliding with atmospheric particles.

In our problem, we took the critical distance x_c as an approximate criterion for the condition "hitting the Earth". If we wish to be more accurate, we would need to write an expression for the distance that the rock passes by the Earth in function of time, calculate its derivative, equate it to zero and solve it with respect to time. The expression should be:

$$d = \sqrt{\left(r_E + x_c - \frac{a \cdot \sin\phi}{2} \cdot t^2\right)^2 + \left((r + r_E) \cdot \cos\theta - \left[v_0 \cdot t + \frac{a \cdot cos\phi}{2} \cdot t^2\right]\right)^2} - r_E$$

After some calculations, we would find that after a time of $t = 161.186$ s, the rock reaches the minimal distance of $d_{min} = 102,965$ m, which is not too far off from our solution, i.e., 104,883 m.

Exercise 6

Problem Statement

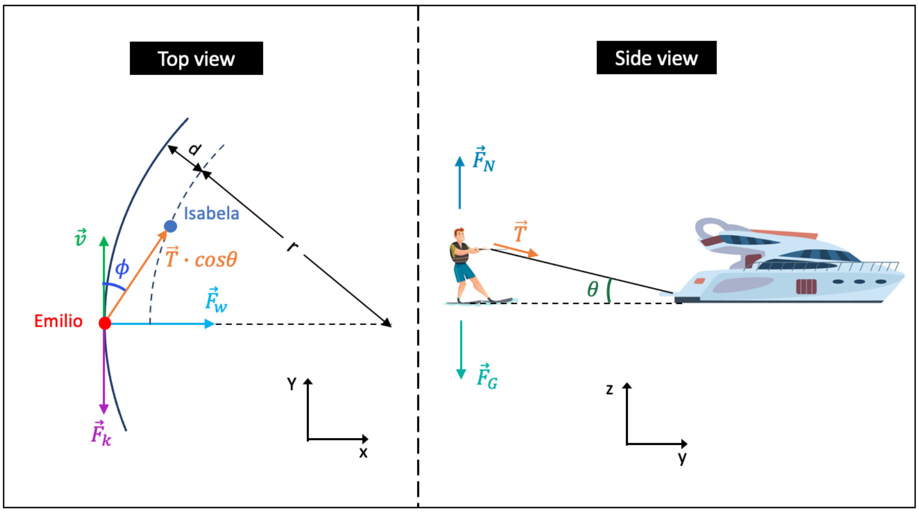

Figure 6

Emilio ($m_E = 72.9$ kg) just drove an hour from his home in Viseu, Portugal, to go water skiing in the Atlantic Ocean along the coast of Aveiro. His friend Isabela agreed to take him onto the water with her boat. At one point, Isabela takes a turn with a radius of $r = 125$ m at a constant speed. Meanwhile, Emilio is firmly holding the tow rope, which makes an angle of $\theta = 13.7°$ with the horizontal, and in the curve, he is following a path (without skidding) that lies radially $d = 5.10$ m more outwards compared to Isabela's position, producing an angle ϕ between the tow rope and his velocity vector $\vec{v} = 18.4 \cdot \vec{i}_y$ m/s. The water that Emilio pushes away sideways in the curve exerts a force of $\vec{F}_w = 150 \cdot \vec{i}_x$ N upon him, and he is also experiencing a kinetic friction force $\vec{F}_k$ opposite to his direction of motion—assume a kinetic friction coefficient of skis on water of $\mu_k = 0.175$. What is the value of the angle ϕ?

Solution

We start with applying Newton's second law to Emilio in the three spatial dimensions, whereby keeping in mind that $F_k = \mu_k \cdot F_N$ and that $\vec{T}$ represents the tension force in the tow rope:

<u>Emilio</u>

$$x: \quad (T \cdot \cos\theta) \cdot \sin\phi + F_w = \frac{m_E \cdot v^2}{r + d}$$

$$y: \quad (T \cdot \cos\theta) \cdot \cos\phi - (\mu_k \cdot F_N) = 0$$

$$z: \quad F_N - m_E \cdot g - T \cdot \sin\theta = 0$$

Plugging the expression for F_N obtained from the equation in the z-direction into that of the y-direction, we can write the following equation for the magnitude of the tension force $\vec{T}$:

$$(T \cdot \cos\theta) \cdot \cos\phi - \mu_k \cdot [m_E \cdot g + T \cdot \sin\theta] = 0$$

$$\Leftrightarrow \; T = \frac{\mu_k \cdot m_E \cdot g}{\cos\theta \cdot \cos\phi - \mu_k \cdot \sin\theta}$$

We now insert the above expression for T into the equation of the x-direction:

$$\left[\frac{\mu_k \cdot m_E \cdot g}{\cos\theta \cdot \cos\phi - \mu_k \cdot \sin\theta}\right] \cdot \cos\theta \cdot \sin\phi + F_w = \frac{m_E \cdot v^2}{r + d}$$

$$\Leftrightarrow \; \left(\frac{m_E \cdot v^2}{r + d} - F_w\right) \cdot \cos\phi - (\mu_k \cdot m_E \cdot g) \cdot \sin\phi = \mu_k \cdot \tan\theta \cdot \left(\frac{m_E \cdot v^2}{r + d} - F_w\right)$$

Remember that the linear combination of a cosine and a sine function, i.e., "$a \cdot \cos\gamma + b \cdot \sin\gamma$", can be replaced by a single cosine function "$c \cdot \cos(\gamma + \delta)$", whereby $c = sgn(a)\sqrt{a^2 + b^2}$ and $\delta = \tan^{-1}(-\frac{b}{a})$. If we apply this to our above expression, we obtain:

$$\sqrt{\left(\frac{m_E \cdot v^2}{r + d} - F_w\right)^2 + (-\mu_k \cdot m_E \cdot g)^2} \cdot \cos\left[\phi + \tan^{-1}\left(\frac{\mu_k \cdot m_E \cdot g}{\frac{m_E \cdot v^2}{r+d} - F_w}\right)\right] = \mu_k \cdot \tan\theta \cdot \left(\frac{m_E \cdot v^2}{r + d} - F_w\right)$$

$$\Leftrightarrow \phi + \tan^{-1}\left(\frac{\mu_k \cdot m_E \cdot g}{\frac{m_E \cdot v^2}{r+d} - F_w}\right) = \cos^{-1}\left[\frac{\mu_k \cdot \tan\theta \cdot \left(\frac{m_E \cdot v^2}{r+d} - F_w\right)}{\sqrt{\left(\frac{m_E \cdot v^2}{r+d} - F_w\right)^2 + (\mu_k \cdot m_E \cdot g)^2}}\right]$$

Let us first calculate the angle $\delta = \tan^{-1}(-\frac{b}{a})$ at the left-hand side of the above equation:

$$\tan^{-1}\left(\frac{\mu_k \cdot m_E \cdot g}{\frac{m_E \cdot v^2}{r+d} - F_w}\right) = \tan^{-1}\left(\frac{0.175 \cdot 72.9 \cdot 9.81}{\frac{72.9 \cdot 18.4^2}{125+5.10} - 150}\right) = 72.4°$$

The right-hand side of our equation is equal to:

$$\cos^{-1}\left[\frac{\mu_k \cdot \tan\theta \cdot \left(\frac{m_E \cdot v^2}{r+d} - F_w\right)}{\sqrt{\left(\frac{m_E \cdot v^2}{r+d} - F_w\right)^2 + (\mu_k \cdot m_E \cdot g)^2}}\right] = \cos^{-1}\left[\frac{0.175 \cdot \tan(13.7°) \cdot \left(\frac{72.9 \cdot 18.4^2}{125+5.10} - 150\right)}{\sqrt{\left(\frac{72.9 \cdot 18.4^2}{125+5.10} - 150\right)^2 + (0.175 \cdot 72.9 \cdot 9.81)^2}}\right]$$

$$= 89.3°$$

Inserting the above two values into our equation, we obtain the value of the angle ϕ:

$$\phi + 72.4° = 89.3° \quad \Leftrightarrow \quad \phi = 16.9°$$

Exercise 7

Problem Statement

Feeling re-energized after a weekend of hiking close to the Bavarian Sea, Grgur is analyzing astronomical data with a pair of fresh eyes on Monday morning in the ESO's headquarters in Garching, Germany. Apparently, the mental and physical recharging over the weekend have payed off as Grgur identifies a pattern between two newly discovered objects after one hour of work. The first object, which he named MS-X52R, is orbiting the planet Mars (in a circular orbit), while the second object, called SN-Y22T, circles the planet Saturn. By the time SN-Y22T has completed 1 revolution around Saturn, MS-X52R has already orbited Mars 7.43 times. Moreover, Grgur also found that the orbital height of SN-Y22T above the planet's surface relative to that of MS-X52R is ten times their respective relative velocity. Grgur is interested in calculating these orbital heights. What values does he find? Remember that the radius and mass of Mars and Saturn are equal to $r_M = 3.39 \times 10^6$ m and $r_S = 60.3 \times 10^6$ m and $M_M = 6.42 \times 10^{23}$ kg and $M_S = 5.69 \times 10^{26}$ kg, respectively.

Solution

For our system "object MS-X52R orbiting planet Mars", the only force acting on that object is the gravitational force $\vec{F}_G = G \cdot \frac{m_o \cdot M_M}{(h_M + r_M)^2} \cdot \vec{i}_r$ in the radial direction (with m_o the mass of the object and h_M the height above the surface). Therefore, when applying Newton's second law to object MS-X52R and keeping in mind that its orbital speed equals $v_M = \frac{2\pi \cdot (h_M + r_M)}{T}$, we obtain the following expression:

$$F_G = m_o \cdot a_r$$

$$\Leftrightarrow G \cdot \frac{m_o \cdot M_M}{(h_M + r_M)^2} = m_o \cdot \frac{v_M^2}{h_M + r_M}$$

$$\Leftrightarrow G \cdot \frac{M_M}{(h_M + r_M)^2} = \frac{[2 \cdot \pi \cdot (h_M + r_M)]^2}{T_M^2 \cdot (h_M + r_M)}$$

$$\Leftrightarrow T_M^2 = \frac{4 \cdot \pi^2}{G \cdot M_M} \cdot (h_M + r_M)^3$$

For the object SN-Y22T we find a similar expression. Based on the observed pattern that the period of the object MS-X52R is 7.43 times shorter compared to the object SN-Y22T ($T_S = 7.43 \cdot T_M$), we can write the following:

$$T_S^2 = 7.43^2 \cdot T_M^2$$

$$\Leftrightarrow \frac{4 \cdot \pi^2}{G \cdot M_S} \cdot (h_S + r_S)^3 = 7.43^2 \cdot \frac{4 \cdot \pi^2}{G \cdot M_M} \cdot (h_M + r_M)^3$$

$$\Leftrightarrow \frac{(h_M + r_M)}{(h_S + r_S)} = \sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}}$$

Based on the second pattern that Grgur observed and given the expression derived from the first pattern together with the equation for the orbital speed obtained from Newton's second law, we find the ratio between the heights h_S and h_M:

$$10 \cdot \frac{v_S}{v_M} = \frac{h_S}{h_M}$$

$$\Leftrightarrow 100 \cdot \frac{v_S^2}{v_M^2} = \frac{h_S^2}{h_M^2}$$

$$\Leftrightarrow 100 \cdot \frac{G \cdot M_S}{(h_S + r_S)} \cdot \frac{(h_M + r_M)}{G \cdot M_M} = \frac{h_S^2}{h_M^2}$$

$$\Leftrightarrow 100 \cdot \frac{M_S}{M_M} \cdot \frac{(h_M + r_M)}{(h_S + r_S)} = \frac{h_S^2}{h_M^2}$$

$$\Leftrightarrow 100 \cdot \frac{M_S}{M_M} \cdot \left[\sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}} \right] = \frac{h_S^2}{h_M^2}$$

$$\Leftrightarrow \frac{h_S}{h_M} = \sqrt{100 \cdot \frac{M_S}{M_M} \cdot \sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}}} = \sqrt{100 \cdot \frac{5.69 \times 10^{26}}{6.42 \times 10^{23}} \cdot \sqrt[3]{\frac{6.42 \times 10^{23}}{7.43^2 \cdot 5.69 \times 10^{26}}}} = 49.2$$

If we plug this ratio into the expression obtained from the first observed pattern, we can calculate the height above the surface h_M at which object MS-X52R orbits the planet Mars:

$$\frac{(h_M + r_M)}{(49.2 \cdot h_M + r_S)} = \sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}}$$

$$\Leftrightarrow h_M = \frac{\left[r_M - r_S \cdot \sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}} \right]}{\left[49.2 \cdot \sqrt[3]{\frac{M_M}{7.43^2 \cdot M_S}} - 1 \right]} = \frac{\left[3.39 \times 10^6 - 60.3 \times 10^6 \cdot \sqrt[3]{\frac{6.42 \times 10^{23}}{7.43^2 \cdot 5.69 \times 10^{26}}} \right]}{\left[49.2 \cdot \sqrt[3]{\frac{6.42 \times 10^{23}}{7.43^2 \cdot 5.69 \times 10^{26}}} - 1 \right]}$$

$$= 5.03 \times 10^3 \text{ km}$$

Finally, the orbital height h_S at which object SN-Y22T circles around the planet Saturn above its surface is equal to $h_S = 49.2 \cdot h_M = 49.2 \cdot 5.04 \times 10^3 = 2.48 \times 10^5$ km.

Exercise 8

Problem Statement

At this time of the year, the volcano Maat Mons on the planet Venus ($M_V = 4.87 \times 10^{24}$ kg) is highly active. Within the atmospheric region close to the planet's surface, where the 8 km-high volcano resides, the air density is immense at a value of approximately $\rho = 67.0$ kg/m^3. At a certain point, a large basaltic rock ($m_r = 1,250$ kg) is being ejected from Maat Mons. When it reaches the highest point in its trajectory it collides with another rock, thereby effectively eliminating any horizontal motion, so that the rock now starts falling vertically. If you know that the drag force $\vec{F}_D$ has the form of $\vec{F}_D = -b \cdot v^2 \cdot \vec{i}_y$ N (with $b = \frac{1}{2} \cdot c_D \cdot \rho \cdot \pi \cdot r^2$, whereby the drag coefficient equals $c_D = 0.635$), that the radius of Venus measures $r_V = 6.05 \times 10^6$ m, and that the basaltic rock has a diameter of about $d = 92.6$ cm, (1) what is the magnitude of the terminal velocity $\vec{v}_T$? (2) Write an expression for the magnitude of the rock's velocity in terms of the time variable t. Assume that $t = 0$ s when the rock is at the highest point of its trajectory. (3) When does the rock reach 95.5% of its terminal velocity $\vec{v}_T$?

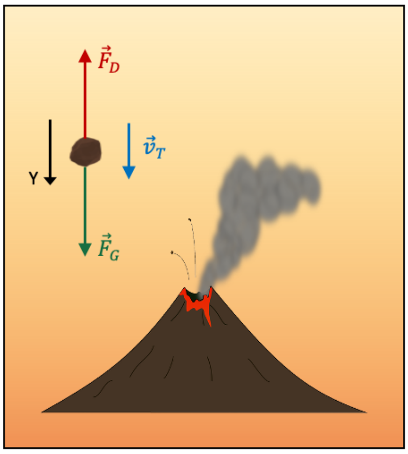

Figure 7

Solution

(1) In a first instance, we want to calculate the magnitude of the gravitational acceleration $\vec{g}$ for the planet Venus:

$$g = \frac{G \cdot M_V}{r_V^2} = \frac{6.67 \times 10^{-11} \cdot 4.87 \times 10^{24}}{(6.05 \times 10^6)^2} = 8.87 \text{ m/s}^2$$

In order to calculate the terminal velocity v_T, we apply Newton's second law to the rock (in terms of the y-direction), keeping in mind that the net acceleration is equal to zero:

$$-F_D + F_G = 0$$

$$\Leftrightarrow -\frac{1}{2} \cdot c_D \cdot \rho \cdot \pi \cdot \left(\frac{d}{2}\right)^2 \cdot v_T^2 + m_r \cdot g = 0$$

$$\Leftrightarrow v_T = \sqrt{\frac{m_r \cdot g}{\frac{1}{2} \cdot c_D \cdot \rho \cdot \pi \cdot \left(\frac{d}{2}\right)^2}} = \sqrt{\frac{1,250 \cdot 8.87}{\frac{1}{2} \cdot 0.635 \cdot 67.0 \cdot \pi \cdot \left(\frac{0.926}{2}\right)^2}} = 27.8 \text{ m/s}$$

(2) Since the basaltic rock is at the highest point in its trajectory when it collides, the magnitude of the y-component of its initial velocity $\vec{v}_0$ with which it starts falling is equal to zero. Based on this information and given that we set $t = 0$ s at that moment, we start with writing Newton's second law under general conditions, i.e., the terminal velocity is not yet reached:

$$-F_D + F_G = m_r \cdot a$$

$$\Leftrightarrow -b \cdot v^2 + m_r \cdot g = m_r \cdot \left(\frac{dv}{dt}\right)$$

$$\Leftrightarrow dt = \frac{dv}{\left(\frac{-b}{m_r}\right) \cdot v^2 + g}$$

$$\Leftrightarrow \int_0^t dt' = \int_0^v \frac{dv'}{\left(\frac{-b}{m_r}\right) \cdot v'^2 + g} = \left(-\frac{m_r}{b}\right) \int_0^v \frac{dv'}{v'^2 - \frac{m_r}{b} \cdot g}$$

We wish to tackle this integral by applying integration by partial fractions. In a first step, we rewrite the fraction in the integral related to the velocity in the following way:

$$\frac{1}{v'^2 - \frac{m_r}{b} \cdot g} = \frac{A}{v' + \sqrt{\frac{g \cdot m_r}{b}}} + \frac{B}{v' - \sqrt{\frac{g \cdot m_r}{b}}}$$

$$= \frac{(A+B) \cdot v' + (B-A) \cdot \sqrt{\frac{g \cdot m_r}{b}}}{v'^2 - \frac{m_r}{b} \cdot g}$$

This means that we have to solve the following two equations:

$$\begin{cases} A + B = 0 \\ (B - A) \cdot \sqrt{\dfrac{g \cdot m_r}{b}} = 1 \end{cases}$$

This gives us the solutions $A = -\frac{1}{2} \cdot \sqrt{\frac{b}{g \cdot m_r}}$ and $B = \frac{1}{2} \cdot \sqrt{\frac{b}{g \cdot m_r}}$. The integral related to the velocity can now be written as follows:

$$\left(-\frac{m_r}{b}\right) \int_0^v \frac{dv'}{v'^2 - \frac{m_r}{b} \cdot g} = \left(-\frac{m_r}{b}\right) \int_0^v \left[\frac{\left(-\frac{1}{2} \cdot \sqrt{\frac{b}{g \cdot m_r}}\right)}{v' + \sqrt{\frac{g \cdot m_r}{b}}} + \frac{\left(\frac{1}{2} \cdot \sqrt{\frac{b}{g \cdot m_r}}\right)}{v' - \sqrt{\frac{g \cdot m_r}{b}}}\right] \cdot dv'$$

$$= \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \int_0^v \left[\frac{1}{v' + \sqrt{\frac{g \cdot m_r}{b}}} - \frac{1}{v' - \sqrt{\frac{g \cdot m_r}{b}}}\right] \cdot dv'$$

$$= \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \left[\int_0^v \frac{dv'}{v' + \sqrt{\frac{g \cdot m_r}{b}}} - \int_0^v \frac{dv'}{v' - \sqrt{\frac{g \cdot m_r}{b}}} \right]$$

We can now solve the integrals from the equation derived from Newton's second law:

$$\int_0^t dt' = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \left[\int_0^v \frac{dv'}{v' + \sqrt{\frac{g \cdot m_r}{b}}} - \int_0^v \frac{dv'}{v' - \sqrt{\frac{g \cdot m_r}{b}}} \right]$$

$$\Leftrightarrow t = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \left[\left(\ln \left| v + \sqrt{\frac{g \cdot m_r}{b}} \right| - \ln \left| \sqrt{\frac{g \cdot m_r}{b}} \right| \right) - \left(\ln \left| v - \sqrt{\frac{g \cdot m_r}{b}} \right| - \ln \left| \sqrt{\frac{g \cdot m_r}{b}} \right| \right) \right]$$

$$\Leftrightarrow t = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \left[\ln \left| v + \sqrt{\frac{g \cdot m_r}{b}} \right| - \ln \left| v - \sqrt{\frac{g \cdot m_r}{b}} \right| \right]$$

$$\Leftrightarrow t = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \ln \left| \frac{v + \sqrt{\frac{g \cdot m_r}{b}}}{v - \sqrt{\frac{g \cdot m_r}{b}}} \right| = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \ln \left(\frac{v + \sqrt{\frac{g \cdot m_r}{b}}}{-v + \sqrt{\frac{g \cdot m_r}{b}}} \right)$$

In the final step of the above equation, we changed the absolute value of the natural logarithm into regular parentheses and thereby switching the sign of the expression in the denominator. From part (1), we have seen that the terminal velocity is equal to $v_T = \sqrt{\frac{g \cdot m_r}{b}}$. Since $v < v_T$ and given that the input value of the natural logarithm must always be greater than zero (keep furthermore in mind that the nominator of the above logarithm is greater than zero), we need to change the sign of the expression in the denominator, if we put regular parentheses.

In a final step, we rewrite the above equation, so that we obtain an expression for the magnitude of the rock's velocity:

$$t = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \ln \left(\frac{v + \sqrt{\frac{g \cdot m_r}{b}}}{-v + \sqrt{\frac{g \cdot m_r}{b}}} \right)$$

$$\Leftrightarrow e^{\left(2 \cdot \sqrt{\frac{g \cdot b}{m_r}}\right) \cdot t} = \frac{v + \sqrt{\frac{g \cdot m_r}{b}}}{-v + \sqrt{\frac{g \cdot m_r}{b}}}$$

$$\Leftrightarrow v = \sqrt{\frac{g \cdot m_r}{b}} \cdot \left[\frac{e^{\left(2 \cdot \sqrt{\frac{g \cdot b}{m_r}}\right) \cdot t} - 1}{e^{\left(2 \cdot \sqrt{\frac{g \cdot b}{m_r}}\right) \cdot t} + 1} \right]$$

The above equation tells us that the speed v of the basaltic rock goes asymptotically to $v_T = \sqrt{\frac{g \cdot m_r}{b}}$ for large values of the time variable t.

(3) Using the formula derived in part (2) for the time t in terms of the velocity v and given that $v = 0.955 \cdot v_T$ and that $v_T = \sqrt{\frac{g \cdot m_r}{b}}$, we calculate the time required for the rock to reach that velocity v as follows:

$$t = \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \ln\left(\frac{v + v_T}{-v + v_T}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\frac{m_r}{g \cdot b}} \cdot \ln\left(\frac{0.955 \cdot v_T + v_T}{-0.955 \cdot v_T + v_T}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\frac{1,250}{8.87 \cdot \left[\frac{1}{2} \cdot 0.635 \cdot 67.0 \cdot \pi \cdot \left(\frac{0.926}{2}\right)^2\right]}} \cdot \ln\left(\frac{1.955}{0.045}\right)$$

$$= 5.91 \text{ s}$$

Exercise 9

Problem Statement

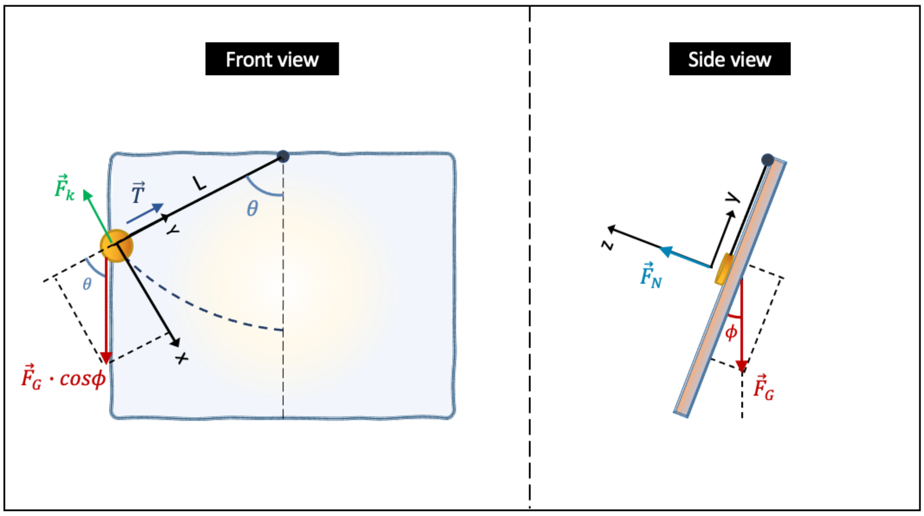

Figure 8

Willow is visiting her grandmother Evie, who lives in Launceston, Tasmania, to spend some quality time with her. During some afternoon tea with traditional Anzac biscuits, Evie tells Willow to go and get an old painting from the attic that she made during her childhood. Once up there, Willow spots the painting on top of a large storage cupboard. While standing on the tips of her toes, she grabs the painting and tilts it away from her by an angle of $\phi = 21.3°$ with the vertical. At that moment, an old golden medallion ($m_m = 0.350$ kg), which was hanging just over the left side of the painting and attached to a chain, which is fixed to the middle of the top edge of the painting, slides from the left side in an arc-like motion towards the middle—initially, the chain, which has a length of $L = 34.4$ cm, was making an angle of $\theta = 63.6°$ with the left side. If the average blink of an eye lasts $t_b = 0.120$ s, how many times can Willow blink before the medallion reaches the middle of the painting? Assume that the kinetic friction coefficient between the metal of the medallion and the canvas is equal to $\mu_k = 0.784$.

Solution

Note that the gravitational acceleration in the xy-plane, i.e., the plane parallel to the surface of the canvas, is equal to $\vec{F}_G \cdot \cos\phi$, given that Willow is tilting the painting with an angle ϕ. Let us apply Newton's second law to the three spatial dimensions, respectively (whereby $\vec{a}_{tan}$ represents the tangential component of the acceleration $\vec{a}$):

$$\begin{cases} x: & -F_k + (m_m \cdot g \cdot \sin\theta) \cdot \cos\phi = m_m \cdot a_{tan} \\ y: & T - (m_m \cdot g \cdot \cos\theta) \cdot \cos\phi = m_m \cdot a_y \\ z: & F_N - m_m \cdot g \cdot \sin\phi = 0 \end{cases}$$

Keep in mind that the centripetal acceleration in the y-direction ($a_y = \frac{v^2}{L}$) is not equal to zero when the medallion is swinging, so that the tension force $\vec{T}$ in the chain does *not* cancel out the respective component of the gravitational force $\vec{F}_G$. This would only be the case when the medallion is hanging still ($v = 0$ m/s).

Given that the magnitude of the kinetic friction force $\vec{F}_k$ is equal to $F_k = \mu_k \cdot F_N$, we combine the equations from the x- and z-direction to obtain an expression for the magnitude of the acceleration $\vec{a}_{tan}$ along the arc-shaped line of the medallion's path of motion:

$$-(\mu_k \cdot F_N) + (m_m \cdot g \cdot \sin\theta) \cdot \cos\phi = m_m \cdot a_{tan}$$

$$\Leftrightarrow\ -(\mu_k \cdot [m_m \cdot g \cdot \sin\phi]) + (m_m \cdot g \cdot \sin\theta) \cdot \cos\phi = m_m \cdot a_{tan}$$

$$\Leftrightarrow a_{tan} = g \cdot (\sin\theta \cdot \cos\phi - \mu_k \cdot \sin\phi)$$

Now we write an expression for the time it takes the medallion to reach the middle of the painting. The distance d that the medallion has to cover is equal to the length of the arc over the angle θ. In other words, $d = L \cdot (\theta \cdot \frac{\pi}{180})$, since θ has to be expressed in radians. Based on the appropriate equation of motion, we calculate the time t it takes the medallion to reach the middle:

$$d = \frac{a_{tan}}{2} \cdot t^2$$

$$\Leftrightarrow t = \sqrt{\frac{2 \cdot d}{a_{tan}}}$$

$$= \sqrt{\frac{2 \cdot L \cdot (\theta \cdot \frac{\pi}{180})}{g \cdot (\sin\theta \cdot \cos\phi - \mu_k \cdot \sin\phi)}}$$

$$= \sqrt{\frac{2 \cdot 0.344 \cdot (63.6^\circ \cdot \frac{\pi}{180})}{9.81 \cdot [\sin(63.6^\circ) \cdot \cos(21.3^\circ) - 0.784 \cdot \sin(21.3^\circ)]}}$$

$$= 0.376 \text{ s}$$

Given that the average blink lasts about $t_b = 0.120$ s, we find the number of times N Willow is able to blink by the time the medallion reaches the centre of the painting with the following calculation:

$$N = \frac{t}{t_b} = \frac{0.376}{0.120} = 3.14$$

Exercise 10

Problem Statement

In the western Pacific Ocean, close to the coast of Tobi Island, Palau, two blacktail damselfish are feeling playful. The heavier of the two (m_1) is swimming right behind the other one ($m_2 = 3.92$ kg), who is moving at a constant speed of $\vec{v}_0 = 1.51 \cdot \vec{i}_x$ m/s under an angle of $\theta = 34.8°$ with the horizontal. At a certain moment, the heavier damselfish is pushing his friend in the same direction of her motion with a constant force of $\vec{F}_{21} = 24.4 \cdot \vec{i}_x$ N. The drag force $\vec{F}_D$ in a viscous medium for lower velocities has the general form of $\vec{F}_D = -(K \cdot \eta) \cdot \vec{v}$, with η the viscosity coefficient with a value of $\eta = 1.787 \times 10^{-3}$ kg/(m·s) for water at 0°C. The parameter K depends on the shape of the object, and if we approximate the fish by a sphere, we obtain $K = 6 \cdot \pi \cdot r$ (in m), with $r = 12.0$ cm. If we ignore the buoyancy force in our problem, how fast is the first blacktail damselfish going after being pushed for $t = 6.25$ s?

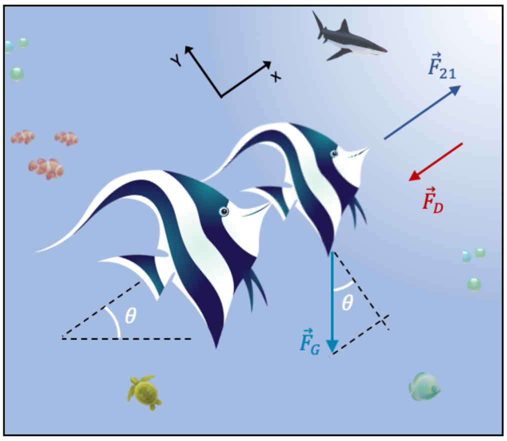

Figure 9

Solution

Applying Newton's second law to the x-direction of the first blacktail damselfish results in the following equation:

$$F_{21} - F_D - F_G \cdot \sin\theta = m_2 \cdot a_x$$

$$\Leftrightarrow F_{21} - (K \cdot \eta) \cdot v_x - m_2 \cdot g \cdot \sin\theta = m_2 \cdot a_x$$

If we wish to find the velocity v_x at which the fish is going, we need to solve the above differential equation for the velocity v_x with respect to the time variable t:

$$F_{21} - (K \cdot \eta) \cdot v_x - m_2 \cdot g \cdot \sin\theta = m_2 \cdot \frac{dv_x}{dt}$$

$$\Leftrightarrow dt = \frac{m_2 \cdot dv_x}{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_x}$$

$$\Leftrightarrow \int_0^t dt' = \int_{v_{0x}}^{v_x} \frac{m_2 \cdot dv'_x}{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v'_x}$$

The general solution of the above integral related to the velocity with *random* parameters is the following:

$$\int_{v_{0x}}^{v_x} \frac{dv'_x}{a - b \cdot v'_x} = \left(-\frac{1}{b}\right) \cdot \left(\ln|a - b \cdot v_x| - \ln|a - b \cdot v_{0x}|\right)$$

Therefore, applying this generic solution to our integral related to the velocity, we can write the following solution:

$$t = \left(-\frac{m_2}{K \cdot \eta}\right) \cdot \left(\ln|(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_x| - \ln|(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_{0x}|\right)$$

$$\Leftrightarrow t = \left(-\frac{m_2}{K \cdot \eta}\right) \cdot \ln\left|\frac{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_x}{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_{0x}}\right|$$

$$\Leftrightarrow e^{\left(-\frac{K \cdot \eta}{m_2}\right) \cdot t} = \frac{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_x}{(F_{21} - m_2 \cdot g \cdot \sin\theta) - (K \cdot \eta) \cdot v_{0x}}$$

$$\Leftrightarrow v_x = \left(\frac{F_{21} - m_2 \cdot g \cdot \sin\theta}{K \cdot \eta}\right) + \left[v_{0x} - \left(\frac{F_{21} - m_2 \cdot g \cdot \sin\theta}{K \cdot \eta}\right)\right] \cdot e^{\left(-\frac{K \cdot \eta}{m_2}\right) \cdot t}$$

Bearing in mind that the parameter K is equal to $K = 6 \cdot \pi \cdot 0.12 = 2.26$ m, we find the following speed v_x of the first blacktail damselfish after a time of $t = 6.25$ s:

$$v_x = \left(\frac{F_{21} - m_2 \cdot g \cdot \sin\theta}{K \cdot \eta}\right) + \left[v_{0x} - \left(\frac{F_{21} - m_2 \cdot g \cdot \sin\theta}{K \cdot \eta}\right)\right] \cdot e^{\left(-\frac{K \cdot \eta}{m_2}\right) \cdot t}$$

$$= \left(\frac{24.4 - 3.92 \cdot 9.81 \cdot \sin(34.8^\circ)}{2.26 \cdot 1.787 \times 10^{-3}}\right) + \left[1.51 - \left(\frac{24.4 - 3.92 \cdot 9.81 \cdot \sin(34.8^\circ)}{2.26 \cdot 1.787 \times 10^{-3}}\right)\right] \cdot e^{\left(-\frac{2.26 \cdot 1.787 \times 10^{-3}}{3.92}\right) \cdot 6.25}$$

$$= 5.40 \text{ m/s}$$

Exercise 11

Problem Statement

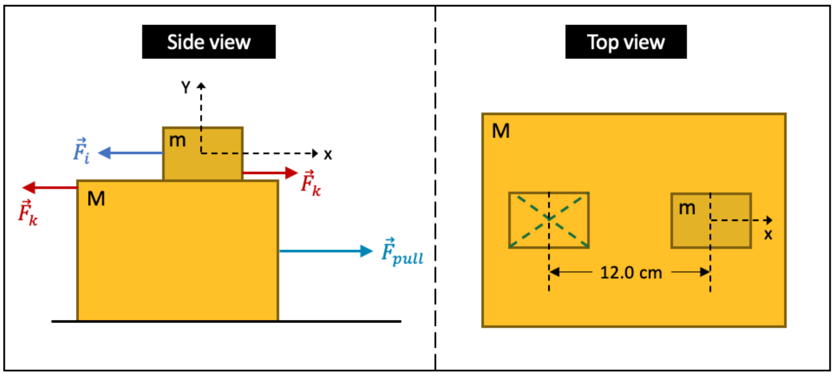

Figure 10

The thirteen-year-old Bahadur is visiting the new science fair with his dad Husani in the Planetarium Science Center in Alexandria, Egypt. In one of the activities, Bahadur has to pull a large block ($M = 8.50$ kg), which is moving on a frictionless rail. On top of the large block, a small block ($m = 4.50$ kg) is positioned precisely 12.0 cm to the right of a marked area. Bahadur is asked to pull the lower block for just the right amount of time t_*, so that the upper block moves to the left and comes to rest precisely within the marked area. Since the top surface of the large block is slightly roughened, the small block needs a minimum amount of force $\vec{F}_s$ before it can start moving (the static friction coefficient is equal to $\mu_s = 0.115$). Once the small block is set in motion, it experiences a slightly lower amount of (kinetic) friction, i.e., $\mu_k = 0.102$. If Bahadur pulls the large block M with a force of $\vec{F}_{pull} = 15.0 \cdot \vec{i}_x$ N, for how long (t_*) should he sustain this force? Assume that the time t_s corresponds to the time needed for block m to overcome the static friction—during this time, block m is not yet moving—and is equal to $t_s = \frac{t_*}{10}$ s.

Solution

Since time t_s corresponds to the time for block m to surmount the static friction, whereby block m still remains stationary, let us define time t_k as the time during which Bahadur is pulling block M *and* whereby block m is actually in motion. Therefore, the total time of pulling block M is equal to $t_* = t_s + t_k$.

When Bahadur pulls block M during a time of t_k seconds, the block m experiences an inertial force $\vec{F}_i$ in the opposite direction of $\vec{F}_{pull}$, as a result of block m being positioned within a non-inertial (accelerating) reference frame, i.e., block M. During time t_k, block m moves a certain distance Δx_1 to the left and at the instant when Bahadur stops pulling, it will have obtained a velocity $\vec{v}_1 = -v_1 \cdot \vec{i}_x$ m/s.

From that moment onwards, the inertial force $\vec{F}_i$ disappears, but block m still undergoes a frictional force $\vec{F}_k$, effectively slowing the block down. If we call the distance over which the block slows down and eventually comes to a halt Δx_2, then the general constraint for our problem is translated as $\Delta x_1 + \Delta x_2 = -0.12$ m (the x-axis points to the right).

Before proceeding, let us first check whether the net acceleration $\vec{a}_n = \frac{\vec{F}_{pull}}{m+M}$ of the system "block m plus block M" that acts on block m is sufficiently large to induce relative motion between both blocks. In other words, the magnitude of the force $\vec{F} = m \cdot \vec{a}_n$ should be larger than that of the static friction force $\vec{F}_s$, i.e., $F_s = \mu_s \cdot F_N = \mu_s \cdot (m \cdot g)$:

$$m \cdot a_n > \mu_s \cdot (m \cdot g) \quad \Leftrightarrow \quad \frac{F_{pull}}{m+M} > \mu_s \cdot g$$

$$\Leftrightarrow \quad \frac{15.0}{4.50+8.50} > 0.115 \cdot 9.81$$

$$\Leftrightarrow \quad 1.15 > 1.13$$

Now that we know that block m will effectively move to the left when Bahadur provides the force $\vec{F}_{pull}$, let us calculate the magnitude of the net acceleration $\vec{a}_p$ of block M once it is set in motion. Applying Newton's second law to block M, we obtain the following value for a_p (keep in mind that block M only experiences kinetic friction at its top side with block m, not with the rail):

$$-\mu_k \cdot (m \cdot g) + F_{pull} = M \cdot a_p \quad \Leftrightarrow \quad a_p = \frac{-\mu_k \cdot (m \cdot g) + F_{pull}}{M} = \frac{-0.102 \cdot (4.50 \cdot 9.81) + 15.0}{8.50} = 1.23 \text{ m/s}^2$$

Applying Newton's second law to block m both during pulling block M and after letting it go, we find the following value for the respective acceleration:

<u>During pull</u>

$$-F_i + F_k = m \cdot a_1$$

$$\Leftrightarrow \quad -(m \cdot a_p) + (\mu_k \cdot m \cdot g) = m \cdot a_1$$

$$\Leftrightarrow \quad a_1 = \mu_k \cdot g - a_p$$

$$= 0.102 \cdot 9.81 - 1.23 = -0.234 \text{ m/s}^2$$

<u>After pull</u>

$$F_k = m \cdot a_2$$

$$\Leftrightarrow \quad (\mu_k \cdot m \cdot g) = m \cdot a_2$$

$$\Leftrightarrow \quad a_2 = \mu_k \cdot g$$

$$= 0.102 \cdot 9.81 = 1.00 \text{ m/s}^2$$

In a next step, we write the following two equations of motion for the moment during and after the pull, respectively:

<u>During pull</u>

$$(-v_1)^2 - 0^2 = 2 \cdot a_1 \cdot \Delta x_1$$

$$\Leftrightarrow \quad \Delta x_1 = \frac{v_1^2}{2 \cdot a_1}$$

<u>After pull</u>

$$0^2 - (-v_1)^2 = 2 \cdot a_2 \cdot \Delta x_2$$

$$\Leftrightarrow \quad \Delta x_2 = -\frac{v_1^2}{2 \cdot a_2}$$

Plugging the above two equations for the displacement into the constraint that $\Delta x_1 + \Delta x_2 = -0.12$, we obtain the following expression:

$$\Delta x_1 + \Delta x_2 = -0.12 \quad \Leftrightarrow \quad \frac{v_1^2}{2 \cdot a_1} - \frac{v_1^2}{2 \cdot a_2} = -0.12$$

$$\Leftrightarrow \quad \frac{v_1^2}{2} \cdot \left(\frac{a_1 - a_2}{a_1 \cdot a_2} \right) = 0.12$$

Now, we insert the equation of motion $v_1 = a_1 \cdot t_k$ into the above expression, which allows us to calculate the time t_k during which Bahadur pulls the block M and whereby block m is accelerating to the left:

$$\frac{(a_1 \cdot t_k)^2}{2} \cdot \left(\frac{a_1 - a_2}{a_1 \cdot a_2} \right) = 0.12 \quad \Leftrightarrow \quad t_k = \sqrt{\frac{0.24}{a_1^2} \cdot \left(\frac{a_1 \cdot a_2}{a_1 - a_2} \right)}$$

$$= \sqrt{\frac{0.24}{(-0.234)^2} \cdot \left(\frac{(-0.234) \cdot 1.00}{-0.234 - 1.00} \right)}$$

$$= 0.911 \text{ s}$$

Finally, the total amount of time t_* that Bahadur pulls the block M, whereby the time t_s needed to overcome the static friction is taken into account, is equal to:

$$t_* = t_s + t_k = \frac{t_*}{10} + t_k \quad \Leftrightarrow \quad t_* = t_k \cdot \frac{10}{9} = 0.911 \cdot \frac{10}{9} = 1.01 \text{ s}$$

Given that the velocity v_1 is equal to $v_1 = a_1 \cdot t_k = (-0.234) \cdot 0.911 = -0.213$ m/s, let us perform a final check to see whether our constraint $\Delta x_1 + \Delta x_2 = -0.12$ m is indeed upheld:

$$\Delta x_1 + \Delta x_2 = \frac{v_1^2}{2 \cdot a_1} - \frac{v_1^2}{2 \cdot a_2}$$

$$= \frac{(-0.213)^2}{2 \cdot (-0.234)} - \frac{(-0.213)^2}{2 \cdot 1.00}$$

$$= -0.12 \text{ m}$$

Exercise 12

Problem Statement

Lagrange points are relatively stable orbits of objects of little mass in the presence of two heavier masses (with one mass (M_1) larger than the other (M_2) for a minimum ratio of $\frac{M_1}{M_2} = 24.96$), which are all orbiting around a common center of mass, i.e., the barycenter. In our Solar System, examples of such massive bodies include the Sun-Jupiter and the Sun-Earth duo as well as the Earth-Moon system. The gravitational interplay within these systems allows for the existence of five Lagrange points, i.e., L_1 up to L_5. From the perspective of a *rotating* reference frame, the relatively stable circular orbit of the object of little mass (say, m_1) is the result of the combined gravitational impact on m_1, due to the large masses M_1 and M_2, being balanced by a pseudo-force, i.e., the centrifugal force, experienced by m_1 from the center of mass. The final effect is such that the period T of the object m_1 is equal to that of both mass M_1 and M_2—the period T is the amount of time during which an object completes one revolution around another object.

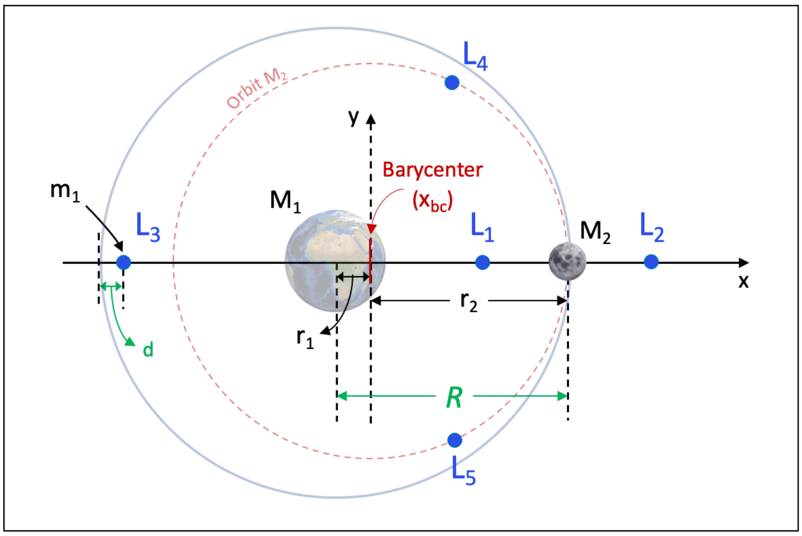

Figure 11

With regard to point L_3 of the Earth-Moon system, the orbit of mass m_1 lies a little bit farther from the barycenter with respect to the Moon (M_2) and there is a small distance d between the position of m_1 and M_2, if both objects would be located at the same side of mass M_1 (with m_1 being closer to M_1). If you know that the mass of the Earth and the Moon are equal to $M_1 = 5.972 \times 10^{24}$ kg and $M_2 = 7.342 \times 10^{22}$ kg, respectively, and that the Earth-Moon distance measures $R = 3.844 \times 10^5$ km, how far lies the Lagrange point L_3 from the center of the Earth?

Solution

First off, let us define the distances r_1 and r_2 from the barycenter x_{bc}, which sits in the origin of our coordinate system ($x_{bc} = 0$). Remember that the definition of the center of mass is equal to $x_{bc} = \frac{x_1 \cdot M_1 + x_2 \cdot M_2}{M_1 + M_2}$.

Distance r_1	Distance r_2
$0 = \dfrac{-r_1 \cdot M_1 + r_2 \cdot M_2}{M_1 + M_2}$	$r_2 = R - r_1$
$\Leftrightarrow \quad r_1 = \dfrac{r_2 \cdot M_2}{M_1}$	$\Leftrightarrow \quad r_2 = R - \dfrac{R \cdot M_2}{M_1 + M_2}$

$$\Leftrightarrow\ r_1 = (R - r_1) \cdot \frac{M_2}{M_1} \qquad\qquad \Leftrightarrow\ r_2 = \frac{R \cdot M_1 + R \cdot M_2 - R \cdot M_2}{M_1 + M_2}$$

$$\Leftrightarrow\ r_1 = \frac{R \cdot M_2}{M_1 + M_2} \qquad\qquad \Leftrightarrow\ r_2 = \frac{R \cdot M_1}{M_1 + M_2}$$

In a next step, we determine the orbital speed v_1 and v_2 of the Earth and the Moon, respectively, around the barycenter x_{bc}. In the isolated Earth-Moon system, whereby our coordinate system (x,y) rotates along with the masses M_1 and M_2 according to their angular velocity $\omega = \frac{2 \cdot \pi}{T}$ (with T the period), the gravitational force, which acts over the distance R between the Earth and the Moon, is balanced by the centrifugal force, which acts from the respective body to the axis of rotation at x_{bc}:

Orbital speed Earth (v_1)

$$G \cdot \frac{M_1 \cdot M_2}{R^2} = M_1 \cdot \frac{v_1^2}{r_1}$$

$$\Leftrightarrow\ v_1 = \sqrt{\frac{G \cdot M_2 \cdot r_1}{R^2}}$$

Orbital speed Moon (v_2)

$$G \cdot \frac{M_1 \cdot M_2}{R^2} = M_2 \cdot \frac{v_2^2}{r_2}$$

$$\Leftrightarrow\ v_2 = \sqrt{\frac{G \cdot M_1 \cdot r_2}{R^2}}$$

As the Lagrange point L_3 rotates around the barycenter x_{bc} with the same period T—and thus the same angular velocity ω—as the Earth and the Moon, we find the following expression for the angular velocity ω of L_3 by using, for instance, the orbital speed v_1 and the definition of r_1:

$$v_1 = \frac{2 \cdot \pi}{T} \cdot r_1 = \omega \cdot r_1 \quad \Leftrightarrow \quad \omega = \frac{v_1}{r_1} = \frac{1}{r_1} \cdot \sqrt{\frac{G \cdot M_2 \cdot r_1}{R^2}} = \sqrt{\frac{G \cdot M_2}{R^2 \cdot r_1}} = \sqrt{\frac{G \cdot (M_1 + M_2)}{R^3}}$$

Before writing the balancing equation between the gravitational forces and the centrifugal force, let us first have a look at L_3's centrifugal force $\vec{F}_{cf,L_3}$. Using the general expression $v = \omega \cdot r$, we write $\vec{F}_{cf,L_3}$'s magnitude as follows (with v_{L_3} representing L_3's orbital speed and "$r_1 + R - d$" the distance between L_3 and the axis of rotation at x_{bc}):

$$F_{cf,L_3} = m_1 \cdot \frac{v_{L_3}^2}{(r_1 + R - d)} = \frac{m_1}{(r_1 + R - d)} \cdot [\omega \cdot (r_1 + R - d)]^2 = m_1 \cdot \omega^2 \cdot (r_1 + R - d)$$

We can now write the balancing equation as follows:

$$G \cdot \frac{M_1 \cdot m_1}{(R - d)^2} + G \cdot \frac{M_2 \cdot m_1}{(2 \cdot R - d)^2} = m_1 \cdot \omega^2 \cdot (r_1 + R - d)$$

Using the above expression for the angular velocity ω and the distance r_1, the equation becomes:

$$G \cdot \frac{M_1 \cdot m_1}{(R-d)^2} + G \cdot \frac{M_2 \cdot m_1}{(2 \cdot R - d)^2} = m_1 \cdot \left[\frac{G \cdot (M_1 + M_2)}{R^3}\right] \cdot \left(\frac{R \cdot M_2}{M_1 + M_2} + R - d\right)$$

$$\Leftrightarrow \frac{M_1}{(R-d)^2} + \frac{M_2}{(2 \cdot R - d)^2} = \left[\frac{(M_1 + M_2)}{R^3}\right] \cdot \left(\frac{R \cdot M_2}{M_1 + M_2} + R - d\right)$$

$$\Leftrightarrow \frac{M_1}{R^2} \cdot \left(\frac{1}{1 - \frac{d}{R}}\right)^2 + \frac{M_2}{4 \cdot R^2} \cdot \left(\frac{1}{1 - \frac{d}{2 \cdot R}}\right)^2 = \frac{M_2}{R^2} + \frac{(M_1 + M_2)}{R^2} - \frac{(M_1 + M_2) \cdot d}{R^3}$$

$$\Leftrightarrow M_1 \cdot \left(1 - \frac{d}{R}\right)^{-2} + \frac{M_2}{4} \cdot \left(1 - \frac{d}{2 \cdot R}\right)^{-2} = M_2 + (M_1 + M_2) - \frac{(M_1 + M_2) \cdot d}{R}$$

Under the assumption that the distance d is much smaller than the distance between the Earth and the Moon R ($d << R$), we can apply the method of binomial expansion (the negative binomial theorem) of the general form "$(1 + x)^{-b} \approx (1 - b \cdot x)$" to our equation and calculate the distance d:

$$M_1 \cdot \left(1 + 2 \cdot \frac{d}{R}\right) + \frac{M_2}{4} \cdot \left(1 + \frac{d}{R}\right) \approx M_2 + (M_1 + M_2) - \frac{(M_1 + M_2) \cdot d}{R}$$

$$\Leftrightarrow \left(2 \cdot M_1 + \frac{M_2}{4} + M_1 + M_2\right) \cdot \frac{d}{R} \approx \frac{7 \cdot M_2}{4}$$

$$\Leftrightarrow d \approx \left(\frac{7 \cdot M_2}{12 \cdot M_1 + 5 \cdot M_2}\right) \cdot R$$

$$\approx \left(\frac{7 \cdot 7.342 \times 10^{22}}{12 \cdot 5.972 \times 10^{24} + 5 \cdot 7.342 \times 10^{22}}\right) \cdot 3.844 \times 10^8$$

$$\approx 2.743 \times 10^3 \text{ km}$$

Finally, the distance between the Lagrange point L_3 and the center of the Earth of the Earth-Moon system is approximately equal to:

$$R - d \approx 3.844 \times 10^8 - 2.743 \times 10^6 \approx 3.817 \times 10^5 \text{ km}$$

Exercise 13

Problem Statement

On a 5.00 m-wide gravel road outside of Balkanabat, Turkmenistan, Melek is going $\vec{v}_0 = 47.2 \cdot \vec{i}_y$ m/s, when she suddenly notices that a 90° left curve is ahead. Melek hits the breaks over a distance of $\Delta x_0 = 156$ m and $t_0 = 4.30$ s later she enters the curve, which has a radius of $r = 85.0$ m, at a velocity $\vec{v}_{in}$ at 1.00 m from the left guardrail. (1) Will Melek skid in the curve? (2) If yes, will she hit the guardrail on the right-hand side? (3) If so, when? If not, at what distance from the right rail does Melek exit the curve? Given that Melek may change her distance from the left rail while going through the curve, apply average values over the width of the road between her point of entry and the right rail when dealing with circular motion. Assume furthermore that Melek maintains her speed v_{in} throughout the curve and that the kinetic friction coefficient between gravel and rubber tires is equal to $\mu_k = 0.718$.

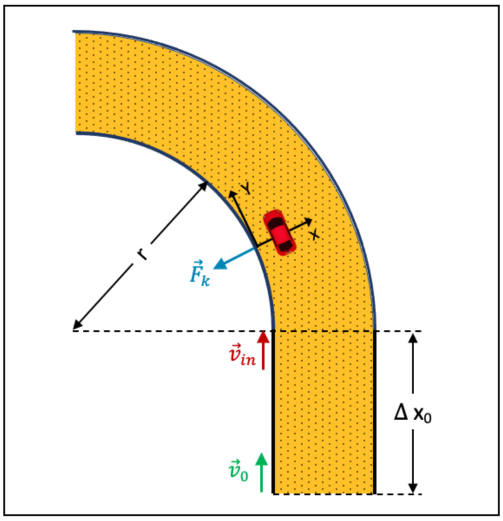

Figure 12

Solution

(1) In order to determine whether Melek will skid in the curve, we have to know the magnitude of her incoming velocity $\vec{v}_{in}$. In a first instance, let us calculate the magnitude of the deceleration $\vec{a}_{dec}$ of Melek's car when she hit the breaks, so that we can find her speed v_{in}:

Deceleration a_{dec}

$$\Delta x_0 = v_0 \cdot t_0 + \frac{a_{dec}}{2} \cdot t_0^2$$

$$\Leftrightarrow \quad a_{dec} = \frac{2 \cdot (\Delta x_0 - v_0 \cdot t_0)}{t_0^2}$$

$$= \frac{2 \cdot (156 - 47.2 \cdot 4.30)}{4.30^2}$$

$$= -5.08 \text{ m/s}^2$$

Incoming speed v_{in}

$$v_{in}^2 - v_0^2 = 2 \cdot a_{dec} \cdot \Delta x_0$$

$$\Leftrightarrow \quad v_{in} = \sqrt{v_0^2 + 2 \cdot a_{dec} \cdot \Delta x_0}$$

$$= \sqrt{47.2^2 + 2 \cdot (-5.08) \cdot 156}$$

$$= 25.4 \text{ m/s}$$

In a next step, we want to know at what maximal speed Melek can go through the curve without skidding. Keep in mind that the kinetic friction force is equal to $\vec{F}_k = -(\mu_k \cdot m_c \cdot g) \cdot \vec{i}_x$ (with

m_c the mass of the car) and that the car in a rotating framework, relative to an inertial reference frame, experiences a centrifugal force $\vec{F}_{cf} = \left(\frac{m_c \cdot v^2}{r}\right) \cdot \vec{i}_x$. However, as Melek enters the curve at 1.00 m from the left guardrail and since she may or may not change her position over the remaining width of the road, we add the average distance $\left(\frac{5.00-1.00}{2}\right) = 2.00$ m as well as the 1.00 m to the radius of the curve. Therefore, the centrifugal force becomes $\vec{F}_{cf} = \left(\frac{m_c \cdot v^2}{r+3.00}\right) \cdot \vec{i}_x$. Applying Newton's second law (in the x-direction) to Melek's car under these conditions, we obtain the following speed v_{max}:

$$-\left(\mu_k \cdot m_c \cdot g\right) + \left(\frac{m_c \cdot v_{max}^2}{r + 3.00}\right) = 0$$

$$\Leftrightarrow v_{max} = \sqrt{\mu_k \cdot g \cdot (r + 3.00)}$$

$$= \sqrt{0.718 \cdot 9.81 \cdot (85.0 + 3.00)}$$

$$= 24.9 \text{ m/s}$$

Since $v_{in} = 25.4$ m/s $> v_{max} = 24.9$ m/s, Melek's car will start skidding in the curve.

(2) As the sideways acceleration is thus not equal to zero, the car will pick up some velocity in the positive x-direction. Let us first establish how much time Melek spends in the curve. Given a constant speed of $v_{in} = 25.4$ m/s and a distance of $s = (r + 3.00) \cdot \frac{\pi}{2} = 138$ m (the angle 90° is expressed in radians), the time is equal to $t = \frac{s}{v_{in}} = \frac{138}{25.4} = 5.45$ s. Next, we wish to know the maximal sideways acceleration a_{max} the car can have before hitting the guardrail on the right:

$$x_{max} = \frac{a_{max}}{2} \cdot t^2 \;\Leftrightarrow\; a_{max} = \frac{2 \cdot x_{max}}{t^2} = \frac{2 \cdot (5.00 - 1.00)}{5.45^2} = 0.269 \text{ m/s}^2$$

We find the sideways acceleration of Melek's car by applying Newton's second law:

$$-(\mu_k \cdot m_c \cdot g) + \left(\frac{m_c \cdot v_{in}^2}{r + 3.00}\right) = m_c \cdot a \;\Leftrightarrow\; a = -(0.718 \cdot 9.81) + \left(\frac{25.4^2}{85.0 + 3.00}\right) = 0.264 \text{ m/s}^2$$

As $a = 0.264$ m/s$^2 < a_{max} = 0.269$ m/s^2, Melek will not hit the guardrail.

(3) Keeping in mind Melek's entry point into the curve at 1.00 m from the left guardrail, the final position in the x-direction of Melek's car at the moment when he exits the curve is equal to:

$$x = x_0 + \frac{a}{2} \cdot t^2 = 1.00 + \frac{0.264}{2} \cdot 5.45^2 = 4.92 \text{ m}$$

Therefore, the distance d from the guardrail at the right-hand side then becomes:

$$d = 5.00 - x = 5.00 - 4.92 = 0.0830 \text{ m or } 8.30 \text{ cm}$$

Exercise 14

Problem Statement

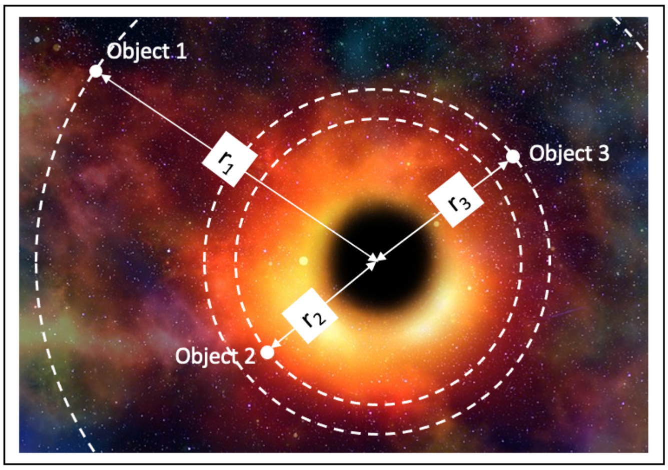

Figure 13

Halima is doing research at the Copperbelt University in Zambia on superclusters, which are aggregate systems of various galaxy groups and smaller clusters, whereby one of them, the Ophiuchus Supercluster, which is located at a distance of roughly 370 million light-years away from us (1 light-year is equal to 9.46×10^{15} m), particularly interests her. Halima suspects to have found a black hole at the edge of the Ophiuchus Supercluster around which three other objects are orbiting in a circular fashion. So far, Halima has managed to retrieve the following information from the orbiting objects: object 1 has a period of $T_1 = 163$ Earth days, the distance from object 2 to the center of the black hole is equal to $r_2 = 5.93 \times 10^7$ km, the distance from object 1 to the black hole is twice as large relative to that of object 3, and the distance from object 3 to the black hole is 1.26 times greater with respect to object 2. (1) Halima wants to calculate the mass of the black hole in terms of the mass of our Sun, which is equal to $M_s = 1.99 \times 10^{30}$ kg. What value does she find? (2) What is the period (in Earth days) for object 2 and 3? (3) What are the orbital velocities of the three objects? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2) and assume that, due to the overwhelmingly strong gravitational influence of the black hole, the gravitational interactions between the three objects are minimal and can therefore be ignored, and that the mass M_{BH} of the black hole remains constant.

Solution

(1) For each of the three isolated subsystems "object plus black hole", the gravitational force $\vec{F}_G = -G \cdot \frac{M \cdot M_{BH}}{r^2} \cdot \vec{i}_r$ between the object and the black hole is counteracted by the existence of a centrifugal force $\vec{F}_{cf} = \frac{M \cdot v^2}{r} \cdot \vec{i}_r$, when viewed from the perspective of the rotating object. As a result, both forces balance each other out and the object follows a relatively stable orbit. Applying Newton's second law to any of the objects, and bearing in mind that the orbital speed v is equal to $v = \frac{2 \cdot \pi}{T} \cdot r$, we obtain the following equation (which is Kepler's third law):

$$- G \cdot \frac{M \cdot M_{BH}}{r^2} + \frac{M \cdot v^2}{r} = 0$$

$$\Leftrightarrow \quad \frac{G \cdot M_{BH}}{r^2} = \frac{4 \cdot \pi^2}{T^2} \cdot r$$

$$\Leftrightarrow \; T^2 = \frac{4 \cdot \pi^2}{G \cdot M_{BH}} \cdot r^3$$

The respective distances between the objects and the center of the black hole are calculated as follows:

$$\begin{cases} r_2 = 5.93 \times 10^{10} \text{ m} \\ r_3 = 1.26 \cdot r_2 = 1.26 \cdot 5.93 \times 10^{10} = 7.47 \times 10^{10} \text{ m} \\ r_1 = 2 \cdot r_3 = 2 \cdot 7.47 \times 10^{10} = 1.49 \times 10^{11} \text{ m} \end{cases}$$

From the equation for object 1, we can now calculate the mass M_{BH} of the black hole (note that the period is expressed in seconds, not Earth days, during the following calculation):

$$T_1^2 = \frac{4 \cdot \pi^2}{G \cdot M_{BH}} \cdot r_1^3$$

$$\Leftrightarrow \; M_{BH} = \frac{4 \cdot \pi^2}{G \cdot T_1^2} \cdot r_1^3$$

$$= \frac{4 \cdot \pi^2}{6.67 \times 10^{-11} \cdot (163 \cdot 24 \cdot 3600)^2} \cdot (1.49 \times 10^{11})^3$$

$$= 9.96 \times 10^{30} \text{ kg}$$

In terms of solar masses, we find the value:

$$M_{BH} = \frac{9.96 \times 10^{30}}{1.99 \times 10^{30}} = 5.00 \; M_s$$

(2) Since the mass of the black hole M_{BH} is constant, we obtain the following expression based on the equation derived in part (1):

$$\frac{4 \cdot \pi^2}{G \cdot M_{BH}} = \frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3} = \frac{T_3^2}{r_3^3}$$

Therefore, the period of object 2 and 3, respectively, can be found as follows:

Period Object 2

$$\frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3}$$

$$\Leftrightarrow \ T_2 = T_1 \cdot \sqrt{\frac{r_2^3}{r_1^3}}$$

$$\Leftrightarrow \ T_2 = 163 \cdot \sqrt{\frac{(5.93 \times 10^{10})^3}{(1.49 \times 10^{11})^3}}$$

$$= 40.7 \text{ Earth days}$$

Period Object 3

$$\frac{T_1^2}{r_1^3} = \frac{T_3^2}{r_3^3}$$

$$\Leftrightarrow \ T_3 = T_1 \cdot \sqrt{\frac{r_3^3}{r_1^3}}$$

$$\Leftrightarrow \ T_3 = 163 \cdot \sqrt{\frac{(7.47 \times 10^{10})^3}{(1.49 \times 10^{11})^3}}$$

$$= 57.6 \text{ Earth days}$$

(3) The expression for the orbital velocity of any object can be derived from the equation obtained in part (1). Therefore, the orbital velocities of the three objects can be calculated as follows:

Orbital speed Object 1

$$v_1 = \sqrt{\frac{G \cdot M_{BH}}{r_1}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \cdot 9.96 \times 10^{30}}{1.49 \times 10^{11}}}$$

$$= 6.67 \times 10^4 \text{ m/s}$$

$$= 2.40 \times 10^5 \text{ km/h}$$

Orbital speed Object 2

$$v_2 = \sqrt{\frac{G \cdot M_{BH}}{r_2}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \cdot 9.96 \times 10^{30}}{5.93 \times 10^{10}}}$$

$$= 1.06 \times 10^5 \text{ m/s}$$

$$= 3.81 \times 10^5 \text{ km/h}$$

Orbital speed Object 3

$$v_3 = \sqrt{\frac{G \cdot M_{BH}}{r_3}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \cdot 9.96 \times 10^{30}}{7.47 \times 10^{10}}}$$

$$= 9.43 \times 10^4 \text{ m/s}$$

$$= 3.39 \times 10^5 \text{ km/h}$$

Alternatively, the orbital speeds can also be calculated based on the formula $v = \frac{2 \cdot \pi}{T} \cdot r$. For instance, the speed for object 2 is equal to $v_2 = \frac{2 \cdot \pi}{T_2} \cdot r_2 = \frac{2 \cdot \pi}{40.7 \cdot 86{,}400} \cdot 5.93 \times 10^{10} = 1.06 \times 10^5$ m/s.

The orbital speeds reveal that the object closest to the black hole, i.e., object 2, is the one with the highest orbital speed and the shortest period. The object farthest away from the black hole, i.e., object 1, has the lowest orbital speed and the longest period.

Exercise 15

Problem Statement

On 23 January 1960, Jacques Piccard and Don Walsh descended in their 18 m-long small submarine, called a bathyscaphe, to a depth of $d = 10,911$ m in the Mariana Trench in the Pacific Ocean. During the descent, both men spent nearly five hours in a 2.16 m-wide pressure sphere. Suppose that at one moment, Jacques was holding a magazine of length $L = 25.00$ cm horizontally with both hands, and on top of it, a set of keys ($m_{sk} = 0.3850$ kg) was resting. The keys were connected to one end of an elastic rubber spring, while the other end was attached to a metal ring through which Jacques had put his index finger of his right hand. Due to a sudden disturbance in the bathyscaphe's balance, Jacques removed his left hand from the magazine to hold on to the side of the pressure sphere. As a result, Jacques tilted the magazine by an angle of $\theta = 46.80°$ with the vertical and the set of keys slid downwards (from the top side of the magazine cover), stretching thereby the rubber spring (the metal ring was still around Jacques' index finger of his right hand). If the restoring force in a spring has the general form of $\vec{F}_r = -k \cdot x \cdot \vec{i}_x$, with k the spring constant, which for this particular rubber spring is equal to $k = 9.450$ N/m, (1) did the set of keys slide off of the bottom of the magazine? (2) If they did, at what distance did the keys dangle from the bottom of the magazine? (3) If Jacques would have held the magazine in the same way when sitting in his living room at home, what would the results have been then? Remember that the universal gravitational constant G is equal to $G = 6.673 \times 10^{-11}$ m^3/(kg$\cdot$ s^2), and the mass and the radius of the Earth to $M_E = 5.9722 \times 10^{24}$ kg and $r_E = 6.3781 \times 10^6$ m, respectively.

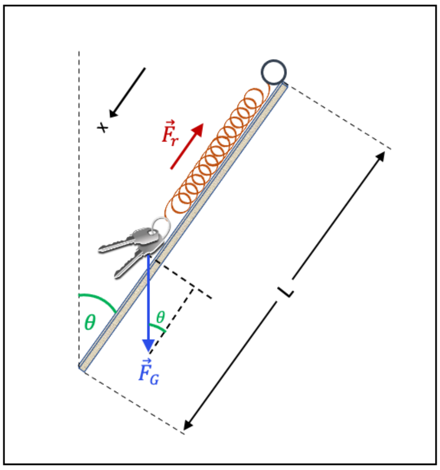

Figure 14

Solution

(1) Let us first determine the value of the gravitational acceleration $\vec{g}_d$ at the depth $d = 10,911$ m:

$$g_d = \frac{G \cdot M_E}{(r_E - d)^2} = \frac{6.673 \times 10^{-11} \cdot 5.9722 \times 10^{24}}{(6.3781 \times 10^6 - 10,911)^2} = 9.830 \text{m/s}^2$$

Due to the friction between the keys and the magazine, the keys would have reached at some point an equilibrium position, whereby the elastic rubber spring was stretched over a distance L_s with respect to its initial position. Applying Newton's second law (in the x-direction) to the set of keys in their equilibrium situation, we find the following value for L_s:

$$-k \cdot L_s + m_{sk} \cdot g_d \cdot \cos\theta = 0 \quad \Leftrightarrow \quad L_s = \frac{m_{sk} \cdot g_d \cdot \cos\theta}{k} = \frac{0.3850 \cdot 9.830 \cdot \cos(46.80°)}{9.450} = 27.42 \text{ cm}$$

Since the length of the magazine $L = 25.00$ cm was shorter than the equilibrium distance $L_s = 27.42$ cm of the set of keys, the keys would have fallen off of the bottom of the magazine.

(2) Under the angle of $\theta = 46.80°$, the remaining distance Δx to the equilibrium position of the keys was equal to $\Delta x = L_s - L = 27.42 - 25.00 = 2.415$ cm. However, at the moment when the keys fell off of the bottom of the magazine, they were subject to the gravitational acceleration g_d and not $g_d \cdot \cos\theta$. Therefore, if we take into account the full acceleration g_d, the remaining part of the rubber string, which was dangling vertically from the bottom of the magazine, stretched over the following distance h:

$$h = \frac{\Delta x}{\cos\theta} = \frac{2.415}{\cos(46.80°)} = 3.528 \text{ cm}$$

(3) At his home, Jacques would have experienced the gravitational acceleration at the surface of the Earth, which is equal to $g = 9.81$ m/s^2. The equilibrium distance L_{sh} would have been equal to:

$$L_{sh} = \frac{m_{sk} \cdot g \cdot \cos\theta}{k} = \frac{0.3850 \cdot 9.81 \cdot \cos(46.80°)}{9.450} = 27.36 \text{ cm}$$

In other words, at the surface of the Earth the rubber spring would stretch over a slightly lesser distance, which is equal to $L_s - L_{sh} = 27.42 - 27.36 = 5.619 \times 10^{-2}$ cm.

The vertical distance h_h at which the keys dangle from the bottom of the magazine would then become:

$$h_h = \frac{(L_{sh} - L)}{\cos\theta} = \frac{27.36 - 25.00}{\cos(46.80°)} = 3.446 \text{ cm}$$

The difference in vertical height in the two scenarios is equal to $h - h_h = 3.528 - 3.446 = 8.208 \times 10^{-2}$ cm. To check our results, if we multiply the difference in vertical height by the cosine of the angle θ, we must obtain the difference between the respective equilibrium distances:

$$(h - h_h) \cdot \cos\theta = L_s - L_{sh} \quad \Leftrightarrow \quad (8.21 \times 10^{-2}) \cdot \cos(46.80°) = 5.62 \times 10^{-2} \text{ cm} = L_s - L_{sh}$$

Exercise 16

Problem Statement

The four largest moons—called the Galilean moons—orbiting (anti-clockwise) around the planet Jupiter ($M_j = 1.898 \times 10^{27}$ kg) are among the largest within our Solar System. Of this quartet, the two innermost moons orbiting Jupiter are Io ($M_{io} = 8.93 \times 10^{22}$ kg) and Europa ($M_{eur} = 4.80 \times 10^{22}$ kg), whereby Io travels at a height of $h_{io} = 350,500$ km above Jupiter's surface. Suppose that about 6.5 years ago the China National Space Administration (CNSA) launched a space probe ($m_{sp} = 2,850$ kg), which just now successfully settled into Europa's orbit at a distance of roughly $s = 526,800$ km behind the moon with an orbital speed of $v_{sp} = 13,739$ m/s. (1) What is the net gravitational force $\vec{F}_G$ experienced by the probe when at the moment of arrival Io is located right above Jupiter whereas Europa makes an angle of $\theta = 45.0°$ with the horizontal? (2) Suppose that, after being in orbit for 21.3 hours, the CNSA decides to bring the probe into Io's orbit. It takes the probe 22.4 hours to reach Io's orbit at an angle of $\phi = 25.0°$ south of west. When the probe arrives at its new location, what angle does Io's position make with the vertical and at what distance is the probe ahead of or behind the moon Io? (3) At that moment, where is Europa located in its orbit with respect to both the vertical and the probe's position? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg· s^2) and that Jupiter's radius measures about $r_j = 7.15 \times 10^7$ m, and assume furthermore circular orbits.

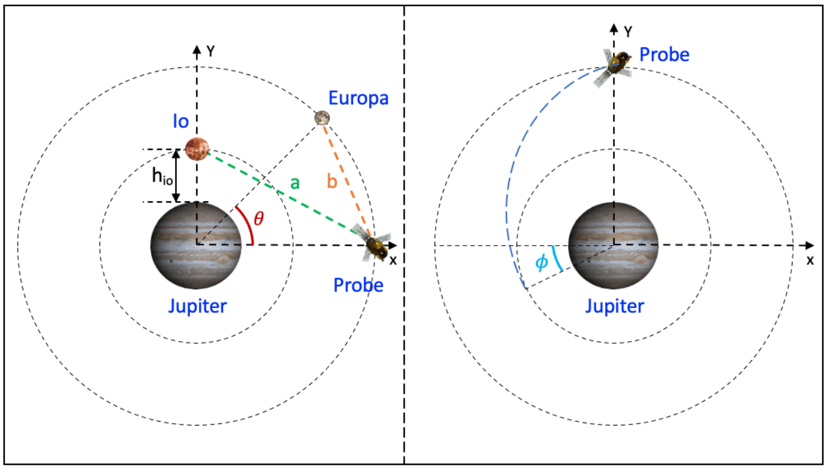

Figure 15

Solution

(1) At the moment when the probe enters into Europa's orbit, it undergoes a gravitational force of Jupiter, Io, and Europa according to our isolated system. In order to determine the net gravitational force $\vec{F}_G$ on the probe, we need to find the distances between the probe and the three masses. Let us start with the distance $d_{j,sp}$ between Jupiter and the space probe. From a rotating frame of reference put at the center of Jupiter, the gravitational force exerted on an orbiting body is balanced by the centrifugal force. As a result, given that the probe's orbital speed is equal to $v_{sp} = 13,739$ m/s, we find $d_{j,sp}$ as follows:

$$G \cdot \frac{M_j \cdot m_{sp}}{d_{j,sp}^2} = \frac{m_{sp} \cdot v_{sp}^2}{d_{j,sp}}$$

$$\Leftrightarrow \; d_{j,sp} = \frac{G \cdot M_j}{v_{sp}^2} = \frac{6.67 \times 10^{-11} \cdot 1.898 \times 10^{27}}{13,739^2} = 670,680 \text{ km}$$

The distance $d_{io,sp}$, which is equal to distance a in Fig. 15, between the moon Io and the probe can be calculated with the help of the Pythagorean theorem applied to the right-angled triangle between Jupiter, Io, and the probe:

$$d_{io,sp} = \sqrt{(r_j + h_{io})^2 + (d_{j,sp})^2} = \sqrt{(7.15 \times 10^7 + 3.505 \times 10^8)^2 + (6.7068 \times 10^8)^2} = 7.92 \times 10^8 \text{ m}$$

Still within the same triangle, the angle at the probe (let's call it γ) is equal to:

$$\gamma = \tan^{-1}\left(\frac{r_j + h_{io}}{d_{j,sp}}\right) = \tan^{-1}\left(\frac{7.15 \times 10^7 + 3.505 \times 10^8}{6.7068 \times 10^8}\right) = 32.2°$$

For the distance $d_{eur,sp}$ between the moon Europa and the probe, which corresponds to distance b in Fig. 15, we can apply the law of sines to the triangle formed between Jupiter, Europa, and the probe $\left(\frac{b}{\sin\theta} = \frac{d_{j,eur}}{\sin\alpha} = \frac{d_{j,sp}}{\sin\beta}\right)$, with $d_{j,eur}$ the distance between Jupiter and the moon Europa and α (β) the angle at the probe (the moon Europa). Since $d_{j,eur} = d_{j,sp}$, it follows from the law of sines that $\sin\alpha = \sin\beta$. As none of the angles within this triangle are larger than 90°, we have that $\alpha = \beta = 67.5°$. As per the law of sines, the distance $d_{eur,sp}$ is then calculated as follows:

$$d_{eur,sp} = b = d_{j,sp} \cdot \frac{\sin\theta}{\sin\beta} = 6.7068 \times 10^8 \cdot \frac{\sin(45.0°)}{\sin(67.5°)} = 5.13 \times 10^8 \text{ m}$$

We can now write the x-and y-components of the net gravitational force $\vec{F}_G$ exerted upon the space probe by the three massive bodies in the probe's vicinity:

x-direction

$$\vec{F}_{G,x} = \left[-\frac{G \cdot M_j \cdot m_{sp}}{d_{j,sp}^2} - \frac{G \cdot M_{io} \cdot m_{sp}}{a^2} \cdot \cos\gamma - \frac{G \cdot M_{eur} \cdot m_{sp}}{b^2} \cdot \cos\alpha\right] \cdot \vec{i}_x$$

$$= (-G \cdot m_{sp}) \cdot \left[\frac{M_j}{d_{j,sp}^2} + \frac{M_{io}}{a^2} \cdot \cos\gamma + \frac{M_{eur}}{b^2} \cdot \cos\alpha\right] \cdot \vec{i}_x$$

$$= (-6.67 \times 10^{-11} \cdot 2,850) \cdot \left[\frac{1.898 \times 10^{27}}{[6.7068 \times 10^8]^2} + \frac{8.93 \times 10^{22}}{[7.92 \times 10^8]^2} \cdot \cos(32.2°) + \frac{4.80 \times 10^{22}}{[5.13 \times 10^8]^2} \cdot \cos(67.5°)\right] \cdot \vec{i}_x$$

$$= -802 \cdot \vec{i}_x \text{ N}$$

y-direction

$$\vec{F}_{G,y} = \left[\frac{G \cdot M_{io} \cdot m_{sp}}{a^2} \cdot \sin\gamma + \frac{G \cdot M_{eur} \cdot m_{sp}}{b^2} \cdot \sin\alpha\right] \cdot \vec{i}_y$$

$$= (G \cdot m_{sp}) \cdot \left[\frac{M_{io}}{a^2} \cdot \sin\gamma + \frac{M_{eur}}{b^2} \cdot \sin\alpha\right] \cdot \vec{i}_y$$

$$= (6.67 \times 10^{-11} \cdot 2,850) \cdot \left[\frac{8.93 \times 10^{22}}{[7.92 \times 10^8]^2} \cdot \sin(32.2^\circ) + \frac{4.80 \times 10^{22}}{[5.13 \times 10^8]^2} \cdot \sin(67.5^\circ)\right] \cdot \vec{i}_y$$

$$= 0.0464 \cdot \vec{i}_y \text{ N}$$

The net gravitational force $\vec{F}_G$ is then equal to:

$$\vec{F}_G = -\left(\sqrt{F_{G,x}^2 + F_{G,y}^2}\right) \cdot \vec{i}_r = -\left(\sqrt{(-802)^2 + 0.0464^2}\right) \cdot \vec{i}_r = -802 \cdot \vec{i}_r \text{ N}$$

The vector $\vec{F}_G$ thereby makes an angle $\delta = \tan^{-1}\left(\frac{F_{G,y}}{|F_{G,x}|}\right) = \tan^{-1}\left(\frac{0.0464}{802}\right) = 0.00331^\circ$ with the horizontal. In other words, the net gravitational force $\vec{F}_G$ basically points in the negative x-direction towards the planet Jupiter, i.e., the center of the probe's orbit. As a check, we can indeed see that the net gravitational force $\vec{F}_G$ is balanced by the centrifugal force $\vec{F}_{cf}$:

$$\vec{F}_{cf} = \frac{m_{sp} \cdot v_{sp}^2}{d_{j,sp}} \cdot \vec{i}_r = \frac{2,850 \cdot 13,739^2}{6.7068 \times 10^8} \cdot \vec{i}_r = 802 \cdot \vec{i}_r \text{ N}$$

(2) One way to find the angular distance $\delta_{d,io}$, i.e., the number of degrees the moon Io has revolved around Jupiter after $t_{io} = 21.3 + 22.4 = 43.7$ hours, is the following (with T_{io} representing the period of Io):

$$\delta_{d,io} = \frac{t_{io}}{T_{io}} \cdot 360^\circ = \frac{t_{io}}{\sqrt{\frac{4 \cdot \pi^2}{G \cdot M_j} \cdot (r_j + h_{io})^3}} \cdot 360^\circ = \frac{43.7 \cdot 3,600}{\sqrt{\frac{4 \cdot \pi^2 \cdot (7.15 \times 10^7 + 3.505 \times 10^8)^3}{6.67 \times 10^{-11} \cdot 1.898 \times 10^{27}}}} \cdot 360^\circ = 369.955^\circ$$

In other words, during the time t_{io}, the moon Io has completed just a bit more than one revolution around Jupiter. Io's angular position $\delta_{p,io}$ with respect to the vertical is then equal to $\delta_{p,io} = 369.955^\circ - 360^\circ = 9.96^\circ$ west of north. This means that the probe is ahead of Io by an angular distance equal to $\Delta\delta_{io} = (90^\circ - \delta_{p,io}) + \phi = (90^\circ - 9.96^\circ) + 25^\circ = 105^\circ$. This corresponds with the following distance d (expressed in meters) between Io and the probe:

$$d = \frac{\Delta\delta_{io}}{360^\circ} \cdot [2 \cdot \pi \cdot (r_j + h_{io})] = \frac{105^\circ}{360^\circ} \cdot [2 \cdot \pi \cdot (7.15 \times 10^7 + 3.505 \times 10^8)] = 7.74 \times 10^5 \text{ km}$$

(3) As the orbital speed v_{eur} of the moon Europa is equal to $v_{eur} = v_{sp} = 13,739$ m/s, the distance d_{eur} covered by Europa during the time t_{io} is equal to:

$$d_{eur} = v_{eur} \cdot t_{io} = 13,739 \cdot (43.7 \cdot 3,600) = 2.16 \times 10^9 \text{ m}$$

The corresponding angular distance $\delta_{d,eur}$ is calculated as follows:

$$\delta_{d,eur} = \frac{d_{eur}}{2 \cdot \pi \cdot d_{j,eur}} \cdot 360° = \frac{2.16 \times 10^9}{2 \cdot \pi \cdot 6.7068 \times 10^8} \cdot 360° = 184.65°$$

In other words, the angular position $\delta_{p,eur}$ of Europa is equal to $\delta_{p,eur} = 45° - (184.65° - 180°) = 40.4°$ west of south. Note that with respect to the probe's angular position, i.e., $\phi = 25.0°$ south of west, Europa's angular position $\delta_{p,eur}$ becomes $\delta_{p,eur} = [(184.65° - 180°) + 45°] - 25° = 24.7°$ ahead of the probe (in the counterclockwise direction).

Exercise 17

Problem Statement

On a sunny Sunday afternoon, Micaela is practicing one of her favourite sport activities, i.e., clay target shooting, at the Club de Cazadores in Tucumán, Argentina. If the target (m_t = 105 g) leaves the shooting station, which is installed at d = 45.5 cm above the ground, with an initial speed of v_0 = 23.6 m/s under an angle of θ = 35.2° with the horizontal, while undergoing a drag force $\vec{F}_D = -b \cdot \vec{v}$ (with a drag coefficient of b = 0.0068 kg/s), at what distance h from the ground does the target find itself when it's at its highest point?

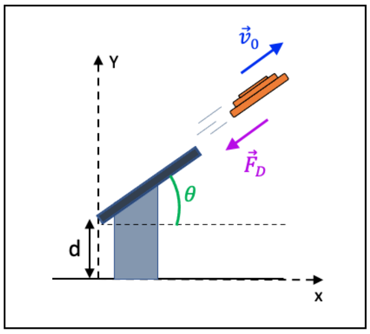

Figure 16

Solution

When applying Newton's second law to the target in the y-direction (the x-direction is irrelevant to our problem), we obtain the following equation:

$$-m_t \cdot g - b \cdot \sin\theta \cdot v = m_t \cdot a_y$$

As we need to find the height h, we must find an expression for h in terms of the time variable t. One way to tackle this problem is to directly integrate the above equation twice. We will apply another method based on differential equations. Keeping in mind that the acceleration a (the speed v) is equal to the second (first) derivative of the position y, the differential equation has the following form:

$$m_t \cdot a_y = -m_t \cdot g - b \cdot \sin\theta \cdot v \quad \Leftrightarrow \quad a_y = -g - \frac{b \cdot \sin\theta}{m_t} \cdot v$$

$$\Leftrightarrow \quad \frac{d^2y}{dt^2} = -g - \frac{b \cdot \sin\theta}{m_t} \cdot \frac{dy}{dt}$$

$$\Leftrightarrow \quad \frac{d^2y}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{dy}{dt} = -g$$

Let us solve this differential equation in two steps. First, we consider the so-called homogeneous solution, whereby we set the right-hand side of the above equation to zero:

$$\frac{d^2y}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{dy}{dt} = 0$$

A possible solution for this equation could be, for instance, $y = e^{\lambda \cdot t}$. Implementing our suggestion, the equation becomes:

$$\begin{aligned}\frac{d^2y}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{dy}{dt} = 0 \quad &\Leftrightarrow \quad \frac{d^2(e^{\lambda \cdot t})}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{d(e^{\lambda \cdot t})}{dt} = 0 \\ &\Leftrightarrow \quad \lambda^2 \cdot e^{\lambda \cdot t} + \frac{\lambda \cdot b \cdot \sin\theta}{m_t} \cdot e^{\lambda \cdot t} = 0 \\ &\Leftrightarrow \quad \lambda^2 + \frac{\lambda \cdot b \cdot \sin\theta}{m_t} = 0 \\ &\Leftrightarrow \quad \lambda_1 = 0 \text{ and } \lambda_2 = -\frac{b \cdot \sin\theta}{m_t}\end{aligned}$$

As both values for λ are a solution, any linear combination of $y = e^{\lambda \cdot t}$ for these two values is also a solution. Therefore, we can write the homogeneous solution y_h to our differential equation in the following general form (with c_1 and c_2 two constants):

$$\begin{aligned}y_h = c_1 \cdot e^{\lambda_1 \cdot t} + c_2 \cdot e^{\lambda_2 \cdot t} \quad &\Leftrightarrow \quad y_h = c_1 \cdot e^{0 \cdot t} + c_2 \cdot e^{-\frac{b \cdot \sin\theta}{m_t} \cdot t} \\ &\Leftrightarrow \quad y_h = c_1 + c_2 \cdot e^{-\frac{b \cdot \sin\theta}{m_t} \cdot t}\end{aligned}$$

The second step involves finding a particular solution y_p, when taking into account that the right-hand side of our original differential equation is not equal to zero (it is equal to "-g"). As a possible solution, we might try the expression $y_p = A \cdot t + B$, with A and B certain constants. Inserting this possible solution into our original differential equation, we obtain the following expression for the constant A:

$$\begin{aligned}\frac{d^2y_p}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{dy_p}{dt} = -g \quad &\Leftrightarrow \quad \frac{d^2(A \cdot t + B)}{dt^2} + \frac{b \cdot \sin\theta}{m_t} \cdot \frac{d(A \cdot t + B)}{dt} = -g \\ &\Leftrightarrow \quad 0 + \frac{b \cdot \sin\theta}{m_t} \cdot A = -g \\ &\Leftrightarrow \quad A = -\frac{m_t \cdot g}{b \cdot \sin\theta}\end{aligned}$$

If we add the particular solution y_p to the homogeneous solution y_h, we get the following complete solution to our differential equation:

$$y = y_h + y_p = \left[c_1 + c_2 \cdot e^{-\frac{b \cdot \sin\theta}{m_t} \cdot t}\right] + \left[-\frac{m_t \cdot g}{b \cdot \sin\theta} \cdot t + B\right]$$

We can find the value of the constants c_1, c_2, and B by taking into account the two initial conditions at $t = 0$ s, whereby the position of the target is equal to $y_0 = d$ and the speed (in the y-direction) is equal to $v_{0y} = v_0 \cdot \sin\theta$. For the first initial condition ($y_0 = d$), we get:

$$d = \left[c_1 + c_2 \cdot e^{-\frac{b\cdot\sin\theta}{m_t}\cdot 0}\right] + \left[-\frac{m_t \cdot g}{b \cdot \sin\theta} \cdot 0 + B\right] \quad \Leftrightarrow \quad d = c_1 + c_2 + B$$

For the second initial condition ($v_{0y} = v_0 \cdot \sin\theta$), we get:

$$v_0 \cdot \sin\theta = \left.\frac{dy}{dt}\right|_{t=0} \quad \Leftrightarrow \; v_0 \cdot \sin\theta = -c_2 \cdot \frac{b \cdot \sin\theta}{m_t} \cdot e^{-\frac{b\cdot\sin\theta}{m_t}\cdot 0} - \frac{m_t \cdot g}{b \cdot \sin\theta}$$

$$\Leftrightarrow \; c_2 = -\frac{m_t}{b \cdot \sin\theta} \cdot \left(\frac{m_t \cdot g}{b \cdot \sin\theta} + v_0 \cdot \sin\theta\right)$$

Given that $d = c_1 + c_2 + B$, the sum of the constants $c_1 + B$ becomes:

$$c_1 + B = d - c_2 = d + \frac{m_t}{b \cdot \sin\theta} \cdot \left(\frac{m_t \cdot g}{b \cdot \sin\theta} + v_0 \cdot \sin\theta\right)$$

If we insert these values for $c_1 + B$ and c_2 into our complete solution to our differential equation, we obtain the following expression:

$$y = d + \frac{m_t}{b \cdot \sin\theta} \cdot \left(\frac{m_t \cdot g}{b \cdot \sin\theta} + v_0 \cdot \sin\theta\right) \cdot \left(1 - e^{-\frac{b\cdot\sin\theta}{m_t}\cdot t}\right) - \frac{m_t \cdot g}{b \cdot \sin\theta} \cdot t$$

When the target is at its highest point during its trajectory, we know that the y-component of its velocity is equal to zero. The time t_{max} at which this occurs is calculated as follows:

$$\left.\frac{dy}{dt}\right|_{t=t_{max}} = 0$$

$$\Leftrightarrow \quad \left(\frac{b \cdot \sin\theta}{m_t}\right) \cdot \left(\frac{m_t}{b \cdot \sin\theta}\right) \cdot \left(\frac{m_t \cdot g}{b \cdot \sin\theta} + v_0 \cdot \sin\theta\right) \cdot e^{-\frac{b\cdot\sin\theta}{m_t}\cdot t_{max}} - \frac{m_t \cdot g}{b \cdot \sin\theta} = 0$$

$$\Leftrightarrow \quad \frac{b \cdot \sin\theta}{m_t \cdot g} \cdot \left(\frac{m_t \cdot g}{b \cdot \sin\theta} + v_0 \cdot \sin\theta\right) = e^{\frac{b\cdot\sin\theta}{m_t}\cdot t_{max}}$$

$$\Leftrightarrow \quad \left(1 + \frac{b \cdot v_0 \cdot \sin^2\theta}{m_t \cdot g}\right) = e^{\frac{b\cdot\sin\theta}{m_t}\cdot t_{max}}$$

$$\Leftrightarrow \quad \ln\left|1+\frac{b\cdot v_0\cdot \sin^2\theta}{m_t\cdot g}\right| = \frac{b\cdot\sin\theta}{m_t}\cdot t_{max}$$

$$\Leftrightarrow \quad t_{max} = \ln\left|1+\frac{b\cdot v_0\cdot \sin^2\theta}{m_t\cdot g}\right|^{\frac{m_t}{b\cdot\sin\theta}} = \ln\left|1+\frac{0.0068\cdot 23.6\cdot\sin^2(35.2^\circ)}{0.105\cdot 9.81}\right|^{\frac{0.105}{0.0068\cdot\sin(35.2^\circ)}} = 1.35 \text{ s}$$

Inserting this value for t_{max} into the above expression for y , we find the distance h from the ground at the time when the target is at its highest point:

$$y = d + \frac{m_t}{b\cdot\sin\theta}\cdot\left(\frac{m_t\cdot g}{b\cdot\sin\theta}+v_0\cdot\sin\theta\right)\cdot\left(1-e^{-\frac{b\cdot\sin\theta}{m_t}\cdot t_{max}}\right) - \frac{m_t\cdot g}{b\cdot\sin\theta}\cdot t_{max}$$

$$\Leftrightarrow \quad h = 0.455 + \frac{0.105}{0.0068\cdot\sin(35.2^\circ)}\cdot\left(\frac{0.105\cdot 9.81}{0.0068\cdot\sin(35.2^\circ)}+23.6\cdot\sin(35.2^\circ)\right)\cdot\left(1-e^{-\frac{0.0068\cdot\sin(35.2^\circ)}{0.105}\cdot 1.35}\right)$$

$$- \frac{0.105\cdot 9.81}{0.0068\cdot\sin(35.2^\circ)}\cdot 1.35$$

$$\Leftrightarrow \quad h = 9.57 \text{ m}$$

Exercise 18

Problem Statement

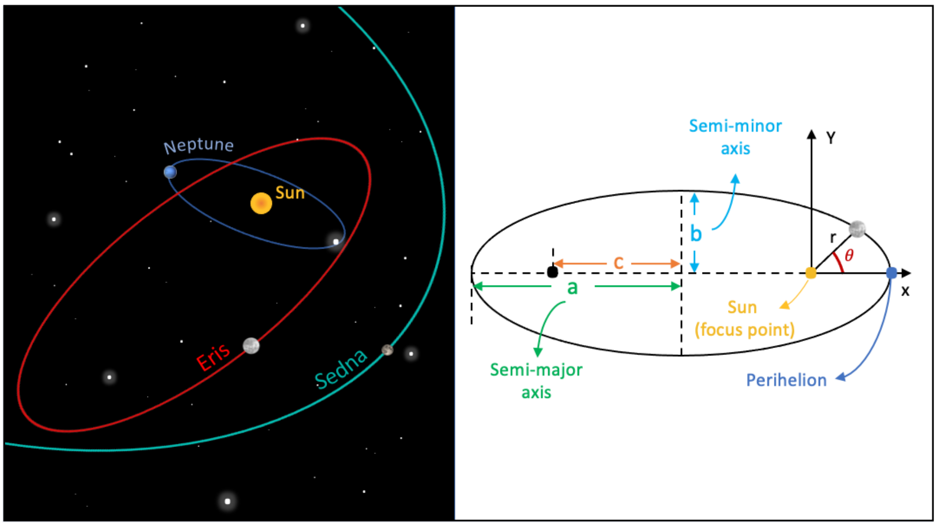

Figure 17

Trans-Neptunian Objects (TNOs) are dwarf planets (or minor planets) in the outer Solar System whereby their average orbiting distance to the Sun ($M_s = 1.99 \times 10^{30}$ kg) is larger than that of Neptune, i.e., the outermost planet within our Solar System. Eris and Sedna are two TNOs following elliptical trajectories around the Sun, whereby the orbit of the dwarf planet Eris, which has an elliptical eccentricity equal to $e_E = 0.436$ and a semi-minor axis of length $b_E = 9.14 \times 10^{12}$ m, is the consequence of historically significant gravitational interactions with Neptune—it is therefore assigned to the sub-classification of "scattered-disk objects". In contrast, due to the much larger orbit of the dwarf planet Sedna, which is most likely the result of a collision with some planet-sized object or star, Sedna is only marginally experiencing Neptune's gravitational influence and therefore belongs, arguably, to the sub-classification of "detached objects". (1) If the eccentricity of Sedna's orbit is 1.95 times greater with respect to Eris and if the distance between one of the foci and the center of Sedna's orbit is 14.54 times larger compared to Eris, how do the orbital velocities of these two dwarf planets compare at their perihelion, i.e., the point on their elliptical orbit closest to the massive body around which they orbit? Use the vis-viva equation, i.e., $v^2 = G \cdot M_s \cdot \left(\frac{2}{r} - \frac{1}{a}\right)$, with G the universal gravitational constant ($G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2)) and a the semi-major axis of the ellipse, to calculate the velocities. (2) What are the periods of the dwarf planets Eris and Sedna? For this problem, put the origin of the respective coordinate system in the focus point to the right of the center of the ellipse and use polar coordinates.

Solution

(1) Let us in the first place recall some of the properties of an elliptical orbit, when expressed in polar coordinates and the origin of the coordinate system is positioned in the right focus point:

General Properties	Elliptical Trajectory
$e^2 = 1 - \dfrac{b^2}{a^2}$	$r = \dfrac{a \cdot (1 - e^2)}{1 + e \cdot \cos\theta}$
$c = e \cdot a$	

In order to calculate the orbital velocities, we need to determine both the length a of the semi-major axis and the distance r_p between the Sun and the perihelion. Based on the general properties of an ellipse, we find the semi-major axis a_E of Eris' orbit as follows:

$$a_E = \frac{b_E}{\sqrt{1-e_E^2}} = \frac{9.14\times10^{12}}{\sqrt{1-0.436^2}} = 1.02\times10^{13}\ \text{m}$$

From Fig. 17 (right-hand side) we can see that the angle θ at the perihelion is equal to $\theta = 0°$. Therefore, using the equation of an elliptical trajectory, we can find the distance $r_{p,E}$ for Eris' orbit:

$$r_{p,E} = \frac{a_E\cdot(1-e_E^2)}{1+e_E} = a_E\cdot(1-e_E) = 1.02\times10^{13}\cdot(1-0.436) = 5.73\times10^{12}\ \text{m}$$

To calculate the length a_S of the semi-major axis of Sedna's orbit, we use the provided relationship between the eccentricities ($e_S = 1.95\cdot e_E$) as well as the distances c ($c_S = 14.54\cdot c_E$) of both dwarf planets. Based on the general properties of ellipses, we find a_S:

$$\begin{aligned}
&c_S = e_S\cdot a_s\\
\Leftrightarrow\ &[14.54\cdot c_E] = [1.95\cdot e_E]\cdot a_s\\
\Leftrightarrow\ &[14.54\cdot(e_E\cdot a_E)] = [1.95\cdot e_E]\cdot a_s\\
\Leftrightarrow\ &a_S = \frac{14.54}{1.95}\cdot a_E = \frac{14.54}{1.95}\cdot 1.02\times10^{13} = 7.57\times10^{13}\ \text{m}
\end{aligned}$$

The distance $r_{p,S}$ between the Sun and the perihelion of Sedna's orbit then becomes:

$$r_{p,S} = a_S\cdot(1-e_S) = a_S\cdot(1-[1.95\cdot e_E]) = 7.57\times10^{13}\cdot(1-[1.95\cdot0.436]) = 1.13\times10^{13}\ \text{m}$$

The orbital speed of Eris and Sedna, respectively, at their perihelion can now be found as follows:

$$\begin{cases}
v_{p,E} = \sqrt{G\cdot M_s\cdot\left(\frac{2}{r_{p,E}}-\frac{1}{a_E}\right)} = \sqrt{6.67\times10^{-11}\cdot1.99\times10^{30}\cdot\left(\frac{2}{5.73\times10^{12}}-\frac{1}{1.02\times10^{13}}\right)} = 5{,}770\ \text{m/s}\\[2ex]
v_{p,S} = \sqrt{G\cdot M_s\cdot\left(\frac{2}{r_{p,S}}-\frac{1}{a_S}\right)} = \sqrt{6.67\times10^{-11}\cdot1.99\times10^{30}\cdot\left(\frac{2}{1.13\times10^{13}}-\frac{1}{7.57\times10^{13}}\right)} = 4{,}650\ \text{m/s}
\end{cases}$$

Let us now consider the orbital speed $v_{p,E}$ of Eris as well as the distance $r_{p,E}$ to its perihelion relative to Sedna:

$$\begin{cases} \dfrac{v_{p,E}}{v_{p,S}} = \dfrac{5,770}{4,650} = 1.24 \\ \\ \dfrac{r_{p,E}}{r_{p,S}} = \dfrac{5.73 \times 10^{12}}{1.13 \times 10^{13}} = 0.505 \end{cases}$$

For circular orbits, we know that the orbital speed increases with a shorter radius $\left(v \sim \sqrt{\frac{1}{r}}\right)$, and from the vis-viva equation we expect a similar trend for elliptical orbits. Indeed, while the distance to the Sun from Eris' perihelion is half that of Sedna, the orbital speed at its perihelion is 24% higher.

(2) The definition of the period T in the case of elliptical orbits is the same as for circular orbits, except that the radius r is replaced by the semi-major axis a. In fact, both definitions of the period become equal to each other when we set the eccentricity in the expression for an elliptical trajectory equal to zero, which is the case for a circle, whereby we find that $r = a$. The period of Eris and Sedna is calculated as follows, respectively (whereby 1 year counts 365.25 days):

$$\begin{cases} T_E = 2 \cdot \pi \cdot \sqrt{\dfrac{a_E^3}{G \cdot M_s}} = 2 \cdot \pi \cdot \sqrt{\dfrac{(1.02 \times 10^{13})^3}{6.67 \times 10^{-11} \cdot 1.99 \times 10^{30}}} = 1.77 \times 10^{10} \text{ s or } 559 \text{ years} \\ \\ T_S = 2 \cdot \pi \cdot \sqrt{\dfrac{a_S^3}{G \cdot M_s}} = 2 \cdot \pi \cdot \sqrt{\dfrac{(7.57 \times 10^{13})^3}{6.67 \times 10^{-11} \cdot 1.99 \times 10^{30}}} = 3.59 \times 10^{11} \text{ s or } 11,400 \text{ years} \end{cases}$$

The dwarf planet Sedna is currently at a distance of $r = 1.26 \times 10^{13}$ m away from the Sun at an angle of $\theta = 37.8°$ south of east, and it will reach its perihelion on 9 March 2076 or, according to other sources, on 18 July 2076. The dwarf planet Eris, on the other hand, will reach its perihelion on 22 December 2259 (or, some say, 7 December 2257), and it is now located at $r = 1.43 \times 10^{13}$ m from the Sun at an angle of $\theta = 12.1°$ south of west. Note that the angles refer to the angle θ of our coordinate system in Fig. 17.

Exercise 19

Problem Statement

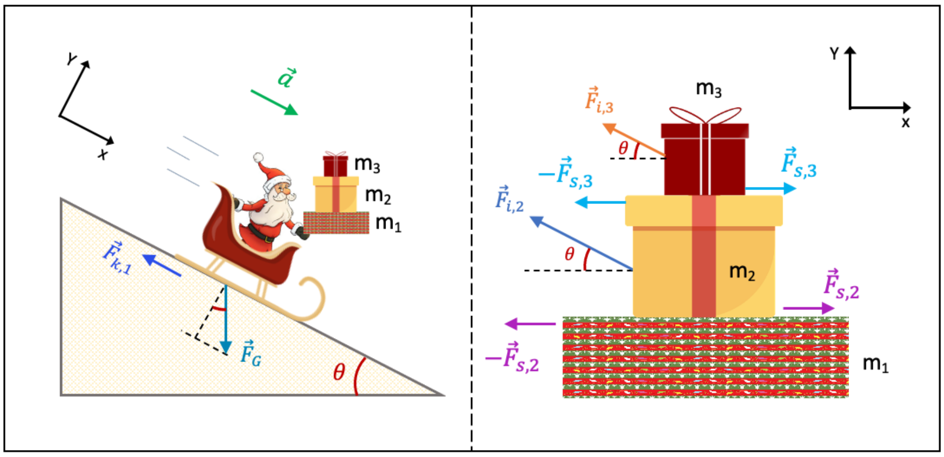

Figure 18

During this cold and snowy month of December in Erzurum, Turkey, Mehmet (m_M = 72.5 kg) has dressed up as Noel Baba to bring his little brother Omer some long-desired gifts. Mehmet wants to do it in style, so he takes his sled (m_s = 5.50 kg) and slides down the incline—which makes an angle of $\theta = 16.4°$ with the horizontal—behind their house while holding three gifts (m_1 = 3.50 kg, m_2 = 2.50 kg, and m_3 = 1.50 kg), all stacked on top of each other. (1) If the kinetic friction coefficient for the sled on snow is equal to $\mu_{k,s} = 0.0455$ and the static friction coefficient for paper on paper to $\mu_s = 0.545$, how will gift 2 and 3 behave relative to gift 1? (2) What is the minimum value that μ_s should have if the gifts have to remain steady? (3) Suppose that μ_s has a value of 95% of the minimum value established in part (2) and that the kinetic friction coefficient $\mu_{k,2}$ between gift 1 and 2 is equal to 75% of this minimum value and the coefficient $\mu_{k,3}$ between gift 2 and 3 to $\mu_{k,3} = \frac{\mu_{k,2}}{2}$. How do the gifts behave now?

Solution

(1) In a first instance, let us calculate the magnitude of the acceleration $\vec{a}$ of the system "Mehmet plus sled plus the three gifts" ($m_{tot} = m_M + m_s + m_1 + m_2 + m_3 = 72.5 + 5.50 + 3.50 + 2.50 + 1.50 = 85.0$ kg) sliding down the incline. Applying Newton's second law in both the x- and y-direction to this system, we obtain the following two equations:

x-direction	y-direction
$F_G \cdot \sin\theta - F_{k,1} = m_{tot} \cdot a$	$F_N - F_G \cdot \cos\theta = 0$
$\Leftrightarrow (m_{tot} \cdot g) \cdot \sin\theta - \mu_{k,s} \cdot F_N = m_{tot} \cdot a$	$\Leftrightarrow F_N - (m_{tot} \cdot g) \cdot \cos\theta = 0$

We then find the magnitude of the acceleration $\vec{a}$ as follows:

$$m_{tot} \cdot a = (m_{tot} \cdot g) \cdot \sin\theta - \mu_{k,s} \cdot [(m_{tot} \cdot g) \cdot \cos\theta]$$

$$\Leftrightarrow a = g \cdot (\sin\theta - \mu_{k,s} \cdot \cos\theta) = 9.81 \cdot [\sin(16.4°) - 0.0455 \cdot \cos(16.4°)] = 2.34 \text{ m/s}^2$$

If we now consider gift 2 and 3 from the perspective of the system "Mehmet plus sled plus gift number 1", which is an accelerating frame of reference, then they will experience an inertial force $\vec{F}_{i,2}$ and $\vec{F}_{i,3}$, respectively, pointing in the opposite direction of the acceleration vector $\vec{a}$. Before we proceed, let us first determine the magnitude of both the normal force $\vec{F}_{N,3}$ and $\vec{F}_{N,2}$ for gift 3 and 2, respectively, by applying Newton's second law in the y-direction (remember that gift 2 experiences a downwards pointing contact force $\vec{F}_c$ (not drawn) due to gift 3 which is, as per Newton's third law, equal in magnitude and opposite in direction to the normal force $\vec{F}_{N,3}$, so that $\vec{F}_c = -\vec{F}_{N,3}$):

$$\begin{cases} F_{N,3} - m_3 \cdot g + F_{i,3} \cdot \sin\theta = 0 \\ \Leftrightarrow F_{N,3} = m_3 \cdot g - (m_3 \cdot a) \cdot \sin\theta = 1.50 \cdot 9.81 - (1.50 \cdot 2.34) \cdot \sin(16.4°) = 13.7 \text{ N} \\ \\ F_{N,2} - F_{N,3} - m_2 \cdot g + F_{i,2} \cdot \sin\theta = 0 \\ \Leftrightarrow F_{N,2} = F_{N,3} + m_2 \cdot g - (m_2 \cdot a) \cdot \sin\theta = 13.7 + 2.50 \cdot 9.81 - (2.50 \cdot 2.34) \cdot \sin(16.4°) = 36.6 \text{ N} \end{cases}$$

To determine whether gift 2 and 3 will start sliding, we have to compare the magnitude of (the x-component of) $\vec{F}_{i,2}$ and $\vec{F}_{i,3}$, respectively, with that of the relevant friction forces (since gift 2 has physical contact with two other gifts, it experiences two friction forces, i.e., one at the bottom and one at the top):

Inertial force (Gift 2)

$$\begin{aligned} F_{i,2,x} &= m_2 \cdot a \cdot \cos\theta \\ &= 2.50 \cdot 2.34 \cdot \cos(16.4°) \\ &= 5.62 \text{ N} \end{aligned}$$

Static friction forces (Gift 2)

$$\begin{aligned} F_{s,2} - F_{s,3} &= \mu_s \cdot F_{N,2} - \mu_s \cdot F_{N,3} \\ &= 0.545 \cdot 36.6 - 0.545 \cdot 13.7 \\ &= 12.5 \text{ N} \end{aligned}$$

Inertial force (Gift 3)

$$\begin{aligned} F_{i,3,x} &= m_3 \cdot a \cdot \cos\theta \\ &= 1.50 \cdot 2.34 \cdot \cos(16.4°) \\ &= 3.37 \text{ N} \end{aligned}$$

Static friction forces (Gift 3)

$$\begin{aligned} F_{s,3} &= \mu_s \cdot F_{N,3} \\ &= 0.545 \cdot 13.7 \\ &= 7.48 \text{ N} \end{aligned}$$

Given that the x-component of both the inertial forces $\vec{F}_{i,2}$ and $\vec{F}_{i,3}$ is unable to overcome the respective static friction forces, neither gift 2 nor gift 3 will slip and therefore both remain in position with respect to gift 1.

(2) The minimum value for the static friction coefficient μ_s can be found by applying Newton's second law in the x-direction to, for instance, gift 2:

$$- F_{i,2} \cdot \cos\theta + F_{s,2} - F_{s,3} = 0$$

$$\Leftrightarrow \; - (m_2 \cdot a) \cdot \cos\theta + \mu_{s,min} \cdot F_{N,2} - \mu_{s,min} \cdot F_{N,3} = 0$$

$$\Leftrightarrow \mu_{s,min} = \frac{(m_2 \cdot a) \cdot \cos\theta}{F_{N,2} - F_{N,3}} = \frac{(2.50 \cdot 2.34) \cdot \cos(16.4°)}{36.6 - 13.7} = 0.246$$

(3) With a new value for μ_s equal to $\mu_{s,*} = 0.95 \cdot \mu_{s,min} = 0.95 \cdot 0.246 = 0.233$, which is lower than the minimum value, we know that gift 2 will now start sliding to the left with net acceleration $\vec{a}_2$. Given that $\mu_{k,2} = 0.75 \cdot \mu_{s,min} = 0.75 \cdot 0.246 = 0.184$ and $\mu_{k,3} = \frac{\mu_{k,2}}{2} = \frac{0.184}{2} = 0.0921$, we can determine the magnitude of the acceleration $\vec{a}_2$ (relative to gift number 1) by applying Newton's second law in the x-direction to gift 2, whereby $\vec{F}_{k,2}$ and $\vec{F}_{k,3}$ represent the kinetic friction forces between gift 1 and 2 and gift 2 and 3, respectively:

$$- F_{i,2} \cdot \cos\theta + F_{k,2} - F_{k,3} = m_2 \cdot a_2$$

$$\Leftrightarrow \; - (m_2 \cdot a) \cdot \cos\theta + \mu_{k,2} \cdot F_{N,2} - \mu_{k,3} \cdot F_{N,3} = m_2 \cdot a_2$$

$$\Leftrightarrow a_2 = -a \cdot \cos\theta + \frac{\mu_{k,2}}{m_2} \cdot F_{N,2} - \frac{\mu_{k,3}}{m_2} \cdot F_{N,3}$$

$$= -2.34 \cdot \cos(16.4°) + \frac{0.184}{2.50} \cdot 36.6 - \frac{0.0921}{2.50} \cdot 13.7$$

$$= -0.0562 \text{ m/s}^2$$

Similarly, the net acceleration $\vec{a}_3$ for gift 3 (with respect to gift 1) is found as follows:

$$- F_{i,3} \cdot \cos\theta + F_{k,3} = m_3 \cdot a_3$$

$$\Leftrightarrow \; - (m_3 \cdot a) \cdot \cos\theta + \mu_{k,3} \cdot F_{N,3} = m_3 \cdot a_3$$

$$\Leftrightarrow a_3 = -a \cdot \cos\theta + \frac{\mu_{k,3}}{m_3} \cdot F_{N,3}$$

$$= -2.34 \cdot \cos(16.4°) + \frac{0.0921}{1.50} \cdot 13.7$$

$$= -1.40 \text{ m/s}^2$$

Relative to gift 1, gift 3 moves faster to the left than gift 2, and under this scenario, gift 3 will fall off of the top of gift 2 quite quickly. If we wish to know the accelerations relative to the ground, we add the term $a_x = a \cdot \cos\theta = 2.34 \cdot \cos(16.4°) = 2.25 \text{ m/s}^2$ to the above values, so that we obtain $a_2 = 2.19 \text{ m/s}^2$ and $a_3 = 0.842 \text{ m/s}^2$.

Exercise 20

Problem Statement

Amina is doing postdoctoral research at the Sultan Qaboos University, in Muscat, Oman, whereby she specializes in binary star systems, i.e., gravitationally bound systems in which two stars orbit around their common center of mass called the barycenter (x_{bc}). Amina is currently studying data from the Lepus constellation, which lies at a declination of 20° south of the celestial equator, and has identified a new binary star system of circular orbits. Star 1 has a mass of $m_1 = 1.45 \cdot M_z$, with M_z the mass of the star Zeta Leporis and equal to $M_z = 1.46 \cdot M_s$ (whereby the mass of the Sun measures $M_s = 1.99 \times 10^{30}$ kg), whereas the mass of star 2 is equal to $m_2 = 3.20 \cdot M_z$. Amina has furthermore calculated that the stars complete one orbit in exactly 166 days. (1) What distance did Amina measure between both stars? Express your answer in terms of the Earth-Sun distance $r_{es} = 1.496 \times 10^8$ km. (2) What value does Amina find for the orbital velocity of each star? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg· s^2).

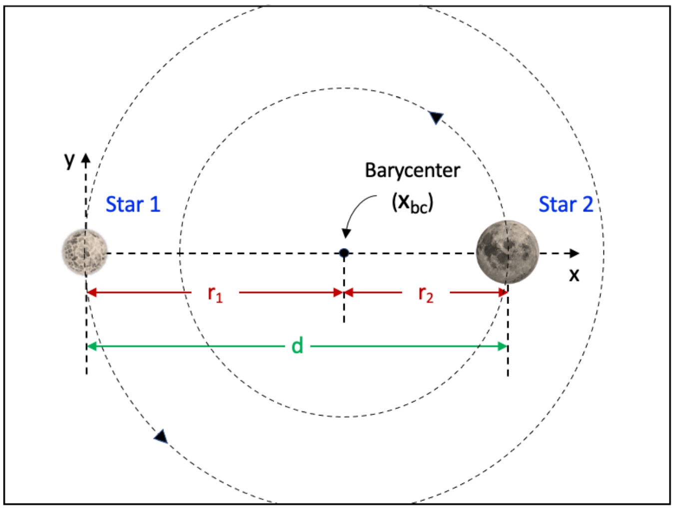

Figure 19

Solution

(1) Let us in a first instance calculate the mass M_z of the star Zeta Leporis:

$$M_z = 1.46 \cdot M_s = 1.46 \cdot 1.99 \times 10^{30} = 2.91 \times 10^{30} \text{ kg}$$

In order to determine the radii r_1 and r_2 of the two stars orbiting around the barycenter, we first apply the definition of the center of mass with the origin of our coordinate system positioned at the center of star 1 to locate the barycenter:

$$x_{bc} = \frac{m_1 \cdot x_1 + m_2 \cdot x_2}{m_1 + m_2}$$

$$= \frac{(1.45 \cdot M_z) \cdot 0 + (3.20 \cdot M_z) \cdot d}{(1.45 \cdot M_z + 3.20 \cdot M_z)}$$

$$= \frac{3.20}{4.65} \cdot d = 0.688 \cdot d$$

As a result, the orbital radius of star 1 and star 2 is equal to $r_1 = 0.688 \cdot d$ and $r_2 = d - r_1 = d - 0.688 \cdot d = 0.312 \cdot d$, respectively.

In a next step, we apply Newton's second law to star 1 from the perspective of a reference frame positioned at the barycenter that is rotating along with the two stars. Within such frame, the gravitational force between both stars is balanced by an inertial force, i.e., the centrifugal force, so that we can write:

$$G \cdot \frac{m_1 \cdot m_2}{d^2} = \frac{m_1 \cdot v_1^2}{r_1} \quad \Leftrightarrow \quad G \cdot \frac{(1.45 \cdot M_z) \cdot (3.20 \cdot M_z)}{d^2} = \frac{(1.45 \cdot M_z) \cdot v_1^2}{0.688 \cdot d}$$

If we now insert the expression for the orbital velocity $v_1 = \frac{2 \cdot \pi \cdot r_1}{T}$ into the above equation, we can calculate the distance d between star 1 and star 2:

$$G \cdot \frac{(1.45 \cdot M_z) \cdot (3.20 \cdot M_z)}{d^2} = \frac{(1.45 \cdot M_z)}{0.688 \cdot d} \cdot \left[\frac{2 \cdot \pi \cdot (0.688 \cdot d)}{T}\right]^2$$

$$\Leftrightarrow \; G \cdot \frac{3.20 \cdot M_z}{d^2} = \frac{4 \cdot \pi^2 \cdot (0.688 \cdot d)}{T^2}$$

$$\Leftrightarrow \; d = \sqrt[3]{\frac{3.20 \cdot G \cdot M_z \cdot T^2}{4 \cdot \pi^2 \cdot 0.688}}$$

$$= \sqrt[3]{\frac{3.20 \cdot 6.67 \times 10^{-11} \cdot 2.91 \times 10^{30} \cdot (166 \cdot 24 \cdot 3,600)^2}{4 \cdot \pi^2 \cdot 0.688}}$$

$$= 1.67 \times 10^{11} \text{ m}$$

$$= \frac{1.67 \times 10^{11}}{r_{es}} \, r_{es} = \frac{1.67 \times 10^{11}}{1.496 \times 10^{11}} \, r_{es} = 1.12 \, r_{es}$$

(2) The orbital velocities of the stars are calculated as follows:

$$\begin{cases} v_1 = \dfrac{2 \cdot \pi \cdot r_1}{T} = \dfrac{2 \cdot \pi \cdot (0.688 \cdot 1.67 \times 10^{11})}{166 \cdot 24 \cdot 3,600} = 5.05 \times 10^4 \text{ m/s} \\[2ex] v_2 = \dfrac{2 \cdot \pi \cdot r_2}{T} = \dfrac{2 \cdot \pi \cdot (0.312 \cdot 1.67 \times 10^{11})}{166 \cdot 24 \cdot 3,600} = 2.29 \times 10^4 \text{ m/s} \end{cases}$$

Since the period is the same for both stars, it indeed makes sense that the (heavier) star with the smaller orbital radius, i.e., star 2, has the lower orbital velocity.