# Physics

Exercises on Work, Energy, and Momentum

# Disclaimer



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# Summary of Exercises

### *Exercise 1*

On a rainy Saturday afternoon, Artem ($m = 69.4$ kg) is feeling rebellious and climbs the $h = 14.0$ m-high, egg-shaped main building of the Pysanka Museum in Kolomyia, Ukraine. Once sitting on top, Artem has difficulties holding on to the wet surface, so he starts to slip and slides down. (1) If the building is approximately elliptical in shape with an elliptical eccentricity of $e = 0.750$ and if Artem loses contact with the side of the egg when the elliptical radius r makes an angle of $\theta = 78.6°$ west of north, what is the magnitude of his velocity $\vec{v}$ in that moment? (2) If the bottom side of the egg sits $b = 2.50$ m below the ground, at what height $h_p$ from the pavement does Artem find himself when he becomes detached from the building? Ignore any kind of friction for this problem.

### *Exercise 2*

During radioactive decay, either the atomic composition of the nucleus of a chemical element is fundamentally altered—this is the case for $\alpha$- and $\beta$-decay—or an element lowers its nuclear energy levels ($\gamma$-emission), emitting thereby high-energy radiation. In one of the several nuclear chain reactions called the Thorium Series, the unstable element thorium ($^{232}_{90}Th$) decays through a series of events until it is transformed into the stable element lead ($^{208}_{82}Pb$). One of the intermediary steps includes an $\alpha$-decay of radon's 220-isotope ($^{220}_{86}Rn$) into polonium's 216-isotope ($^{216}_{84}Po$) whereby an $\alpha$-particle, i.e., the nucleus of a helium (He) atom, is emitted. Suppose now that two $\alpha$-particles ($m_\alpha = 6.64 \times 10^{-27}$ kg) elastically collide at a speed of $v_{1,i} = 14.2 \times 10^6$ m/s and $v_{2,i} = 14.8 \times 10^6$ m/s, respectively. The velocity $\vec{v}_{1,i}$ of the first particle $\alpha_1$ is initially making an angle of $\theta_{1,i} = 135°$ with the z-axis, whereby the angle $\phi_{1,i}$, i.e., the angle between the projection onto the xy-plane and the x-axis, is equal to $\phi_{1,i} = 65.4°$. Regarding the second particle $\alpha_2$, the respective angles are equal to $\theta_{2,i} = 66.0°$ and $\phi_{2,i} = 153°$. (1) If you know that after the collision the angle with the z-axis is equal to $\theta_{1,f} = 49.13°$ and $\theta_{2,f} = 123.21°$, respectively, what is the final speed of both particles, i.e., $v_{1,f}$ and $v_{2,f}$? (2) In which direction are $\alpha_1$ and $\alpha_2$ now headed?

### *Exercise 3*

You own the private company called Satplans Science Ltd., dedicated to gathering and processing scientific data from the four innermost planets in our Solar System. As a result of healthy working capital levels, you are able to install the satellite Suzy 3 ($m = 4,630$ kg) in a perfectly circular areosynchronous equatorial orbit (AEO) around the planet Mars—an AEO is the Martian equivalent of a geostationary orbit around Earth—despite the significant orbital station keeping costs due to the gravitational impact of the planet's two moons, i.e., Phobos and Deimos. If you consider the system "Suzy 3", (1) what is the work done on the satellite? (2) Is linear momentum conserved? (3) Write a general formula for the work done by the satellite's engine when changing orbit. (4) Suppose that Suzy 3 is guided towards a new orbit with a radius 60% of its original. How much work has Suzy 3's engine performed? (5) What is the total amount of work done on the system? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$) and the mass, the radius, and the rotation period of Mars to $M = 6.417 \times 10^{23}$ kg, $r = 3.396 \times 10^6$ m, and $T = 24\text{h } 37\text{min } 22.7\text{s}$, respectively.

## *Exercise 4*

It's mid-June and Estée is currently taking the final exam of her AP Physics 1 course at the American School of Madrid in Pozuelo de Alarcón, Spain. The weather is particularly hot today and she has 20 minutes left to answer the final question of her exam. Luckily, as it is one of her most favourite courses, Estée prepared thoroughly for this exam and with her acute sense of focus she finishes the question under 10 minutes, despite the oppressive heat. The final question was the following. A pulley system with three blocks A ($m_A = 4.75$ kg), B ($m_B = 3.50$ kg), and C ($m_C$) is presented in Fig. 5, whereby mass C is hanging $d = 75.4$ cm from the top of the incline, whose length is equal to $L = 2.52$ m and makes an angle of $\theta = 64.2°$ with the horizontal. The surface under mass A generates a kinetic friction coefficient of $\mu_{k1} = 0.453$ with the block, whereas the incline has a rougher surface and therefore produces a higher kinetic friction coefficient of $\mu_{k2} = 0.678$ with mass B. Initially, someone is preventing block A from moving and when they release the block, mass C is accelerating downwards until it hits the ground. If you know that the total work done on block C during its displacement is equal to $W = 5.54$ J, (1) determine the mass of block C and (2) its speed when it hits the ground. What answers did Estée find?

## *Exercise 5*

Duško ($m_D$) is enjoying his winter holidays in the Kamnik-Savinja Alps in the north of Slovenia. The seasoned skier that he is, Duško loves going off-piste to explore and carve out new paths. At a certain point, he is standing on top of a hill and notices that further down the ski run is interrupted by a large gap, after which the path continues. Just before the gap, the slope goes back up and at the end of the upward slope, at the very edge of the gap, there are three naturally formed ramps, which make an angle of $\theta_1 = \frac{7\cdot\pi}{36}$, $\theta_2 = \frac{\pi}{4}$, and $\theta_3 = \frac{11\cdot\pi}{36}$ with the horizontal, respectively. If you know that the distance between the bottom of the hill and the point where Duško is currently standing is the minimal height required to gain sufficient speed to cross the gap, which one of the three ramps should Duško choose to safely reach the other side? Assume that the edges at both sides of the gap are at the same height and ignore any friction or drag forces for this problem.

## *Exercise 6*

For the past two hours, Chanmony ($m_C = 74.8$ kg) has been guiding her paraglider over the rural outskirts of the Kampong Chhnang province, Cambodia, enjoying the undulating paddy fields, the meandering Tonle Sap River, and the hilly landscapes in the west. Meanwhile, Ponnleu ($m_P = 69.6$ kg) is taking up the beautiful scenery from a lower altitude, steering her mountain bike across several dusty village roads. At one point, Chanmony is descending at a constant velocity with a magnitude of $v_C = 7.82$ m/s in the southeastern direction ($\phi = 40.8°$ south of east) under an angle of $\theta = 15.5°$ with the horizontal, and is about to land near the roadside on the opposite side of a village road that lies parallel to the east-west axis. However, right at the moment when Chanmony flies over the road, Ponnleu, who was initially going at a speed of $v_0 = 4.25$ m/s and has been accelerating ($a = 0.507$ m/s$^2$) for the past $\Delta x = 200$ m, is all caught up in her own world, not paying attention to her surroundings, and fails to see Chanmony coming from the northwestern direction. Both collide, but somehow still manage to hold on to each other and roll entangled for a distance $d$ in the field near the road until they come to a halt, thanks to the kinetic friction with the grass ($\mu_k = 0.439$). (1) What is the velocity of Chanmony and Ponnleu rolling together right after the collision? (2) What distance do they need to come to a stop?

## ***Exercise 7***

Most of the leftover debris from the days when the Solar System was being formed is orbiting in a large torus-shaped disk either between the planets Mars and Jupiter—called the asteroid belt— or beyond the outermost planet Neptune—this disk is referred to as the Kuiper belt, whose main region's width is about 20 times the distance between the Earth and the Sun. The largest and most massive object that belongs to the Kuiper belt is the dwarf planet Pluto with a mass of $M = 1.30 \times 10^{22}$ kg and a radius of $r = 1.19 \times 10^6$ m. Its flimsy gaseous atmosphere mainly consists of nitrogen ($N_2$), methane ($CH_4$), and carbon monoxide (CO), stretching out at some places as high as 1,600 km. Suppose that a massive rock ($m = 2,750$ kg) is knocked out from its orbit within the Kuiper belt and is headed straight towards Pluto. When it is $h = 25.0$ km away from Pluto's surface, the rock has a velocity of $\vec{v}_1 = -339 \cdot \vec{i}_y$ m/s, and despite the thin atmosphere, the rock experiences a drag force, which has the form of $\vec{F}_D = b \cdot v^2 \cdot \vec{i}_y$. A little over a minute later, the rock hits Pluto's surface at a velocity of $\vec{v}_2 = -368 \cdot \vec{i}_y$ m/s. (1) Use calculus to derive an expression for the work $W_D$ done by the drag force $\vec{F}_D$ on the rock. (2) What is the value of $W_D$? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$).

## ***Exercise 8***

As her parents had to go and do some errands for about an hour, Semira is babysitting her three-year-old baby brother Jemal in their home right behind Fiat Tagliero in Asmara, Eritrea. Jemal loves to play with the colourful rubber toy spring (with a length of $L = 45.0$ cm) that his sister bought him for his recent birthday, and Semira wants to show him a new trick. She places the spring horizontally on the kitchen floor with one end leaning against the plinth of a cupboard and presses the spring together over a distance $\Delta x_1$. Semira then places two small plastic blocks of mass $m_1 = 0.15$ kg and $m_2 = 0.35$ kg on top of each other (the lightest one goes on top) and puts them in front of the compressed spring. When Semira lets go of the blocks, the spring shoots them forward across the kitchen floor to the great amusement of Jemal.

(1) What is the maximum distance that Semira should compress the spring so that the upper block stays put when being released? (2) What is the total work done on the two blocks combined during this displacement? (3) When the spring reaches its equilibrium position, the two blocks become detached from the spring. What is their speed at that moment? (4) How far ($\Delta x_2$) do the blocks slide across the kitchen floor? (5) Does the upper block still remain steady during $\Delta x_2$? The spring constant k is equal to $k = 10.3$ N/m, and assume that the kinetic friction coefficient between the lower block and the kitchen floor is equal to $\mu_{k1} = 0.065$, and that the kinetic (static) friction coefficient between the two plastic blocks equals $\mu_{k2} = 0.115$ ($\mu_s = 0.225$).

## *Exercise 9*

In Nur-Sultan, the capital city of Kazakhstan, Sarsen ($m = 57.5$ kg) is trying out the professional skatepark of the newly created green urban area, which also includes city parks, an outdoor cinema, a beach area, bike lanes and pedestrian esplanades. One of the skatepark's main attractions is a looping installed at the end of a long inclined run-up track, which makes an angle of $\theta = 12.5°$ with the ground. As a safety measure, an elastic rubber rope is hanging from the top at both sides of the looping. If the wheels of the skateboard only create kinetic friction with the track ($\mu_k = 0.112$)—the friction with the surface of the looping is negligible—and given an inner radius of the looping equal to $R = 3.55$ m, (1) what minimum distance $L$ should Sarsen walk up the track in order to successfully go through the looping? (2) Suppose that Sarsen did not attain sufficient speed and loses contact with the surface of the looping when he is a horizontal distance of $d = 1.00$ m away from the rubber rope. Luckily, he manages to get hold of the bottom end of the rubber rope, which has a length of $s = 1.00$ m and stretches according to the spring force $\vec{F}_x = -k \cdot \vec{x}$ ($k = 357$ N/m), and falls down vertically. How far from the ground is Sarsen when the rope is maximally stretched right after his fall?

## *Exercise 10*

Elina is a promising young billiard player who is currently participating in a local tournament in Brèst, Belarus. She managed to reach the finals and is now in a position where she can win the tournament. That is, only if Elina is able to pocket the last two billiard balls with just one single stroke. The (white) cue ball is located near the right edge, whereas the last object ball, i.e., the solid red number 3, finds itself in front of the top right pocket, i.e., $d_1 = 7.60$ cm from the right edge and $d_2 = 27.3$ cm from the top edge. The (black) 8 ball is positioned close to the top left pocket, i.e., $d_3 = 5.85$ cm from the top edge and $d_4 = 20.4$ cm from the left edge. Elina holds the cue stick in such a way that it makes an angle $\theta_1$ with the right edge and gives the cue ball an initial speed of $v_{c,i}$. If Elina first pockets ball number 3 and subsequently the 8 ball with just one shot, (1) what speed $v_{c,i}$ should she give the cue ball? (2) What angle $\theta_1$ does Elina's cue stick make with the right edge? Assume that the solid red ball and the 8 ball enter their respective pocket with a speed of $v_{3,f} = 1.25$ m/s and $v_{8,f} = 0.86$ m/s and that the three billiard balls all have the same mass $m = 165$ g. Ignore any kind of friction.

## *Exercise 11*

In the experimental classroom of the University of Costa Rica in the capital city of San José, Samuel is putting his knowledge on the laws of physics into practice. One of the experiments consists of two large, differently shaped, frictionless ramps put right next to each other, whereby two small steel bearing balls ($m = 0.354$ kg) are released simultaneously from the top of the slope (one ball for each ramp). The purpose of this particular experimental design is to demonstrate pratically how the law of energy conservation is at work. One of the ramps follows a straight path, whereas the other has an elliptical shape—in fact, it is the bottom left segment of an ellipse when dividing a full ellipse into four equal parts. Samuel wishes to figure out what the exact position is of each bearing ball when the speed $v_e$ of the ball on the elliptical trajectory is twice that of the ball on the straight path ($v_s$). If Samuel has already calculated that the ball on the straight trajectory needs $t = 1.91$ s to reach the bottom, at which moment it possesses a speed of $v_{s,f} = 5.83$ m/s, and if he has now installed one of the measuring devices next to the straight path at a height of $y_s = 1.44$ m, what coordinates—with respect to the coordinate system (x,y)—does Samuel find for both bearing balls?

## *Exercise 12*

According to one report of the Umeå University in Umeå, Sweden, the tenuous atmosphere of the planet Mercury is mainly composed of oxygen (42%), sodium (29%), hydrogen (22%), helium (6%), and minor traces of other elements, among which potassium (0.5%). Suppose now that a sodium atom ($m_{Na} = 22.9898$ amu) is whizzing through Mercury's atmosphere relatively close to its surface with a speed of $v_{Na,i} = 1,252$ m/s and under an angle of $\phi = 65.4°$ with the horizontal. A potassium atom ($m_K = 39.0983$ amu), traveling at a speed $v_{K,i}$, is right behind the sodium atom and collides with it, sending the sodium atom straight up. (1) What should be the minimum incoming speed of the potassium atom so that the sodium atom is able to exit Mercury's atmosphere? (2) After the collision, is the kinetic energy of the potassium atom still sufficient to make it out of Mercury's well of gravitational potential energy? Ignore any solar radiation pressure or drag forces for this problem and assume that the collision is perfectly elastic. Remember that 1 atomic mass unit (amu) is equal to 1 amu $= 1.661 \times 10^{-24}$ g, that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$), and that the mass and radius of Mercury is equal to $M = 3.30 \times 10^{23}$ kg and $r = 2.44 \times 10^6$ m, respectively.

## *Exercise 13*

You are sitting at gate 15 of the Shenyang Taoxian International Airport, which is located at the capital city of Shenyang in the province Liaoning in China, waiting to board your flight CZ3602 to Guangzhou in the south of China. Being the astute engineer that you are, you immediately spot that the airplane model is the Airbus A320 Neo and since you have some free time on your hands, you decide to do some off-hand calculations. The flight attendant mentioned earlier that $n = 161$ passengers booked a seat, and you estimate that each person weighs about $m_{pas} = 75.0$ kg and that they carry $m_{hl} = 5.00$ kg of hand luggage and checked in a suitcase of $m_{sc} = 16.5$ kg. You further know that an empty A320 Neo model has a mass of $m_{pl} = 44.3$ t and that the fuel tanks contain approximately 27,500 L of jet fuel (with a density of $d = 692$ g/L). This specific model is furthermore equipped with two Pratt & Whitney PW1127G engines that each give a thrust of $T = 27,000$ lbf. As it is raining, you estimate that the tires create a slightly lower kinetic friction ($\mu_k = 0.135$) with the runway. (1) You're interested in finding the speed $v_h$ of the airplane halfway the runway, which has a total length of $L = 1,982$ m. What value for $v_h$ do you write down in your notebook? (2) If you estimate that the average power of the plane at that moment is equal to $P_h = 3.97$ MW and that the plane requires 70.7 % of the total takeoff time t to get to that point, how much time does it still need to accelerate before taking off? (3) What value do you find for the speed $v_f$ at lift-off? Remember that the pound-force is equal to 1 lbf = 1 lb × g, with $g$ the acceleration due to gravity, and you assume that the pilot needs the entire length of the runway to take off.

## *Exercise 14*

At high school, Hilde enjoyed studying mathematics and during her two final years she chose the advanced course option whereby she was taught 8 hours of mathematics per week. After high school, Hilde wanted to combine her interest in mathematics with her fascination for the natural laws that explain how the physical world works. As a result, she decided to pursue a master's degree in physics and astronomy at the Free University of Brussels, in Belgium. After a semester of hard work, Hilde is ready to tackle her first exam, which, according to her schedule, is that of the course "Classical Mechanics". The opening question consists of three parts and reads as follows. A particle is undergoing a force $\vec{F}(\vec{r})$, which is equal to $\vec{F}(\vec{r}) = \left[\frac{xy^2z^2}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_x + \left[\frac{x^2yz^2}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_y + \left[\frac{x^2y^2z}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_z$, with k a constant equal to $k = 0.453$. (1) Show that the force $\vec{F}(\vec{r})$ is conservative. (2) Determine the potential energy function $V(\vec{r})$, whereby $V(\vec{0}) = 0$. (3) Calculate the work done on the particle by this force as it moves from $\vec{r}_1 = (2, 2, 2)$ to $\vec{r}_2 = (1, -3, 5)$. How did Hilde answer this opening question?

## ***Exercise 15***

For the past 2 years, Toivo has held the maximum score on the Cyclone pinball machine in the local pub Kurva Kodu in Rakvere, Estonia. However, last night, his best friend Kaarli broke his record, and since then Toivo has been trying non-stop to regain his leader position on the Hall of Fame Scoreboard. The pinball machine has a length of $L = 1.25$ m and the playfield makes an angle of $\theta = 9.65°$ with the horizontal. The top left and top right corners of the playfield are round in shape and on the right-hand side, there is a long isolated compartment from where the metal ball (with a mass and radius equal to $m = 0.252$ kg and $r = 0.550$ cm) is launched. The compartment has a width of $d = 8.00$ cm and its left edge stops at a distance $d$ from the top edge of the pinball machine. The launch mechanism is a spring ($k = 155$ N/m), which compresses when being pulled backwards from outside of the machine. In resting mode, the equilibrium length of the spring is equal to $s = 14.0$ cm. Toivo has also figured out that it greatly benefits his game if the ball enters the playfield when it still "sticks" to the top edge of the pinball machine as it exits the top right rounded corner. If you know that the metal ball produces kinetic friction ($\mu_k = 0.228$) with the bottom surface of the playfield, how far back, at a minimum, should Toivo pull the external handle so that, upon release, the metal ball enters the playfield with the greatest odds of beating Kaarli's maximum score?

## ***Exercise 16***

Suppose that 30,000 years ago, at a distance of $d = 2.50$ light years away from the center of the Sun, two massive rocks collided. The first rock, with a mass of $m_1 = 5.95 \times 10^5$ kg, smashed with a speed of $v_{1,i} = 95,400$ km/h into a heavier second rock ($m_2 = 1.22 \times 10^6$ kg), which was traveling slower at $v_{2,i} = 10,200$ km/h. After the collision, which happened to be perfectly elastic, rock 1 deviated from its original path by an angle of $\alpha = 33.2°$ and headed straight towards our Solar System, which we consider, for practical purposes, to be equal to the system "the Sun plus planet Earth". Moreover, as soon as it followed its new course, rock 1 became sensitive to the gravitational influence of our Solar System (ignore the gravitational pull by rock 2). Today, rock 1 finally reached our Solar system and is about to hit the surface of the Sun. If you know that at that moment the Earth is in an orbital position at 90° with respect to the line of trajectory of rock 1, what is the rock's speed as it crashes into the Sun? Remember that the universal gravitational constant $G$ is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$), that the mass and radius of the Sun are equal to $M_s = 1.99 \times 10^{30}$ kg and $r_s = 6.96 \times 10^5$ km, respectively, that the mass of the Earth is equal to $M_E = 5.97 \times 10^{24}$ kg, and that 1 light year measures $9.46 \times 10^{12}$ km. Also take into account that at a distance $d$ the Earth-Sun distance ($r_{ES} = 1.496 \times 10^8$ km) becomes, relatively speaking, very small and can be ignored in the calculations.

## ***Exercise 17***

After spending their entire morning attending classes at the Norbuling Central School in Gelephu, Bhutan, Sangay ($m = 57.4$ kg) and her friends Sherab and Kim rush to one of the nearby tributary streams of the Manas River. On one side of the river bank, a couple of indigenous trees called Ehretia acuminata are standing tall next to each other and Sangay has attached a rope of length $L = 8.55$ m to one of their branches, whereby one end of the rope is a distance $\Delta y = 1.75$ m short from touching the ground. Sangay runs up to the rope with an initial speed $v_0$, grabs it and subsequently swings on it until she briefly comes to a halt, at which moment the rope is making an angle of $\theta_{max} = 28.4°$ with the vertical. After a couple of swings, Sangay feels adventurous and she quickly estimates that when releasing the rope at an angle $\theta$, whereby $\theta < \theta_{max}$, she will make it to the other riverbank. Sangay is right in her calculations and she indeed just reaches the other side of the stream. (1) If you know that the tension in the rope right before the moment when Sangay releases it is equal to $T = 565$ N, what is the value of the angle $\theta$? (2) How wide is the river? (3) With what speed does Sangay hit the ground on the other side?

## ***Exercise 18***

On 14 December 2007, the Earth surveillance satellite RADARSAT-2 ($m = 2,250$ kg) was launched with the assistance of a Soyuz launch vehicle from the Baikonur Cosmodrome in the south of Kazakhstan (45°58′42.3″N 63°17′31.9″E). The data gathered during its observation is used for research as well as for developing applications and services in a wide range of areas, including pollution monitoring, ice monitoring, agricultural crop monitoring, geological mapping, and disaster management. RADARSAT-2 has been put in a near polar heliosynchronous orbit with an orbital period roughly equal to $T_R = 101$ min at an inclination angle of $\theta_i = 98.6°$. In a heliosynchronous orbit, a satellite crosses the equator always at the same local time, which in the case of RADARSAT-2 is about 18:00 hrs (when moving from south to north). The inclination angle $\theta_i$ is the angle between the equator and the orbital plane of the satellite, whereby 0° corresponds to a satellite orbiting along the equator in the same direction as the Earth's spin. If you estimate that the average power of the engines combined was about $P = 10.6$ MW, how long did it take the Soyuz launch vehicle to place RADARSAT-2 into orbit? Remember that the universal gravitational constant $G$ is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$) and that the mass and radius of the Earth are equal to $M_E = 5.97 \times 10^{24}$ kg and $r_E = 6.38 \times 10^3$ km, respectively. Assume a circular orbit for the satellite.

## ***Exercise 19***

Luan and Annika are spending their Saturday afternoon improving their shooting skills at the Long Range Shooting Club in Leandra, South Africa. Luan owns a .380 ACP gun, while Annika brought her .40 Smith & Wesson to the shooting range. Luan's .380 ACP fires its bullets of 9.0 mm caliber with a muzzle velocity of $\vec{v}_{01} = v_{01} \cdot \vec{i}_x$, whereas the muzzle velocity of the 10.2 mm caliber bullets of

Annika's .40 S&W is equal to $\vec{v}_{02} = v_{02} \cdot \vec{i}_x$. They are each standing at a distance of $d = 45.5$ m away from a small wooden block ($M = 2.55$ kg), suspended from a rope with length $L = 0.750$ m. When firing their gun, aimed at their respective wooden block, a drag force $\vec{F}_D = -b \cdot v^2 \cdot \vec{i}_x$ has slowed the bullet's muzzle velocity (in the x-direction) by 5% by the time the bullet hits the block. Upon impact, the block swings slightly backwards until it reaches a height h (with respect to the top edge of the block) and makes an angle $\theta$ with the vertical. If you know that the muzzle speed $v_{02}$, the mass $m_2$ of the .40 S&W's bullets, and the maximum swinging height $h_2$ of Annika' wooden block relative to Luan are equal to 1.122, 1.722, and 3.721, respectively, and that Luan's block makes a $\theta_1 = 13.95°$ angle when at its maximum swinging height $h_1$, (1) what is the mass $m_1$ and $m_2$ of the 9.0 mm and 10.2 mm caliber bullets, respectively, expressed in grains, whereby 1 grain $= 6.48 \times 10^{-2}$ g? (2) What is the magnitude of the muzzle velocity $\vec{v}_{01}$ (.380 ACP) and $\vec{v}_{02}$ (.40 S&W)? (3) Which angle $\theta_2$ does Annika's block make with the vertical at its maximum height $h_2$? Assume that between the moment when the bullets enter the wooden block and until they come to a halt within the block, the block is not experiencing any major changes in its motion.

## ***Exercise 20***

Last week, Olivia and Adam have been watching a dozen of Youtube videos on how to make your own water bottle rocket. They even went the extra mile and figured out some of the basics of the underlying physics of their new project. Earlier this morning, they went to buy all the required equipment and with the assembled bottle rocket under their arm, Olivia and Adam are now headed to the nearby Blatherskite Park in Alice Springs, Australia, to try out their first design. Their rocket consists of three empty 2.00 L plastic soda water bottles ($m_b = 44.9$ g per unit) firmly taped together and designed in such a way that combined they make one cylindrical container. The rocket is filled with heated soda water at a temperature of 42.5°C (for some extra kinetic energy) for a total volume of a little under one third per bottle ($V_b = 0.63$ L per unit). Once Olivia and Adam start pumping air into the rocket, the growing pressure increasingly pushes on the water until at one point the water will come rushing out of the nozzle at the bottom of the rocket, providing the bottle rocket with upwards thrust and sending it flying through the air. Olivia and Adam estimate that about 459 g of soda water will shoot out of the rocket every second at a constant speed of $v_w = 36.5$ m/s, relative to the rocket. They furthermore take into account an average drag force of $\vec{F}_D = -25.5 \cdot \vec{i}_y$ N during the rocket's ascent. If you know that the density of soda water is equal to $\rho = 1.01442$ kg/L, (1) how high will the bottle rocket go? (2) How much time did the rocket spend in the air (ignore air friction during the rocket's descent)? (3) What is the total power supplied by the thrust force $\vec{F}_T$ of the bottle rocket, expressed in horsepower (hp)? Remember that 1 hp = 745.7 W.

# Exercise 1

## *Problem Statement*

On a rainy Saturday afternoon, Artem ($m = 69.4$ kg) is feeling rebellious and climbs the $h = 14.0$ m-high, egg-shaped main building of the Pysanka Museum in Kolomyia, Ukraine. Once sitting on top, Artem has difficulties holding on to the wet surface, so he starts to slip and slides down. (1) If the building is approximately elliptical in shape with an elliptical eccentricity of $e = 0.750$ and if Artem loses contact with the side of the egg when the elliptical radius r makes an angle of $\theta = 78.6°$ west of north, what is the magnitude of his velocity $\vec{v}$ in that moment? (2) If the bottom side of the egg sits $b = 2.50$ m below the ground, at what height $h_p$ from the pavement does Artem find himself when he becomes detached from the building? Ignore any kind of friction for this problem.

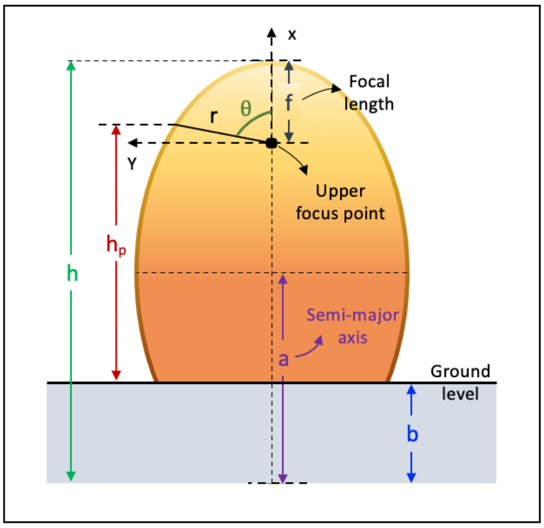


***Figure 1***

## *Solution*

(1) If we use polar coordinates, then we find the following expressions for the radius $r$ and the focal length $f$ of an ellipse for the specific choice of our coordinate system (with a and e representing the semi-major axis and the elliptical eccentricity, respectively):

$$\begin{cases} r = \dfrac{a \cdot (1 - e^2)}{1 + e \cdot \cos\theta} \\ \\ f = a \cdot (1 - e) \end{cases}$$

Given that the height of the egg-shaped building is equal to $h = 14.0$ m, we find that the semi-major axis $a$ equals $a = \frac{h}{2} = \frac{14.0}{2} = 7.00$ m. The radius $r$ at the angle of $\theta = 78.6°$ and the focal length $f$ then become:

$$\begin{cases} r = \dfrac{a \cdot (1 - e^2)}{1 + e \cdot \cos\theta} = \dfrac{7.00 \cdot (1 - 0.750^2)}{1 + 0.750 \cdot \cos(78.6°)} = 2.76 \text{ m} \\ \\ f = a \cdot (1 - e) = 7.00 \cdot (1 - 0.750) = 1.75 \text{ m} \end{cases}$$

Because the only force acting upon Artem, i.e., the gravitational force, is a conservative force—this means that the work done by such a force between two points is independent of the particular path followed between those points—we know that the total mechanical energy, i.e., the sum of the kinetic energy $E_k$ and the potential energy $E_p$, of a system is conserved. Put differently, the change in the total mechanical energy for such conservative systems is always equal to zero.

Applied to our problem, the conservation of energy implies that the total mechanical energy of Artem at the top of the egg-shaped building must be equal to the mechanical energy at the moment when he loses contact with the side of the egg (at $\theta = 78.6°$). We then find the magnitude of Artem's velocity $\vec{v}$ as follows (with the indices $i$ and $f$ referring to the initial and final position of the system, respectively):

$$\begin{aligned} E_{k,i} + E_{p,i} = E_{k,f} + E_{p,f} \quad &\Leftrightarrow \; 0 + m \cdot g \cdot f = \frac{m \cdot v^2}{2} + m \cdot g \cdot (r \cdot \cos\theta) \\ &\Leftrightarrow \; v = \sqrt{2 \cdot g \cdot (f - r \cdot \cos\theta)} \\ &\phantom{\Leftrightarrow \; v} = \sqrt{2 \cdot 9.81 \cdot (1.75 - 2.76 \cdot \cos(78.6°))} \\ &\phantom{\Leftrightarrow \; v} = 4.90 \text{ m/s} \end{aligned}$$

(2) The height $h_p$ from the ground at which Artems becomes detached from the egg-shaped building is found in the following manner:

$$h_p = h - (f - r \cdot \cos\theta) - b = 14.0 - [1.75 - 2.76 \cdot \cos(78.6°)] - 2.50 = 10.3 \text{ m}$$

Luckily, his backpack as well as the nearby bushes broke his fall, so that Artem did not sustain any major injuries.

# Exercise 2

## *Problem Statement*

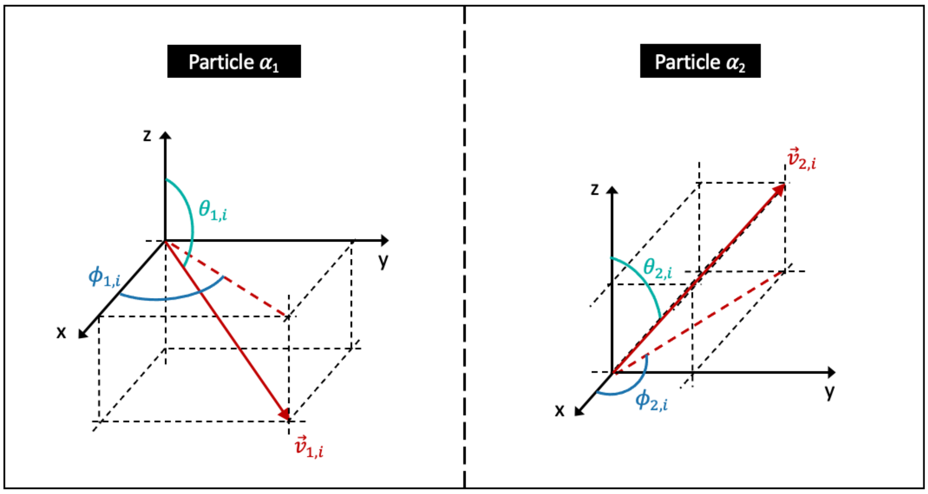


***Figure 2***

During radioactive decay, either the atomic composition of the nucleus of a chemical element is fundamentally altered—this is the case for $\alpha$- and $\beta$-decay—or an element lowers its nuclear energy levels ($\gamma$-emission), emitting thereby high-energy radiation. In one of the several nuclear chain reactions called the Thorium Series, the unstable element thorium ($^{232}_{90}$Th) decays through a series of events until it is transformed into the stable element lead ($^{208}_{82}$Pb). One of the intermediary steps includes an $\alpha$-decay of radon's 220-isotope ($^{220}_{86}$Rn) into polonium's 216-isotope ($^{216}_{84}$Po) whereby an $\alpha$-particle, i.e., the nucleus of a helium (He) atom, is emitted. Suppose now that two $\alpha$-particles ($m_\alpha = 6.64 \times 10^{-27}$ kg) elastically collide at a speed of $v_{1,i} = 14.2 \times 10^6$ m/s and $v_{2,i} = 14.8 \times 10^6$ m/s, respectively. The velocity vector $\vec{v}_{1,i}$ of the first particle $\alpha_1$ is initially making an angle of $\theta_{1,i} = 135°$ with the z-axis, whereby the angle $\phi_{1,i}$, i.e., the angle between the projection onto the xy-plane and the x-axis, is equal to $\phi_{1,i} = 65.4°$. Regarding the second particle $\alpha_2$, the respective angles are equal to $\theta_{2,i} = 66.0°$ and $\phi_{2,i} = 153°$. (1) If you know that after the collision the angles with the z-axis are equal to $\theta_{1,f} = 49.13°$ and $\theta_{2,f} = 123.21°$, respectively, what is the final speed of both particles, i.e., $v_{1,f}$ and $v_{2,f}$? (2) In which direction are $\alpha_1$ and $\alpha_2$ now headed?

## *Solution*

(1) Using spherical coordinates, the velocity vector $\vec{v}$ takes on the following general form:

$$\vec{v} = (v \cdot \sin\theta \cdot \cos\phi) \cdot \vec{i}_x + (v \cdot \sin\theta \cdot \sin\phi) \cdot \vec{i}_y + (v \cdot \cos\theta) \cdot \vec{i}_z$$

Since in the isolated system "particle $\alpha_1$ plus particle $\alpha_2$" the total linear momentum $\vec{p} = m \cdot \vec{v}$ is conserved, we can write the following expression for the x-direction:

$$p_i = p_f \quad \Leftrightarrow \quad (m_\alpha \cdot v_{1,i,x}) + (m_\alpha \cdot v_{2,i,x}) = (m_\alpha \cdot v_{1,f,x}) + (m_\alpha \cdot v_{2,f,x})$$

$$\Leftrightarrow \quad (v_{1,i} \cdot \sin\theta_{1,i} \cdot \cos\phi_{1,i}) + (v_{2,i} \cdot \sin\theta_{2,i} \cdot \cos\phi_{2,i}) = (v_{1,f} \cdot \sin\theta_{1,f} \cdot \cos\phi_{1,f}) + (v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos\phi_{2,f})$$

Applying the law of conservation of linear momentum for our isolated system to the y- and z-direction, we can summarize the three equations as follows:

$$\begin{cases} x: \ (v_{1,i} \cdot \sin\theta_{1,i} \cdot \cos\phi_{1,i}) + (v_{2,i} \cdot \sin\theta_{2,i} \cdot \cos\phi_{2,i}) = (v_{1,f} \cdot \sin\theta_{1,f} \cdot \cos\phi_{1,f}) + (v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos\phi_{2,f}) \\ \\ y: \ (v_{1,i} \cdot \sin\theta_{1,i} \cdot \sin\phi_{1,i}) + (v_{2,i} \cdot \sin\theta_{2,i} \cdot \sin\phi_{2,i}) = (v_{1,f} \cdot \sin\theta_{1,f} \cdot \sin\phi_{1,f}) + (v_{2,f} \cdot \sin\theta_{2,f} \cdot \sin\phi_{2,f}) \\ \\ z: \ (v_{1,i} \cdot \cos\theta_{1,i}) + (v_{2,i} \cdot \cos\theta_{2,i}) = (v_{1,f} \cdot \cos\theta_{1,f}) + (v_{2,f} \cdot \cos\theta_{2,f}) \end{cases}$$

Another quantity that is conserved in an isolated system is the total kinetic energy $E_k = \frac{m \cdot v^2}{2}$, for which we can write the following expression:

$$E_{k,i} = E_{k,f} \ \Leftrightarrow \ \left(\frac{m_\alpha \cdot v_{1,i}^2}{2}\right) + \left(\frac{m_\alpha \cdot v_{2,i}^2}{2}\right) = \left(\frac{m_\alpha \cdot v_{1,f}^2}{2}\right) + \left(\frac{m_\alpha \cdot v_{2,f}^2}{2}\right)$$

$$\Leftrightarrow \ v_{1,i}^2 + v_{2,i}^2 = v_{1,f}^2 + v_{2,f}^2$$

To make our further calculations more transparent, let us calculate the left-hand side of the above four equations and replace them by the letters a to d. With regard to the three equations of the conservation of linear momentum, we then obtain:

$$\begin{cases} x: \ a = (v_{1,i} \cdot \sin\theta_{1,i} \cdot \cos\phi_{1,i}) + (v_{2,i} \cdot \sin\theta_{2,i} \cdot \cos\phi_{2,i}) \\ \quad = [14.2 \times 10^6 \cdot \sin(135°) \cdot \cos(65.4°)] + [14.8 \times 10^6 \cdot \sin(66.0°) \cdot \cos(153°)] = -7.87 \times 10^6 \text{ m/s} \\ \\ y: \ b = (v_{1,i} \cdot \sin\theta_{1,i} \cdot \sin\phi_{1,i}) + (v_{2,i} \cdot \sin\theta_{2,i} \cdot \sin\phi_{2,i}) \\ \quad = [14.2 \times 10^6 \cdot \sin(135°) \cdot \sin(65.4°)] + [14.8 \times 10^6 \cdot \sin(66.0°) \cdot \sin(153°)] = 15.3 \times 10^6 \text{ m/s} \\ \\ z: \ c = (v_{1,i} \cdot \cos\theta_{1,i}) + (v_{2,i} \cdot \cos\theta_{2,i}) \\ \quad = [14.2 \times 10^6 \cdot \cos(135°)] + [14.8 \times 10^6 \cdot \cos(66.0°)] = -4.02 \times 10^6 \text{ m/s} \end{cases}$$

Regarding the conservation of kinetic energy, we find the following value:

$$d = v_{1,i}^2 + v_{2,i}^2 = (14.2 \times 10^6)^2 + (14.8 \times 10^6)^2 = 4.21 \times 10^{14} \text{ m}^2/\text{s}^2$$

To determine the final velocities of the two $\alpha$-particles, we will use the equation of the conservation of linear momentum in the z-direction. Rearranging and squaring the equation and then replacing $v_{2,f}^2$ by the expression obtained from the conservation of kinetic energy, we obtain the following quadratic equation:

$$
\begin{aligned}
& (v_{2,f} \cdot \cos\theta_{2,f})^2 = [c - (v_{1,f} \cdot \cos\theta_{1,f})]^2 \\
\Leftrightarrow\ & v_{2,f}^2 \cdot \cos^2\theta_{2,f} = c^2 - 2 \cdot c \cdot (v_{1,f} \cdot \cos\theta_{1,f}) + v_{1,f}^2 \cdot \cos^2\theta_{1,f} \\
\Leftrightarrow\ & [d - v_{1,f}^2] \cdot \cos^2\theta_{2,f} = c^2 - 2 \cdot c \cdot (v_{1,f} \cdot \cos\theta_{1,f}) + v_{1,f}^2 \cdot \cos^2\theta_{1,f} \\
\Leftrightarrow\ & (\cos^2\theta_{1,f} + \cos^2\theta_{2,f}) \cdot v_{1,f}^2 - (2 \cdot c \cdot \cos\theta_{1,f}) \cdot v_{1,f} + (c^2 - d \cdot \cos^2\theta_{2,f}) = 0 \\
\Leftrightarrow\ & [\cos^2(49.13°) + \cos^2(123.21°)] \cdot v_{1,f}^2 - [2 \cdot (-4.02 \times 10^6) \cdot \cos(49.13°)] \cdot v_{1,f} \ + \\
& \qquad [(-4.02 \times 10^6)^2 - 4.21 \times 10^{14} \cdot \cos^2(123.21°)] = 0
\end{aligned}
$$

Solving the above quadratic equation provides two solutions, i.e., $v_{1,f} = 9.20 \times 10^6$ m/s and $v_{1,f} = -16.4 \times 10^6$ m/s. For the remainder of this exercise, we will only focus on the first solution. The equation in the z-direction of the conservation of linear momentum, for instance, then allows us to find the value of $v_{2,f}$:

$$v_{2,f} = \frac{c - v_{1,f} \cdot \cos\theta_{1,f}}{\cos\theta_{2,f}} = \frac{(-4.02 \times 10^6) - 9.20 \times 10^6 \cdot \cos(49.13°)}{\cos(123.21°)} = 18.3 \times 10^6 \text{ m/s}$$

(2) To determine the final direction of both particles, we need to find the value of the angles $\phi_{1,f}$ and $\phi_{2,f}$. Let us start with squaring the equation related to the conservation of linear momentum in the y-direction and subsequently apply the trigonometric identity "$\cos^2\alpha + \sin^2\alpha = 1$":

$$
\begin{aligned}
& b^2 = (v_{1,f}^2 \cdot \sin^2\theta_{1,f} \cdot \sin^2\phi_{1,f}) + (2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \sin\phi_{1,f} \cdot \sin\phi_{2,f}) + (v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot \sin^2\phi_{2,f}) \\
\Leftrightarrow\ & b^2 = [v_{1,f}^2 \cdot \sin^2\theta_{1,f} \cdot (1 - \cos^2\phi_{1,f})] + (2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \sin\phi_{1,f} \cdot \sin\phi_{2,f}) + [v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot (1 - \cos^2\phi_{2,f})] \\
\Leftrightarrow\ & v_{1,f}^2 \cdot \sin^2\theta_{1,f} \cdot \cos^2\phi_{1,f} = (v_{1,f}^2 \cdot \sin^2\theta_{1,f} - b^2) + (2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \sin\phi_{1,f} \cdot \sin\phi_{2,f}) + [v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot (1 - \cos^2\phi_{2,f})]
\end{aligned}
$$

In a next step, we square the equation with respect to the conservation of linear momentum in the x-direction and replace the term "$v_{1,f}^2 \cdot \sin^2\theta_{1,f} \cdot \cos^2\phi_{1,f}$" with the expression established above, after which we make use of the angle subtraction theorem "$\cos(\beta - \alpha) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$":

$$a^2 = (v_{1,f}^2 \cdot \sin^2\theta_{1,f} \cdot \cos^2\phi_{1,f}) + (2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \cos\phi_{1,f} \cdot \cos\phi_{2,f}) + (v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot \cos^2\phi_{2,f})$$

$$= \left[(v_{1,f}^2 \cdot \sin^2\theta_{1,f} - b^2) + (2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \sin\phi_{1,f} \cdot \sin\phi_{2,f}) + [v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot (1 - \cos^2\phi_{2,f})]\right] +$$

$$(2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \cos\phi_{1,f} \cdot \cos\phi_{2,f}) + (v_{2,f}^2 \cdot \sin^2\theta_{2,f} \cdot \cos^2\phi_{2,f})$$

$$= (v_{1,f}^2 \cdot \sin^2\theta_{1,f} - b^2) + [2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f} \cdot \cos(\phi_{2,f} - \phi_{1,f})] + v_{2,f}^2 \cdot \sin^2\theta_{2,f}$$

$$\Leftrightarrow \cos(\phi_{2,f} - \phi_{1,f}) = \frac{(a^2+b^2) - [v_{1,f}^2 \cdot \sin^2\theta_{1,f} + v_{2,f}^2 \cdot \sin^2\theta_{2,f}]}{2 \cdot v_{1,f} \cdot v_{2,f} \cdot \sin\theta_{1,f} \cdot \sin\theta_{2,f}}$$

$$= \frac{[(-7.87\times10^6)^2 + (15.3\times10^6)^2] - [(9.20\times10^6)^2 \cdot \sin^2(49.13°) + (18.3\times10^6)^2 \cdot \sin^2(123.21°)]}{2 \cdot (9.20\times10^6) \cdot (18.3\times10^6) \cdot \sin(49.13°) \cdot \sin(123.21°)}$$

$$= 5.32 \times 10^{-2}$$

$$\Leftrightarrow \phi_{2,f} - \phi_{1,f} = \cos^{-1}(5.32 \times 10^{-2}) = 86.9°$$

Inserting the above relationship between the two angles into, for instance, the equation related to the conservation of linear momentum in the x-direction and making use of the angle addition theorem as well as the fact that the expression "$a \cdot \cos\alpha + b \cdot \sin\alpha$" can be replaced by the single cosine function "$c_1 \cdot \cos(\alpha + \delta)$", whereby $c_1 = sgn(a)\sqrt{a^2 + b^2}$ and $\delta = \tan^{-1}(-\frac{b}{a})$, we find the following expression for the value a:

$$a = (v_{1,f} \cdot \sin\theta_{1,f} \cdot \cos\phi_{1,f}) + [v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos(\phi_{1,f} + 86.9°)]$$

$$= (v_{1,f} \cdot \sin\theta_{1,f} \cdot \cos\phi_{1,f}) + (v_{2,f} \cdot \sin\theta_{2,f} \cdot [\cos\phi_{1,f} \cdot \cos(86.9°) - \sin\phi_{1,f} \cdot \sin(86.9°)])$$

$$= [v_{1,f} \cdot \sin\theta_{1,f} + v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos(86.9°)] \cdot \cos\phi_{1,f} - [v_{2,f} \cdot \sin\theta_{2,f} \cdot \sin(86.9°)] \cdot \sin\phi_{1,f}$$

$$= c_1 \cdot \cos(\phi_{1,f} + \delta)$$

whereby $c_1$ and $\delta$ are equal to:

$$c_1 = \sqrt{[v_{1,f} \cdot \sin\theta_{1,f} + v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos(86.9°)]^2 + [-v_{2,f} \cdot \sin\theta_{2,f} \cdot \sin(86.9°)]^2}$$

$$= \sqrt{[9.20 \times 10^6 \cdot \sin(49.13°) + 18.3 \times 10^6 \cdot \sin(123.21°) \cdot \cos(86.9°)]^2 + [-18.3 \times 10^6 \cdot \sin(123.21°) \cdot \sin(86.9°)]^2}$$

$$= 17.2 \times 10^6 \text{ m/s}$$

$$\delta = \tan^{-1}\left[-\frac{-v_{2,f} \cdot \sin\theta_{2,f} \cdot \sin(86.9°)}{v_{1,f} \cdot \sin\theta_{1,f} + v_{2,f} \cdot \sin\theta_{2,f} \cdot \cos(86.9°)}\right]$$

$$= \tan^{-1}\left[-\frac{-18.3\times 10^6\cdot\sin(123.21°)\cdot\sin(86.9°)}{9.20\times 10^6\cdot\sin(49.13°)+18.3\times 10^6\cdot\sin(123.21°)\cdot\cos(86.9°)}\right]$$

$$= 63.1°$$

Plugging these values back into the expression for a, we can calculate the angle $\phi_{1,f}$:

$$a = c_1\cdot\cos(\phi_{1,f}+\delta) \quad\Leftrightarrow\quad \phi_{1,f} = \cos^{-1}\left(\frac{a}{c_1}\right)-\delta$$

$$= \cos^{-1}\left(\frac{-7.87\times 10^6}{17.2\times 10^6}\right)-63.1°$$

$$= 54.2°$$

The angle $\phi_{2,f}$ is then equal to $\phi_{2,f} = \phi_{1,f} + 86.9° = 54.2° + 86.9° = 141°$.

Fig. 3 provides a 3D view of the collision between particle $\alpha_1$ and $\alpha_2$. The second particle hits the first particle, which is initially following a downward trajectory ($\theta_{1,i} = 135°$), from below and pushes the first particle upwards ($\theta_{1,f} = 49.13°$). Due to the collision, the second particle's initial upwards trajectory ($\theta_{2,i} = 66.0°$) is converted into a downward path ($\theta_{2,f} = 123.21°$).

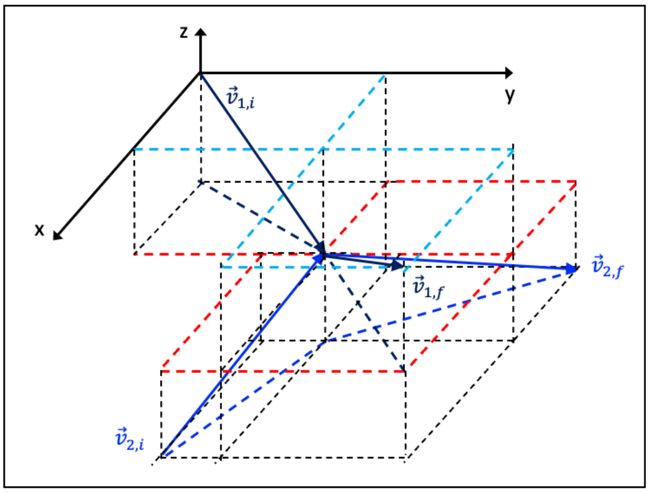


***Figure 3***

# Exercise 3

## *Problem Statement*

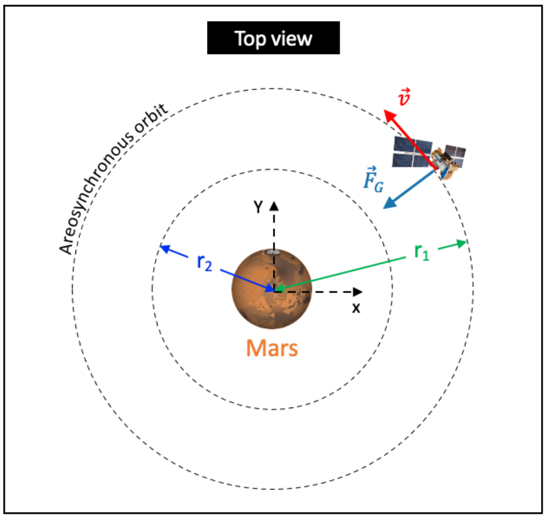


***Figure 4***

You own the private company called Satplans Science Ltd., dedicated to gathering and processing scientific data from the four innermost planets in our Solar System. As a result of healthy working capital levels, you are able to install the satellite Suzy 3 ($m = 4,630$ kg) in a perfectly circular areosynchronous equatorial orbit (AEO) around the planet Mars—an AEO is the Martian equivalent of a geostationary orbit around Earth—despite the significant orbital station keeping costs due to the gravitational impact of the planet's two moons, i.e., Phobos and Deimos. If you consider the system "Suzy 3", (1) what is the work done on the satellite? (2) Is linear momentum conserved? (3) Write a general formula for the work done by the satellite's engine when changing orbit. (4) Suppose that Suzy 3 is guided towards a new orbit with a radius 60% of its original. How much work has Suzy 3's engine performed? (5) What is the total amount of work done on the system? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$) and the mass, the radius, and the rotation period of Mars to $M = 6.417 \times 10^{23}$ kg, $r = 3.396 \times 10^6$ m, and $T =$ 24h 37min 22.7s, respectively.

## *Solution*

(1) In the circular AEO whereby Suzy 3 is traveling at a *constant* orbital speed v, there is no net force in the *tangential* direction. As a result, no work (W) is being done on Suzy 3. If we consider the radial direction, even though Suzy 3 experiences a net force equal to the gravitational force $\vec{F}_G$ (viewed from the *stationary* reference frame at the center of Mars), this force is oriented perpendicular to the satellite's direction of motion, so that, according to the expression "$W = F \cdot \Delta x \cdot \cos\theta$", the work done by gravity on Suzy 3 is zero. Put another way still, since there is no displacement of the satellite in the radial direction, the gravitational force is not performing any work.

(2) If the linear momentum $\vec{p} = m \cdot \vec{v}$ is conserved, i.e., $m_1 \cdot v_1 = m_2 \cdot v_2$, then as per Newton's first law, i.e., $\vec{F}_{net} = \frac{\Delta \vec{p}}{\Delta t} = \vec{0}$, the net force within the system under consideration must be equal to zero. If we consider the system "Mars plus Suzy 3", the total net force is equal to zero, since the gravitational force experienced by Suzy 3 due to Mars is equal in magnitude and opposite in direction to the gravitational force experienced by Mars due to the satellite (as stated by Newton's third law). Both forces cancel each other out, so the linear momentum for the system is conserved. However, if we only take into account the satellite, i.e., the subsystem "Suzy 3", then there actually is a net force within this subsystem, i.e., $\vec{F}_G$, so that the linear momentum is not conserved.

(3) As gravity is not doing any work on a satellite traveling in a circular orbit, it cannot cause the satellite to change its orbit, as long as the satellite maintains the same orbital speed. As a result,

any change in orbit must be brought about by an external force $\vec{F}_{ext}$, which in this case is provided by the satellite's boosters.

Before proceeding, let us express the orbital speed $v$ only in terms of the variables $G$, $M$, and $r$. We know that the gravitational force $\vec{F}_G$ gives rise to the centripetal force $\vec{F}_{cp} = \frac{m \cdot v^2}{r} \cdot \vec{i}_r$, so that we can write the following equation for an object in a circular orbit, whereby the axis of rotation coincides with the center of mass, and derive from it an expression for the orbital speed $v$:

$$F_G = F_{cp} \quad \Leftrightarrow \quad \frac{G \cdot m \cdot M}{r^2} = \frac{m \cdot v^2}{r}$$

$$v = \sqrt{\frac{G \cdot M}{r}}$$

Next, we insert the above expression into the definition of total mechanical energy $E_{me}$:

$$\begin{aligned} E_{me} = E_k + E_p \quad \Leftrightarrow \quad E_{me} &= \frac{m \cdot v^2}{2} - \frac{G \cdot m \cdot M}{r} \\ &= \frac{m \cdot \left(\sqrt{\frac{G \cdot M}{r}}\right)^2}{2} - \frac{G \cdot m \cdot M}{r} \\ &= \frac{G \cdot m \cdot M}{2 \cdot r} - \frac{G \cdot m \cdot M}{r} \\ &= -\frac{G \cdot m \cdot M}{2 \cdot r} \end{aligned}$$

As an external force $\vec{F}_{ext}$ (provided by the boosters of the satellite) has to do work on the system "Suzy 3" in order to change its orbit, the total mechanical energy of the satellite in its new orbit ($E_{me,f}$) is now equal to the total mechanical energy in its initial orbit ($E_{me,i}$) plus the work performed by the boosters ($W_{ext}$). Based on this information, we can write a general formula for $W_{ext}$ (with $r_1$ ($r_2$) the radius of the initial (final) orbit):

$$\begin{aligned} E_{me,i} + W_{ext} = E_{me,f} \quad \Leftrightarrow \quad W_{ext} &= E_{me,f} - E_{me,i} \\ &= -\frac{G \cdot m \cdot M}{2 \cdot r_2} - \left(-\frac{G \cdot m \cdot M}{2 \cdot r_1}\right) \\ &= -\frac{G \cdot m \cdot M}{2} \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \\ &= -\frac{G \cdot m \cdot M}{2} \left(\frac{r_1 - r_2}{r_1 \cdot r_2}\right) \end{aligned}$$

(4) Using Kepler's third law, we first calculate the orbital radius of Suzy 3's AEO:

$$T^2 = \frac{4 \cdot \pi^2}{G \cdot M} \cdot r_1^3 \quad \Leftrightarrow \quad r_1 = \sqrt[3]{\frac{G \cdot M \cdot T^2}{4 \cdot \pi^2}}$$

$$= \sqrt[3]{\frac{6.67 \times 10^{-11} \cdot 6.417 \times 10^{23} \cdot (24 \cdot 3,600 + 37 \cdot 60 + 22.7)^2}{4 \cdot \pi^2}}$$

$$= 20,400 \text{ km}$$

The new orbit lies at a distance from Mars' center whose value corresponds to 60% of its original radius. The new orbital radius is therefore equal to $r_2 = 0.60 \cdot r_1 = 0.60 \cdot 2.04 \times 10^4$=12,300 km. Using the formula derived in part (3), the work performed by Suzy 3's engine during its approach towards an orbit closer to Mars is calculated as follows:

$$W_{ext} = -\frac{G \cdot m \cdot M}{2} \left( \frac{r_1 - r_2}{r_1 \cdot r_2} \right) = -\frac{6.67 \times 10^{-11} \cdot 4,630 \cdot 6.417 \times 10^{23}}{2} \left( \frac{2.04 \times 10^7 - 1.23 \times 10^7}{2.04 \times 10^7 \cdot 1.23 \times 10^7} \right)$$

$$= -3.23 \times 10^9 \text{ J}$$

(5) The work done by the satellite's engine is negative because the system is losing energy in the form of gravitational potential energy. That is, as Suzy travels towards a smaller orbit around Mars, the gravitational potential energy is being reduced, i.e., it becomes more negative. On the other hand, given that the gravitational force $\vec{F}_G$ points into the same direction as the displacement of the satellite, gravity is performing positive work, as it converts gravitational potential energy into kinetic energy. To see what the net value is of the total work $W_{tot}$ done by these two forces on Suzy, we perform the following calculation (with $W_c$ representing the work done by the conservative force $\vec{F}_G$):

$$W_{tot} = W_{ext} + W_c = W_{ext} + (-\Delta E_p) = W_{ext} + \left[ -\left( -\frac{G \cdot m \cdot M}{r_2} \right) + \left( -\frac{G \cdot m \cdot M}{r_1} \right) \right]$$

$$\Leftrightarrow \quad W_{tot} = W_{ext} + G \cdot m \cdot M \cdot \left( \frac{r_1 - r_2}{r_1 \cdot r_2} \right)$$

$$= -3.23 \times 10^9 + 6.67 \times 10^{-11} \cdot 4,630 \cdot 6.417 \times 10^{23} \cdot \left( \frac{2.04 \times 10^7 - 1.23 \times 10^7}{2.04 \times 10^7 \cdot 1.23 \times 10^7} \right)$$

$$= 3.23 \times 10^9 \text{ J}$$

Given that $W_{tot} = \Delta E_k > 0$, we expect that Suzy 3's orbital speed in the lower orbit is greater with respect to its original orbit, i.e., the AEO. Based on the expression for the orbital speed derived in part (3), we indeed find that $v_2 = 1.87 \times 10^3$ m/s $>$ $v_1 = 1.45 \times 10^3$ m/s.

# Exercise 4

## *Problem Statement*

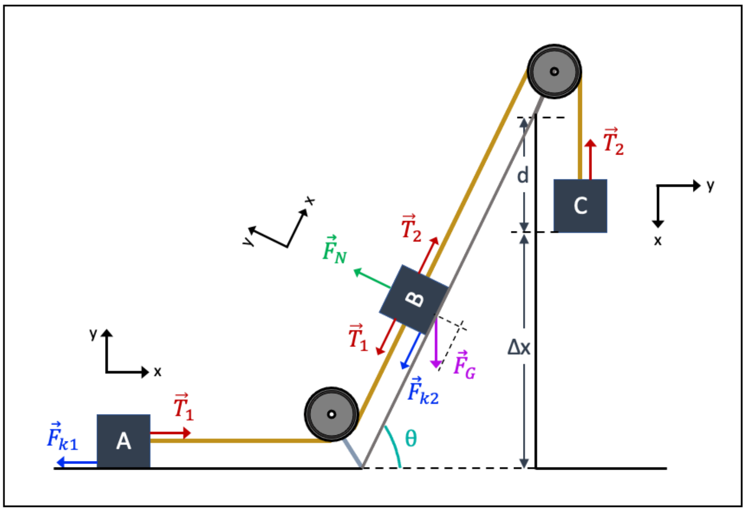


***Figure 5***

It's mid-June and Estée is currently taking the final exam of her AP Physics 1 course at the American School of Madrid in Pozuelo de Alarcón, Spain. The weather is particularly hot today and she has 20 minutes left to answer the final question of her exam. Luckily, as it is one of her most favourite courses, Estée prepared thoroughly for this exam and with her acute sense of focus she finishes the question under 10 minutes, despite the oppressive heat. The final question was the following. A pulley system with three blocks A ($m_A = 4.75$ kg), B ($m_B = 3.50$ kg), and C ($m_C$) is presented in Fig. 5, whereby mass C is hanging $d = 75.4$ cm from the top of the incline, whose length is equal to $L = 2.52$ m and makes an angle of $\theta = 64.2°$ with the horizontal. The surface under mass A generates a kinetic friction coefficient of $\mu_{k1} = 0.453$ with the block, whereas the incline has a rougher surface and therefore produces a higher kinetic friction coefficient of $\mu_{k2} = 0.678$ with mass B. Initially, someone is preventing block A from moving and when they release the block, mass C is accelerating downwards until it hits the ground. If you know that the total work done on block C during its displacement is equal to $W = 5.54$ J, (1) determine the mass of block C and (2) its speed when it hits the ground. What answers did Estée find?

## *Solution*

In a first step, we calculate the displacement $\Delta x$ of block C:

$$\Delta x = L \cdot \sin\theta - d = 2.52 \cdot \sin(64.2°) - 0.754 = 1.51 \text{ m}$$

Next, based on the definition of work, we find an expression for the acceleration $a$ of mass C:

$$W = F \cdot \Delta x = (m_C \cdot a) \cdot \Delta x \quad \Leftrightarrow \quad a = \frac{W}{m_C \cdot \Delta x}$$

Since the three blocks are connected via the pulley system, the acceleration of each of the blocks is equal to $a$. Applying Newton's second law to the three masses, we find the following equations (with the magnitude of the kinetic friction force $\vec{F}_k$ equal to $F_k = \mu_k \cdot F_N$):

$$\begin{cases} A: \ -\mu_{k1} \cdot (m_A \cdot g) + T_1 = m_A \cdot a \\ \\ B: \ -T_1 - \mu_{k2} \cdot (m_B \cdot g \cdot \cos\theta) - (m_B \cdot g \cdot \sin\theta) + T_2 = m_B \cdot a \\ \\ C: \ -T_2 + m_C \cdot g = m_C \cdot a \end{cases}$$

If we insert both the expression for $T_1$ of equation A and that for $T_2$ of equation C into the equation B, we obtain the following expression:

$$-\left[\mu_{k1} \cdot (m_A \cdot g) + m_A \cdot a\right] - \mu_{k2} \cdot (m_B \cdot g \cdot \cos\theta) - (m_B \cdot g \cdot \sin\theta) + \left[m_C \cdot g - m_C \cdot a\right] = m_B \cdot a$$

$$\Leftrightarrow \quad -\mu_{k1} \cdot m_A \cdot g - (\mu_{k2} \cdot \cos\theta + \sin\theta) \cdot m_B \cdot g + m_C \cdot g - m_C \cdot a = (m_A + m_B) \cdot a$$

Using the above derived expression for the acceleration a, we find the following quadratic equation:

$$-\mu_{k1} \cdot m_A \cdot g - (\mu_{k2} \cdot \cos\theta + \sin\theta) \cdot m_B \cdot g + m_C \cdot g - m_C \cdot \left[\tfrac{W}{m_C \cdot \Delta x}\right] = (m_A + m_B) \cdot \left[\tfrac{W}{m_C \cdot \Delta x}\right]$$

$$\Leftrightarrow \quad g \cdot m_C^2 - \left[(\mu_{k1} \cdot m_A + (\mu_{k2} \cdot \cos\theta + \sin\theta) \cdot m_B) \cdot g + \frac{W}{\Delta x}\right] \cdot m_C - (m_A + m_B) \cdot \frac{W}{\Delta x} = 0$$

$$\Leftrightarrow \quad 9.81 \cdot m_C^2 - \left[(0.453 \cdot 4.75 + [0.678 \cdot \cos(64.2°) + \sin(64.2°)] \cdot 3.50) \cdot 9.81 + \tfrac{5.54}{1.51}\right] \cdot m_C - (4.75 + 3.50) \cdot \tfrac{5.54}{1.51} = 0$$

Solving the above quadratic equation produces one physically sensible solution ($m_C > 0$): $m_C = 7.14$ kg.

(2) The speed with which block C reaches the ground can be calculated as follows:

$$v^2 - v_0^2 = 2 \cdot a \cdot \Delta x \quad \Leftrightarrow \quad v^2 - 0^2 = 2 \cdot \left[\frac{W}{m_C \cdot \Delta x}\right] \cdot \Delta x$$

$$\Leftrightarrow \quad v = \sqrt{\frac{2 \cdot W}{m_C}}$$

$$= \sqrt{\frac{2 \cdot 5.54}{7.14}}$$

$$= 1.25 \text{ m/s}$$

# Exercise 5

## *Problem Statement*

Duško ($m_D$) is enjoying his winter holidays in the Kamnik-Savinja Alps in the north of Slovenia. The seasoned skier that he is, Duško loves going off-piste to explore and carve out new paths. At a certain point, he is standing on top of a hill and notices that further down the ski run is interrupted by a large gap, after which the path continues. Just before the gap, the slope goes back up and at the end of the upward slope, at the very edge of the gap, there are three naturally formed ramps, which make an angle of $\theta_1 = \frac{7 \cdot \pi}{36}$, $\theta_2 = \frac{\pi}{4}$, and $\theta_3 = \frac{11 \cdot \pi}{36}$ with the horizontal, respectively. If you know that the distance between the bottom of the hill and the point where Duško is currently standing is the minimal height required to gain sufficient speed to cross the gap, which one of the three ramps should Duško choose to safely reach the other side? Assume that the edges at both sides of the gap are at the same height and ignore any friction or drag forces for this problem.

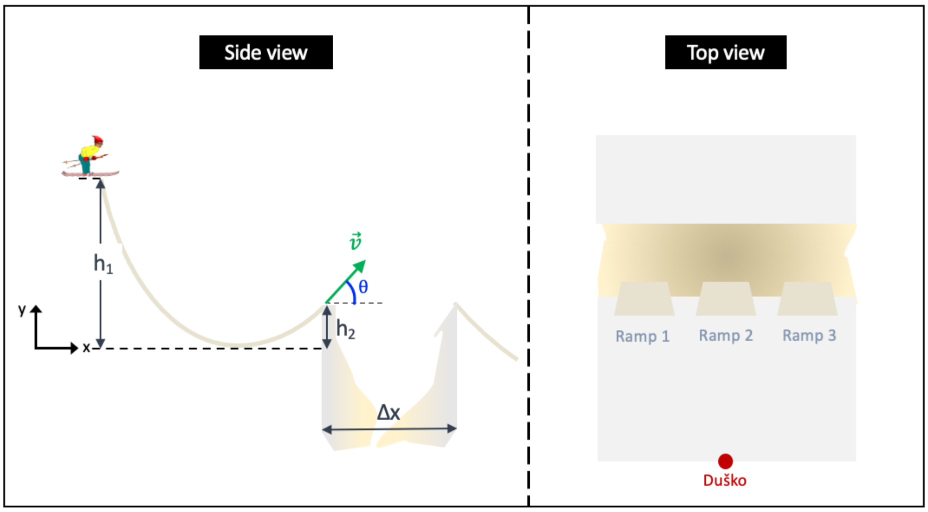


***Figure 6***

## *Solution*

Since the only force present in the system "Duško" is the gravitational force $\vec{F}_G$, which is a conservative force, we know that his total mechanical energy remains constant. In a first instance, let us determine the magnitude of the velocity $\vec{v}$ with which Duško reaches one of the ramps, using the fact that the energy is constant (whereby $E_1$ and $E_2$ represent the total energy at his initial position and at the edge of the gap, respectively):

$$E_1 = E_2 \quad \Leftrightarrow \quad E_{k1} + E_{p1} = E_{k2} + E_{p2} \quad \Leftrightarrow \quad \frac{m_D \cdot v_0^2}{2} + m_D \cdot g \cdot h_1 = \frac{m_D \cdot v^2}{2} + m_D \cdot g \cdot h_2$$

$$\Leftrightarrow \quad \frac{m_D \cdot 0^2}{2} + m_D \cdot g \cdot h_1 = \frac{m_D \cdot v^2}{2} + m_D \cdot g \cdot h_2$$

$$\Leftrightarrow \quad v = \sqrt{2 \cdot g \cdot (h_1 - h_2)}$$

In a next step, we need to find the time that Duško spends in the air while crossing the gap.

Given that the two edges at both sides of the gap are at the same height, the highest point during his parabolic trajectory coincides with the point in the middle of the gap. Considering the y-direction, the time $t_{1/2}$ it takes to reach the highest point is equal to:

$$v_{f,y} = v_{i,y} + a_y \cdot t_{1/2} \quad \Leftrightarrow \quad 0 = (v \cdot \sin\theta) - g \cdot t_{1/2} \quad \Leftrightarrow \quad t_{1/2} = \frac{v}{g} \cdot \sin\theta$$

The total time $t$ to cross the gap is then equal to $t = 2 \cdot t_{1/2} = \frac{2 \cdot v}{g} \cdot \sin\theta$. If we now look at the x-direction, we obtain the following expression for the horizontal distance $\Delta x$:

$$\Delta x = v_x \cdot t = (v \cdot \cos\theta) \cdot t = (v \cdot \cos\theta) \cdot \left[\frac{2 \cdot v}{g} \cdot \sin\theta\right] = \frac{v^2}{g} \cdot \sin(2\theta)$$

If we now insert the earlier derived expression for the speed $v$ in the above equation, we find a formula for the height $h_1$ in terms of the angle $\theta$:

$$\Delta x = \frac{[\sqrt{2 \cdot g \cdot (h_1 - h_2)}]^2}{g} \cdot \sin(2\theta) \quad \Leftrightarrow \quad \Delta x = 2 \cdot \sin(2\theta) \cdot (h_1 - h_2)$$

$$\Leftrightarrow \quad h_1 = h_2 + \frac{\Delta x}{2 \cdot \sin(2\theta)}$$

To find the angle $\theta$ that gives the minimum required height $h_1$ we take the derivative of $h_1$ with respect to $\theta$ and equate the expression to zero:

$$\frac{dh_1}{d\theta} = -\Delta x \cdot \frac{\cos(2\theta)}{\sin^2(2\theta)} = 0 \quad \Leftrightarrow \quad \cos(2\theta) = 0$$

$$\Leftrightarrow \quad 2\theta = \frac{\pi}{2}$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{4}$$

If Duško wishes to safely reach the other side of the gap, he is strongly advised to take the middle ramp, i.e., ramp 2, which makes an angle of $\theta_2 = \frac{\pi}{4}$ with the horizontal.

# Exercise 6

## *Problem Statement*

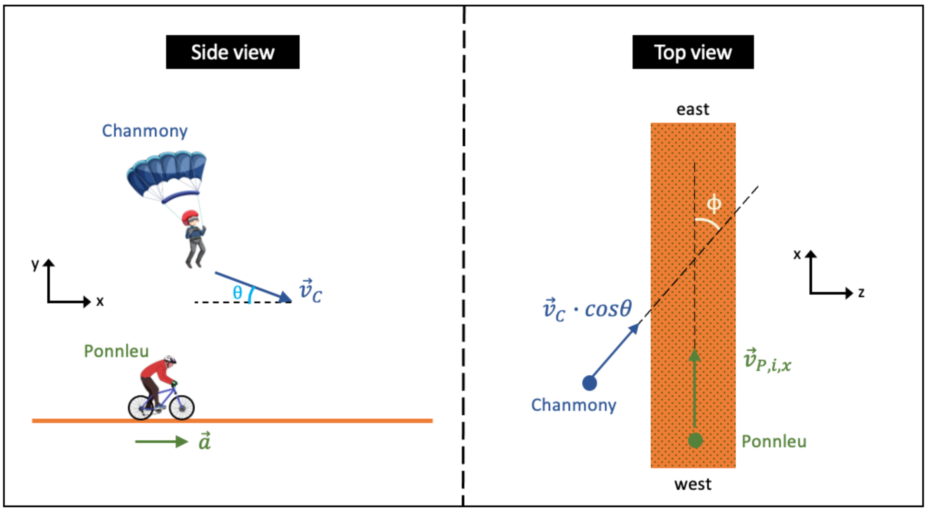


***Figure 7***

For the past two hours, Chanmony ($m_C$ = 74.8 kg) has been guiding her paraglider over the rural outskirts of the Kampong Chhnang province, Cambodia, enjoying the undulating paddy fields, the meandering Tonle Sap River, and the hilly landscapes in the west. Meanwhile, Ponnleu ($m_P$ = 69.6 kg) is taking up the beautiful scenery from a lower altitude, steering her mountain bike across several dusty village roads. At one point, Chanmony is descending at a constant velocity with a magnitude of $v_C = 7.82$ m/s in the southeastern direction ($\phi = 40.8°$ south of east) under an angle of $\theta = 15.5°$ with the horizontal, and is about to land near the roadside on the opposite side of a village road that lies parallel to the east-west axis. However, right at the moment when Chanmony flies over the road, Ponnleu, who was initially going at a speed of $v_0 = 4.25$ m/s and has been accelerating ($a = 0.507$ m/s$^2$) for the past $\Delta x = 200$ m, is all caught up in her own world, not paying attention to her surroundings, and fails to see Chanmony coming from the northwestern direction. Both collide, but somehow still manage to hold on to each other and roll entangled for a distance $d$ in the field near the road until they come to a halt, thanks to the kinetic friction with the grass ($\mu_k = 0.439$). (1) What is the velocity of Chanmony and Ponnleu rolling together right after the collision? (2) What distance do they need to come to a stop?

## *Solution*

(1) Since the entangled motion of Chanmony and Ponnleu after the collision occurs entirely within the xz-plane, let us determine the magnitude of their incoming velocities $\vec{v}_{C,i}$ and $\vec{v}_{P,i}$ both in the x- and z-direction:

$$\text{Chanmony} \begin{cases} x: \; v_{C,i,x} = (v_C \cdot \cos\theta) \cdot \cos\phi = [7.82 \cdot \cos(15.5°)] \cdot \cos(40.8°) = 5.70 \text{ m/s} \\ z: \; v_{C,i,z} = (v_C \cdot \cos\theta) \cdot \sin\phi = [7.82 \cdot \cos(15.5°)] \cdot \sin(40.8°) = 4.92 \text{ m/s} \end{cases}$$

$$\text{Ponnleu} \begin{cases} x: \; v_{P,i,x} = \sqrt{v_0^2 + 2 \cdot a \cdot \Delta x} = \sqrt{4.25^2 + 2 \cdot 0.507 \cdot 200} = 14.9 \text{ m/s} \\ z: \; v_{P,i,z} = 0 \text{ m/s} \end{cases}$$

The type of collision we are dealing with in this problem is a perfectly inelastic collision, given that the two colliding objects continue as one object right after the collision rather than bouncing off of each other, each in a separate direction. This means that their mechanical energy is not conserved—some of the kinetic energy is transformed into heat (due to internal friction) and sound. However, the total linear momentum $\vec{p} = m \cdot \vec{v}$ of the system "Chanmony plus Ponnleu" remains constant, because the magnitude of the net force of this system is equal to zero (the kinetic friction with the grass only comes into the picture a moment later).

With respect to the x-direction, we can therefore calculate the magnitude of final velocity $\vec{v}_{f,x}$ as follows:

$$p_{C,i,x} + p_{P,i,x} = p_{tot,f,x} \quad \Leftrightarrow \quad m_C \cdot v_{C,i,x} + m_P \cdot v_{P,i,x} = m_{tot} \cdot v_{f,x} = (m_C + m_P) \cdot v_{f,x}$$

$$\Leftrightarrow \quad v_{f,x} = \frac{m_C \cdot v_{C,i,x} + m_P \cdot v_{P,i,x}}{m_C + m_P}$$

$$= \frac{74.8 \cdot 5.70 + 69.6 \cdot 14.9}{74.8 + 69.6} = 10.1 \text{ m/s}$$

Regarding the z-direction, the magnitude of final velocity $\vec{v}_{f,z}$ is equal to:

$$p_{C,i,z} + p_{P,i,z} = p_{tot,f,z} \quad \Leftrightarrow \quad m_C \cdot v_{C,i,z} + m_P \cdot v_{P,i,z} = (m_C + m_P) \cdot v_{f,z}$$

$$\Leftrightarrow \quad v_{f,z} = \frac{m_C \cdot v_{C,i,z} + m_P \cdot v_{P,i,z}}{m_C + m_P}$$

$$= \frac{74.8 \cdot 4.92 + 69.6 \cdot 0}{74.8 + 69.6} = 2.55 \text{ m/s}$$

The magnitude of the final velocity $\vec{v}_f$ is therefore equal to $v_f = \sqrt{v_{f,x}^2 + v_{f,z}^2} = \sqrt{10.1^2 + 2.55^2} = 10.4$ m/s at an angle of $\beta = \tan^{-1}\left(\frac{v_{f,z}}{v_{f,x}}\right) = \tan^{-1}\left(\frac{2.55}{10.1}\right) = 14.1°$ south of east.

(2) To find the distance d, we first have to determine the magnitude of the deceleration $\vec{a}_s$ of Chanmony and Ponnleu. Applying Newton's second law to the two of them combined in their direction of motion, we obtain:

$$-\mu_k \cdot (m_C + m_P) \cdot g = (m_C + m_P) \cdot a_s \quad \Leftrightarrow \quad a_s = -\mu_k \cdot g = -0.439 \cdot 9.81 = -4.31 \text{ m/s}^2$$

The distance d that Chanmony and Ponnleu roll together until they come to a halt is then calculated in the following way:

$$v^2 - v_f^2 = 2 \cdot a_s \cdot d \quad \Leftrightarrow \quad d = \frac{v^2 - v_f^2}{2 \cdot a_s} = \frac{0^2 - 10.4^2}{2 \cdot (-4.31)} = 12.6 \text{ m}$$

# Exercise 7

## *Problem Statement*

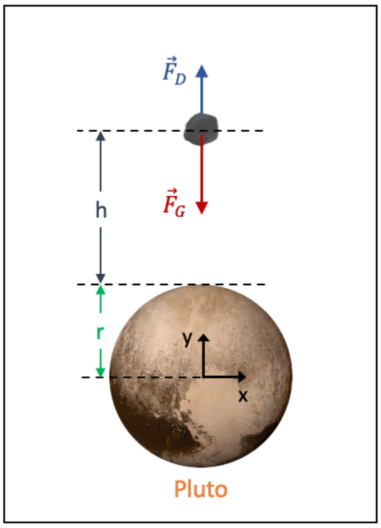


***Figure 8***

Most of the leftover debris from the days when the Solar System was being formed is orbiting in a large torus-shaped disk either between the planets Mars and Jupiter—called the asteroid belt— or beyond the outermost planet Neptune—this disk is referred to as the Kuiper belt, whose main region's width is about 20 times the distance between the Earth and the Sun. The largest and most massive object that belongs to the Kuiper belt is the dwarf planet Pluto with a mass of $M = 1.30 \times 10^{22}$ kg and a radius of $r = 1.19 \times 10^6$ m. Its flimsy gaseous atmosphere mainly consists of nitrogen ($N_2$), methane ($CH_4$), and carbon monoxide (CO), stretching out at some places as high as 1,600 km. Suppose that a massive rock ($m = 2,750$ kg) is knocked out from its orbit within the Kuiper belt and is headed straight towards Pluto. When it is $h = 25.0$ km away from Pluto's surface, the rock has a velocity of $\vec{v}_1 = -339 \cdot \vec{i}_y$ m/s, and despite the thin atmosphere, the rock experiences a drag force, which has the form of $\vec{F}_D = b \cdot v^2 \cdot \vec{i}_y$. A little over a minute later, the rock hits Pluto's surface at a velocity of $\vec{v}_2 = -368 \cdot \vec{i}_y$ m/s. (1) Use calculus to derive an expression for the work $W_D$ done by the drag force $\vec{F}_D$ on the rock. (2) What is the value of $W_D$? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$).

## *Solution*

(1) Let us start with writing the definition of work done by the force $\vec{F}_D$:

$$W_D = \int_{y_1}^{y_2} F_D \cdot dy' = \int_{y_1}^{y_2} b \cdot v^2 \cdot dy'$$

Since we have no information about the value of the drag coefficient b, let us rewrite the above integral by applying Newton's second law to the rock (in the y-direction), which gives the following equation:

$$F_D - F_G = m \cdot a \quad \Leftrightarrow \quad b \cdot v^2 - \frac{G \cdot m \cdot M}{y^2} = m \cdot a$$

$$\Leftrightarrow \quad b \cdot v^2 = \frac{G \cdot m \cdot M}{y^2} + m \cdot a$$

Inserting this last expression into the above integral, we can then write:

$$W_D = \int_{y_1}^{y_2} \left( \frac{G \cdot m \cdot M}{y'^2} + m \cdot a \right) \cdot dy' = \int_{y_1}^{y_2} \left( \frac{G \cdot m \cdot M}{y'^2} \right) \cdot dy' + \int_{y_1}^{y_2} m \cdot a \cdot dy'$$

The first integral can be solved as follow:

$$\int_{y_1}^{y_2} \left( \frac{G \cdot m \cdot M}{y'^2} \right) \cdot dy' = G \cdot m \cdot M \cdot \int_{y_1}^{y_2} \left( \frac{1}{y'^2} \right) \cdot dy' = -G \cdot m \cdot M \cdot \left[ \left( \frac{1}{y} \right) \Bigg|_{y_1 = r+h}^{y_2 = r} \right]$$

$$= -G \cdot m \cdot M \cdot \left[ \frac{1}{r} - \frac{1}{r+h} \right]$$

Next, using the fact that the acceleration is equal to the second derivate of the position with respect to time and transforming the limits of integration from position-based variables into time-based variables, we can write the second integral as follows:

$$\int_{y_1}^{y_2} m \cdot a \cdot dy' = \int_{y_1}^{y_2} m \cdot \frac{d^2 y'}{dt^2} \cdot dy' = \int_{t_1}^{t_2} m \cdot \frac{d^2 y}{dt'^2} \cdot \frac{dy}{dt'} \cdot dt'$$

Now, based on the Leibniz rule, we can write the following expression:

$$\frac{d}{dt'} \left( \frac{dy}{dt'} \right)^2 = 2 \cdot \frac{dy}{dt'} \cdot \frac{d^2 y}{dt'^2} \quad \Leftrightarrow \quad \frac{1}{2} \cdot \frac{d}{dt'} \left( \frac{dy}{dt'} \right)^2 = \frac{dy}{dt'} \cdot \frac{d^2 y}{dt'^2}$$

If we insert this last expression into our second integral, we can solve the second integral in the following way:

$$\int_{t_1}^{t_2} m \cdot \left[ \frac{1}{2} \cdot \frac{d}{dt'} \left( \frac{dy}{dt'} \right)^2 \right] \cdot dt' = \frac{m}{2} \cdot \int_{t_1}^{t_2} d \left( \frac{dy}{dt'} \right)^2 = \frac{m}{2} \cdot \left[ \left( \frac{dy}{dt} \right)^2 \Bigg|_{t_1}^{t_2} \right]$$

$$= \frac{m}{2} \cdot \left[ \left( \frac{dy}{dt_2} \right)^2 - \left( \frac{dy}{dt_1} \right)^2 \right]$$

$$= \frac{m}{2} \cdot (v_2^2 - v_1^2)$$

In a final step, we insert the solutions for the two integrals into the initial definition of the work $W_D$ performed by the drag force $\vec{F}_D$, so that we obtain the following expression:

$$W_D = -G \cdot m \cdot M \cdot \left[\frac{1}{r} - \frac{1}{r+h}\right] + \frac{m}{2} \cdot (v_2^2 - v_1^2)$$

This is precisely equal to the work-energy relationship "$W_{tot} = W_{ext} + W_c \quad \Leftrightarrow \quad W_{ext} = W_{tot} - W_c = \Delta E_k + \Delta E_p$", with $W_{tot}$ the total work done on the rock, $W_{ext}$ the work done by external forces, i.e., the drag force $W_D$, $W_c$ the work done by conservative forces, i.e., the gravitational force $\vec{F}_G$, and $\Delta E_k$ ($\Delta E_p$) the difference in kinetic (potential) energy of the rock, whereby:

$$\begin{cases} W_{tot} = \Delta E_k = \dfrac{m}{2} \cdot (v_2^2 - v_1^2) \\ \\ -W_c = \Delta E_p = \left[\left(-\dfrac{G \cdot m \cdot M}{r}\right) - \left(-\dfrac{G \cdot m \cdot M}{r+h}\right)\right] \end{cases}$$

(2) The work $W_D$ performed by the drag force $\vec{F}_D$ on the rock that is approaching the dwarf planet Pluto can be calculated as follows:

$$W_D = -G \cdot m \cdot M \cdot \left[\frac{1}{r} - \frac{1}{r+h}\right] + \frac{m}{2} \cdot (v_2^2 - v_1^2)$$

$$= -6.67 \times 10^{-11} \cdot 2,750 \cdot 1.30 \times 10^{22} \cdot \left[\tfrac{1}{1.19 \times 10^6} - \tfrac{1}{1.19 \times 10^6 + 25.0 \times 10^3}\right] + \tfrac{2,750}{2} \cdot \left[(-368)^2 - (-339)^2\right]$$

$$= -41.2 \times 10^6 \text{ J} + 28.2 \times 10^6 \text{ J}$$

$$= -13.0 \times 10^6 \text{ J}$$

The work $W_D$ done by the drag force $\vec{F}_D$ on the rock is negative since the force points into the opposite direction of the rock's motion, i.e., it tries to slow the rock down. The work $W_c$ done by gravity is positive ($W_c = 41.2 \times 10^6$ J) as the gravitational force $\vec{F}_G$ pumps energy into the rock by converting gravitational potential energy into kinetic energy, i.e., the rock gains speed as it approaches Pluto.

# Exercise 8

## *Problem Statement*

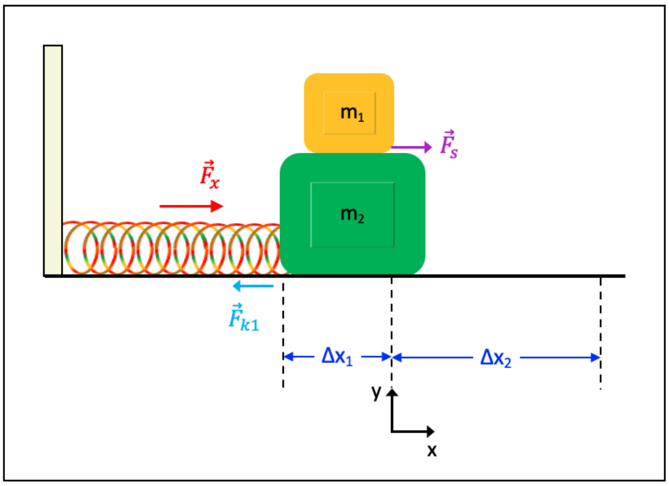


***Figure 9***

As her parents had to go and do some errands for about an hour, Semira is babysitting her three-year-old baby brother Jemal in their home right behind Fiat Tagliero in Asmara, Eritrea. Jemal loves to play with the colourful rubber toy spring (with a length of $L =$ 45.0 cm) that his sister bought him for his recent birthday, and Semira wants to show him a new trick. She places the spring horizontally on the kitchen floor with one end leaning against the plinth of a cupboard and presses the spring together over a distance $\Delta x_1$. Semira then places two small plastic blocks of mass $m_1 = 0.15$ kg and $m_2 = 0.35$ kg on top of each other (the lightest one goes on top) and puts them in front of the compressed spring. When Semira lets go of the blocks, the spring shoots them forward across the kitchen floor to the great amusement of Jemal.

(1) What is the maximum distance that Semira should compress the spring so that the upper block stays put when being released? (2) What is the total work done on the two blocks combined during this displacement? (3) When the spring reaches its equilibrium position, the two blocks become detached from the spring. What is their speed at that moment? (4) How far ($\Delta x_2$) do the blocks slide across the kitchen floor? (5) Does the upper block still remain steady during $\Delta x_2$? The spring constant k is equal to $k = 10.3$ N/m, and assume that the kinetic friction coefficient between the lower block and the kitchen floor is equal to $\mu_{k1} = 0.065$, and that the kinetic (static) friction coefficient between the two plastic blocks equals $\mu_{k2} = 0.115$ ($\mu_s = 0.225$).

## *Solution*

(1) Given that we want the upper block to remain stationary, let us in a first instance determine the magnitude of the acceleration $\vec{a}_1$ of the two blocks combined at the moment when Semira releases them. Applying Newton's second law in the x-direction provides us with the following equation (whereby $\vec{F}_x = -k \cdot \Delta\vec{x}$ and $\vec{F}_{k1}$ represent the restoring spring force and the kinetic friction force, respectively):

$$-F_{k1} + F_x = (m_1 + m_2) \cdot a_1 \quad \Leftrightarrow \quad -\mu_{k1} \cdot (m_1 + m_2) \cdot g - k \cdot \Delta x = (m_1 + m_2) \cdot a_1$$

$$\Leftrightarrow \; a_1 = -\frac{k \cdot \Delta x}{(m_1 + m_2)} - \mu_{k1} \cdot g$$

Since the upper block $m_1$ is positioned within an accelerating frame of reference, i.e., block $m_2$, it experiences an inertial force $\vec{F}_{i1}$ opposite to the direction of motion of the reference frame, i.e., towards the negative x-direction. The only other force in the x-direction that is able to counterbalance this inertial force is the static friction force $\vec{F}_s$, so that the requirement that the upper block remain stationary can be translated in mathematical terms in the following way:

$$F_{i1} = F_s \quad \Leftrightarrow \quad m_1 \cdot a_1 = \mu_s \cdot m_1 \cdot g \quad \Leftrightarrow \quad m_1 \cdot \left[ -\frac{k \cdot \Delta x}{(m_1 + m_2)} - \mu_{k1} \cdot g \right] = \mu_s \cdot m_1 \cdot g$$

$$\Leftrightarrow \quad \Delta x = -\frac{g}{k} \cdot (\mu_{k1} + \mu_s) \cdot (m_1 + m_2)$$

$$= -\frac{9.81}{10.3} \cdot (0.065 + 0.225) \cdot (0.15 + 0.35)$$

$$= -13.8 \text{ cm}$$

Given that the equilibrium position of the spring is located at the origin of our coordinate system, the maximum distance $\Delta x_1$ that Semira should compress the spring is equal to $\Delta x_1 = 0 - \Delta x = 0 - (-13.8) = 13.8$ cm.

(2) Given that the total work $W_{tot}$ done on the two blocks is equal to the work $W_{ext}$ done by external forces, i.e., the friction force $\vec{F}_{k1}$, plus the work $W_c$ done by conservative forces, i.e., the restoring spring force $\vec{F}_x$, we can calculate $W_{tot}$ as follows:

$$W_{tot} = W_{ext} + W_c = \int_{x_0}^{x_1} F_{k1} \cdot dx' + \int_{x_0}^{x_1} F_x \cdot dx'$$

$$= \int_{x_0}^{x_1} [-\mu_{k1} \cdot (m_1 + m_2) \cdot g] \cdot dx' + \int_{x_0}^{x_1} (-k \cdot x') \cdot dx'$$

$$= [-\mu_{k1} \cdot (m_1 + m_2) \cdot g] \cdot \int_{x_0}^{x_1} dx' - k \cdot \int_{x_0}^{x_1} x' \cdot dx'$$

$$= [-\mu_{k1} \cdot (m_1 + m_2) \cdot g] \cdot \left[ \ (x)\Big|_{x_0=-0.138}^{x_1=0} \ \right] - k \cdot \left[ \left( \frac{x^2}{2} \right) \Bigg|_{x_0=-0.138}^{x_1=0} \right]$$

$$= [-0.065 \cdot (0.15 + 0.35) \cdot 9.81] \cdot [0 - (-0.138)] - 10.3 \cdot \left[ 0 - \frac{(-0.138)^2}{2} \right]$$

$$= 5.42 \times 10^{-2} \text{ J}$$

(3) The net work $W_{tot}$ done on the blocks during the displacement $\Delta x_1$ is equal to the difference in kinetic energy $\Delta E_k$, so that we can calculate the speed of the two blocks combined at the origin of the coordinate system—which is the point at which the spring is at its equilibrium position and whereby the blocks lose contact with the spring—as follows:

$$W_{tot} = \Delta E_k = \frac{(m_1 + m_2) \cdot v_2^2}{2} - \frac{(m_1 + m_2) \cdot v_1^2}{2} \quad \Leftrightarrow \quad v_2 = \sqrt{\frac{2 \cdot W_{tot}}{(m_1 + m_2)} + v_1^2}$$

$$= \sqrt{\frac{2 \cdot 5.42 \times 10^{-2}}{(0.15 + 0.35)} + 0^2}$$

$$= 0.466 \text{ m/s}$$

(4) To determine the distance $\Delta x_2$, we have to know the magnitude of the deceleration $\vec{a}_2$ of the two blocks combined from the moment they become detached from the spring until they come to a halt. The only force acting on the blocks in the x-direction is the kinetic friction force $\vec{F}_{k1}$, so that applying Newton's second law gives us the following deceleration:

$$-\mu_{k1} \cdot (m_1 + m_2) \cdot g = (m_1 + m_2) \cdot a_2 \quad \Leftrightarrow \quad a_2 = -\mu_{k1} \cdot g = -0.065 \cdot 9.81 = -0.638 \text{ m/s}^2$$

The distance $\Delta x_2$ is then calculated as follows:

$$v_f^2 - v_i^2 = 2 \cdot a_2 \cdot \Delta x_2 \quad \Leftrightarrow \quad \Delta x_2 = \frac{v_f^2 - v_i^2}{2 \cdot a_2} = \frac{0^2 - 0.466^2}{2 \cdot (-0.638)} = 17.0 \text{ cm}$$

(5) The upper block remains stationary during the displacement $\Delta x_2$ if the static friction force $\vec{F}_s$, which points into the negative x-direction, is able to overcome the inertial force $\vec{F}_{i2}$, which is the result of the upper block being positioned within an accelerating frame of reference (the lower block) and is directed towards the positive x-direction:

$$F_s \geq F_{i2} \quad \Leftrightarrow \quad \mu_s \cdot m_1 \cdot g \geq m_1 \cdot |a_2|$$

$$\Leftrightarrow \quad 0.225 \cdot 0.15 \cdot 9.81 \geq 0.15 \cdot 0.638$$

$$\Leftrightarrow \quad 0.331 \text{ N} \geq 9.56 \times 10^{-2} \text{ N}$$

Since the static friction is greater than the inertial force acting on the upper block, the block $m_2$ remains steady also during the displacement $\Delta x_2$.

# Exercise 9

## *Problem Statement*

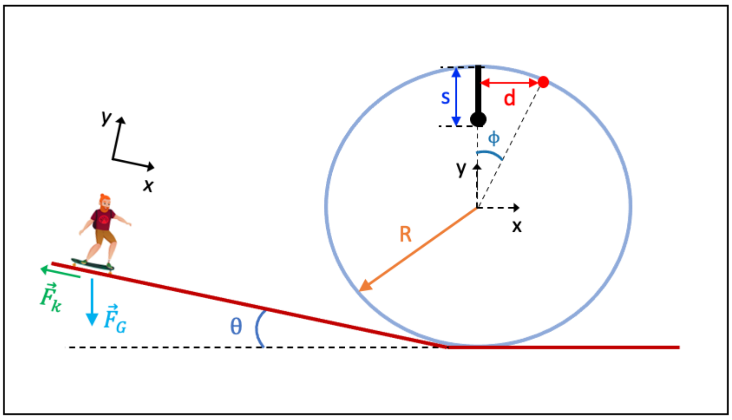


***Figure 10***

In Nur-Sultan, the capital city of Kazakhstan, Sarsen ($m = 63.5$ kg) is trying out the professional skatepark of the newly created green urban area, which also includes city parks, an outdoor cinema, a beach area, bike lanes and pedestrian esplanades. One of the skatepark's main attractions is a looping installed at the end of a long inclined run-up track, which makes an angle of $\theta = 12.5°$ with the ground. As a safety measure, an elastic rubber rope is hanging from the top at both sides of the looping. If the wheels of the skateboard only create kinetic friction with the track ($\mu_k = 0.112$)—the friction with the surface of the looping is negligible—and given an inner radius of the looping equal to $R = 3.55$ m, (1) what minimum distance $L$ should Sarsen walk up the track in order to successfully go through the looping? (2) Suppose that Sarsen did not attain sufficient speed and loses contact with the surface of the looping when he is a horizontal distance of $d = 1.00$ m away from the rubber rope. Luckily, he manages to get hold of the bottom end of the rubber rope, which has a length of $s = 1.00$ m and stretches according to the spring force $\vec{F}_x = -k \cdot \vec{x}$ ($k = 357$ N/m), and falls down vertically. How far from the ground is Sarsen when the rope is maximally stretched right after his fall?

## *Solution*

(1) To determine the speed $v_1$ at the end of the run-up track, we could either use Newton's second law or the work-energy relationship. If we go for the latter option, let us write out the three components of work done on the system "Sarsen", i.e., the net work done $W_{to}$, the work $W_{ext}$ done by external forces, i.e., the friction force $\vec{F}_k$, and the work $W_c$ done by conservative forces, i.e., the gravitational force $\vec{F}_G$:

$$\begin{cases} W_{tot} = \Delta E_k = \dfrac{m \cdot v_1^2}{2} - \dfrac{m \cdot v_0^2}{2} = \dfrac{m \cdot v_1^2}{2} - \dfrac{m \cdot 0^2}{2} = \dfrac{m \cdot v_1^2}{2} \\[2ex] W_{ext} = -F_k \cdot L = -\mu_k \cdot F_N \cdot L = -\mu_k \cdot (m \cdot g \cdot \cos\theta) \cdot L \\[2ex] W_c = F_{G,x} \cdot L = m \cdot g \cdot \sin\theta \cdot L \end{cases}$$

We then find the following expression for the speed $v_1^2$ at the end of the track:

$$W_{tot} = W_{ext} + W_c \quad \Leftrightarrow \quad \frac{m \cdot v_1^2}{2} = -\mu_k \cdot (m \cdot g \cdot \cos\theta) \cdot L + m \cdot g \cdot \sin\theta \cdot L$$

$$\Leftrightarrow \quad v_1^2 = 2 \cdot g \cdot L \cdot (\sin\theta - \mu_k \cdot \cos\theta)$$

In a next step, we know that the looping is basically frictionless, so that only conservative forces are at play. As a result, the total mechanical energy remains constant and we can use this fact to find an expression for the velocity $v_2^2$ at the top of the looping. Comparing the position at the bottom when Sarsen enters the looping with that at the top provides us with the following equation:

$$E_{bottom} = E_{top} \quad \Leftrightarrow \quad E_{k,b} + E_{p,b} = E_{k,t} + E_{p,t} \quad \Leftrightarrow \quad \frac{m \cdot v_1^2}{2} + 0 = \frac{m \cdot v_2^2}{2} + m \cdot g \cdot (2 \cdot R)$$

$$\Leftrightarrow \quad v_2^2 = v_1^2 - 4 \cdot g \cdot R$$

There is another constraint that must be taken into account and it is related to the circular motion within the looping. In order to go through the looping *successfully*, the centripetal force $\vec{F}_{cp}$ must overcome the gravitational force $\vec{F}_G$ at the top of the looping. At a minimum, they must be equal in magnitude, which gives us the following constraint:

$$\frac{m \cdot v_2^2}{R} = m \cdot g \quad \Leftrightarrow \quad v_2^2 = g \cdot R$$

Using this expression for $v_2^2$ in the above energy condition, we find the following expression for $v_1^2$:

$$[g \cdot R] = v_1^2 - 4 \cdot g \cdot R \quad \Leftrightarrow \quad v_1^2 = 5 \cdot g \cdot R$$

If we equate the above expression for $v_1^2$ to that found earlier with respect to the work-energy relationship, we can calculate the minimum required distance L that Sarsen must ride on the track to go successfully through the looping:

$$5 \cdot g \cdot R = 2 \cdot g \cdot L \cdot (\sin\theta - \mu_k \cdot \cos\theta) \quad \Leftrightarrow \quad L = \frac{5 \cdot R}{2 \cdot (\sin\theta - \mu_k \cdot \cos\theta)}$$

$$= \frac{5 \cdot 3.55}{2 \cdot [\sin(12.5°) - 0.112 \cdot \cos(12.5°)]}$$

$$= 82.9 \text{ m}$$

(2) When Sarsen loses contact with the surface of the looping, he starts free-falling due to the gravitational force $\vec{F}_G$. Therefore, when grabbing the bottom end of the rubber rope, he will have obtained an initial speed $v_y$. The vertical distance $\Delta y$ of free fall is equal to the difference in height between the red dot in Fig. 10 and the bottom end of the rubber rope. To calculate the distance $\Delta y$, we first have to find the angle $\phi$:

$$d = R \cdot \sin\phi \quad \Leftrightarrow \quad \phi = \sin^{-1}\left(\frac{d}{R}\right) = \sin^{-1}\left(\frac{1.00}{3.55}\right) = 16.4°$$

The distance $\Delta y$ is now found as follows:

$$R - s + \Delta y = R \cdot \cos\phi \quad \Leftrightarrow \quad \Delta y = s - R \cdot (1 - \cos\phi) = 1.00 - 3.55 \cdot [1 - \cos(16.4°)] = 85.6 \text{ cm}$$

The speed $v_y$ when Sarsen grabs the bottom end of the rubber rope is then equal to (note that although the distance $\Delta y$ is positive, the displacement $\Delta y_d$ is negative ($\Delta y_d = -\Delta y$)):

$$v_y^2 - 0^2 = 2 \cdot (-g) \cdot \Delta y_d \quad \Leftrightarrow \quad v_y^2 = 2 \cdot (-g) \cdot \Delta y_d = 2 \cdot g \cdot \Delta y$$

Between the moment when Sarsen holds on to the rubber rope and when the rope becomes maximally stretched over a distance $\Delta y_{max}$, only conservative forces, i.e., the gravitational force $\vec{F}_G$ and the spring force $\vec{F}_x$, are present, so that the total mechanical energy of Sarsen at both moments is equal (whereby the above expression for $v_y^2$ is used):

$$E_1 = E_2 \quad \Leftrightarrow \quad \frac{m \cdot v_y^2}{2} + m \cdot g \cdot (R - s) = \frac{k \cdot (\Delta y_{max})^2}{2} + m \cdot g \cdot (-[\Delta y_{max} - (R - s)])$$

$$\Leftrightarrow \quad \frac{k \cdot (\Delta y_{max})^2}{2} - m \cdot g \cdot \Delta y_{max} - \frac{m \cdot v_y^2}{2} = 0$$

$$\Leftrightarrow \quad \frac{k \cdot (\Delta y_{max})^2}{2} - m \cdot g \cdot \Delta y_{max} - \frac{m \cdot [2 \cdot g \cdot \Delta y]}{2} = 0$$

$$\Leftrightarrow \quad \frac{357 \cdot (\Delta y_{max})^2}{2} - 63.5 \cdot 9.81 \cdot \Delta y_{max} - 63.5 \cdot 9.81 \cdot 0.856 = 0$$

The above quadratic equation provides a physically sensible solution ($\Delta y_{max} \geq 0$) equal to $\Delta y_{max} = 4.20$ m. The distance $\Delta y_{ground}$ above the ground then becomes:

$$\Delta y_{ground} = 2 \cdot R - s - \Delta y_{max} = 2 \cdot 3.55 - 1.00 - 4.20 = 1.90 \text{ m}$$

# Exercise 10

## *Problem Statement*

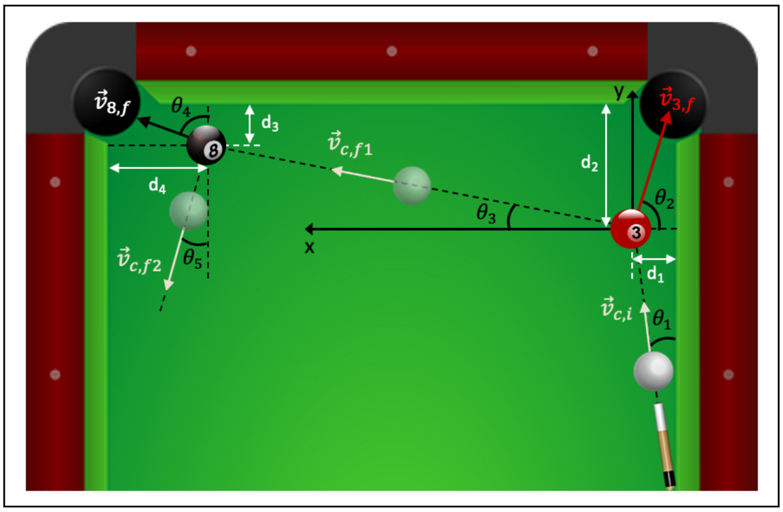


***Figure 11***

Elina is a promising young billiard player who is currently participating in a local tournament in Brèst, Belarus. She managed to reach the finals and is now in a position where she can win the tournament. That is, only if Elina is able to pocket the last two billiard balls with just one single stroke. The (white) cue ball is located near the right edge, whereas the last object ball, i.e., the solid red number 3, finds itself in front of the top right pocket, i.e., $d_1 = 7.60$ cm from the right edge and $d_2 = 27.3$ cm from the top edge. The (black) 8 ball is positioned close to the top left pocket, i.e., $d_3 = 5.85$ cm from the top edge and $d_4 = 20.4$ cm from the left edge. Elina holds the cue stick in such a way that it makes an angle $\theta_1$ with the right edge and gives the cue ball an initial speed of $v_{c,i}$. If Elina first pockets ball number 3 and subsequently the 8 ball with just one shot, (1) what speed $v_{c,i}$ should she give the cue ball? (2) What angle $\theta_1$ does Elina's cue stick make with the right edge? Assume that the solid red ball and the 8 ball enter their respective pocket with a speed of $v_{3,f} = 1.25$ m/s and $v_{8,f} = 0.86$ m/s and that the three billiard balls all have the same mass $m = 165$ g. Ignore any kind of friction.

## *Solution*

(1) Let us refer to the collision between the cue ball and the solid red ball as system 1, whereas system 2 refers to the subsequent collision between the cue ball and the 8 ball. Both systems are considered isolated systems, so that the total linear momentum $\vec{p} = m \cdot \vec{v}$ is conserved. Moreover, since we are dealing with perfectly elastic collisions, also the total kinetic energy $E_k = \frac{m \cdot v^2}{2}$ remains constant. As a result, we can write the following three equations for system 1 (as the mass of the three balls is the same, it is canceled in the three equations):

$$\begin{cases} x: v_{c,i} \cdot \sin\theta_1 = v_{c,f1} \cdot \cos\theta_3 - v_{3,f} \cdot \cos\theta_2 \\ \\ y: v_{c,i} \cdot \cos\theta_1 = v_{c,f1} \cdot \sin\theta_3 + v_{3,f} \cdot \sin\theta_2 \\ \\ \quad v_{c,i}^2 = v_{c,f1}^2 + v_{3,f}^2 \end{cases}$$

Similarly, we have the following three equations for system 2:

$$\begin{cases} x : v_{c,f1} \cdot \cos\theta_3 = v_{c,f2} \cdot \sin\theta_5 + v_{8,f} \cdot \sin\theta_4 \\ \\ y : v_{c,f1} \cdot \sin\theta_3 = -v_{c,f2} \cdot \cos\theta_5 + v_{8,f} \cdot \cos\theta_4 \\ \\ \quad v_{c,f}^2 = v_{c,f2}^2 + v_{8,f}^2 \end{cases}$$

Based on trigonometry, Fig. 11 tells us that the angles $\theta_2$ and $\theta_4$ are equal to:

$$\begin{cases} \theta_2 = \tan^{-1}\left(\dfrac{d_2}{d_1}\right) = \tan^{-1}\left(\dfrac{27.3}{7.60}\right) = 74.4° \\ \\ \theta_4 = \tan^{-1}\left(\dfrac{d_4}{d_3}\right) = \tan^{-1}\left(\dfrac{20.4}{5.85}\right) = 74.0° \end{cases}$$

To determine the angle $\theta_3$, we first square the equation of the conservation of linear momentum in both the x- and y-direction with respect to system 1 and subsequently add them together. In a next step, we make use of the angle addition theorem "$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$" and the trigonometric identity "$\cos^2\alpha + \sin^2\alpha = 1$". Finally, we apply the equation of conservation of kinetic energy so that some terms cancel out:

$$\begin{cases} x : \quad v_{c,i}^2 \cdot \sin^2\theta_1 = v_{c,f1}^2 \cdot \cos^2\theta_3 - 2 \cdot v_{c,f1} \cdot v_{3,f} \cdot \cos\theta_2 \cdot \cos\theta_3 + v_{3,f}^2 \cdot \cos^2\theta_2 \\ \\ y : \quad v_{c,i}^2 \cdot \cos^2\theta_1 = v_{c,f1}^2 \cdot \sin^2\theta_3 + 2 \cdot v_{c,f1} \cdot v_{3,f} \cdot \sin\theta_2 \cdot \sin\theta_3 + v_{3,f}^2 \cdot \sin^2\theta_2 \end{cases}$$

$$\Rightarrow \; v_{c,i}^2 \cdot (\cos^2\theta_1 + \sin^2\theta_1) = v_{c,f1}^2 \cdot (\cos^2\theta_3 + \sin^2\theta_3) - 2 \cdot v_{c,f1} \cdot v_{3,f} \cdot (\cos\theta_2 \cdot \cos\theta_3 - \sin\theta_2 \cdot \sin\theta_3)$$

$$+ \, v_{3,f}^2 \cdot (\cos^2\theta_2 + \sin^2\theta_2)$$

$$\Leftrightarrow \; v_{c,i}^2 = v_{c,f1}^2 - 2 \cdot v_{c,f1} \cdot v_{3,f} \cdot \cos(\theta_2 + \theta_3) + v_{3,f}^2$$

$$\Leftrightarrow \; [v_{c,f1}^2 + v_{3,f}^2] = v_{c,f1}^2 - 2 \cdot v_{c,f1} \cdot v_{3,f} \cdot \cos(\theta_2 + \theta_3) + v_{3,f}^2$$

$$\Leftrightarrow \; 0 = -2 \cdot v_{c,f1} \cdot v_{3,f} \cdot \cos(\theta_2 + \theta_3)$$

$$\Leftrightarrow \; \theta_2 + \theta_3 = 90°$$

$$\Leftrightarrow \; \theta_3 = 90° - \theta_2 = 90° - 74.4° = 15.6°$$

If we apply the above procedure to system 2, we obtain the following value for the angle $\theta_5$:

$$\begin{cases} x: \quad v_{c,f}^2 \cdot \cos^2\theta_3 = v_{c,f2}^2 \cdot \sin^2\theta_5 + 2 \cdot v_{c,f2} \cdot v_{8,f} \cdot \sin\theta_4 \cdot \sin\theta_5 + v_{8,f}^2 \cdot \sin^2\theta_4 \\ \\ y: \quad v_{c,f}^2 \cdot \sin^2\theta_3 = v_{c,f2}^2 \cdot \cos^2\theta_5 - 2 \cdot v_{c,f2} \cdot v_{8,f} \cdot \cos\theta_4 \cdot \cos\theta_5 + v_{8,f}^2 \cdot \cos^2\theta_4 \end{cases}$$

$$\Rightarrow \ v_{c,f}^2 = v_{c,f2}^2 - 2 \cdot v_{c,f2} \cdot v_{8,f} \cdot \cos(\theta_4 + \theta_5) + v_{8,f}^2$$

$$\Leftrightarrow \ [v_{c,f2}^2 + v_{8,f}^2] = v_{c,f2}^2 - 2 \cdot v_{c,f2} \cdot v_{8,f} \cdot \cos(\theta_4 + \theta_5) + v_{8,f}^2$$

$$\Leftrightarrow \ 0 = -2 \cdot v_{c,f2} \cdot v_{8,f} \cdot \cos(\theta_4 + \theta_5)$$

$$\Leftrightarrow \ \theta_4 + \theta_5 = 90^\circ$$

$$\Leftrightarrow \ \theta_5 = 90^\circ - \theta_4 = 90^\circ - 74.0^\circ = 16.0^\circ$$

To obtain the value for the speed $v_{c,f1}$ we formulate an expression for $v_{c,f2}$ based on the equation of conservation of linear momentum in the y-direction (system 2) and insert it into the equation in the x-direction:

$$v_{c,f2} = v_{8,f} \cdot \frac{\cos\theta_4}{\cos\theta_5} - v_{c,f1} \cdot \frac{\sin\theta_3}{\cos\theta_5} \ \Rightarrow v_{c,f1} \cdot \cos\theta_3 = [v_{8,f} \cdot \frac{\cos\theta_4}{\cos\theta_5} - v_{c,f1} \cdot \frac{\sin\theta_3}{\cos\theta_5}] \cdot \sin\theta_5 + v_{8,f} \cdot \sin\theta_4$$

$$\Leftrightarrow v_{c,f1} \cdot (\cos\theta_3 + \tan\theta_5 \cdot \sin\theta_3) = v_{8,f} \cdot (\sin\theta_4 + \tan\theta_5 \cdot \cos\theta_4)$$

$$\Leftrightarrow v_{c,f1} = v_{8,f} \cdot \left[\frac{\sin\theta_4 + \tan\theta_5 \cdot \cos\theta_4}{\cos\theta_3 + \tan\theta_5 \cdot \sin\theta_3}\right]$$

$$= 0.86 \cdot \left[\frac{\sin(74.0^\circ) + \tan(16.0^\circ) \cdot \cos(74.0^\circ)}{\cos(15.6^\circ) + \tan(16.0^\circ) \cdot \sin(15.6^\circ)}\right]$$

$$= 0.86 \text{ m/s}$$

Finally, based on the equation of conservation of kinetic energy (system 1), we find the value for $v_{c,i}$:

$$v_{c,i} = \sqrt{v_{c,f1}^2 + v_{3,f}^2} = \sqrt{0.86^2 + 1.25^2} = 1.52 \text{ m/s}$$

(2) The angle $\theta_1$ can be found by using, for instance, the equation of conservation of linear momentum in the x-direction (system 1):

$$\theta_1 = \sin^{-1}\left[\frac{v_{c,f1}}{v_{c,i}} \cdot \cos\theta_3 - \frac{v_{3,f}}{v_{c,i}} \cdot \cos\theta_2\right] = \sin^{-1}\left[\frac{0.86}{1.52} \cdot \cos(15.6^\circ) - \frac{1.25}{1.52} \cdot \cos(74.4^\circ)\right] = 19.0^\circ$$

# Exercise 11

## *Problem Statement*

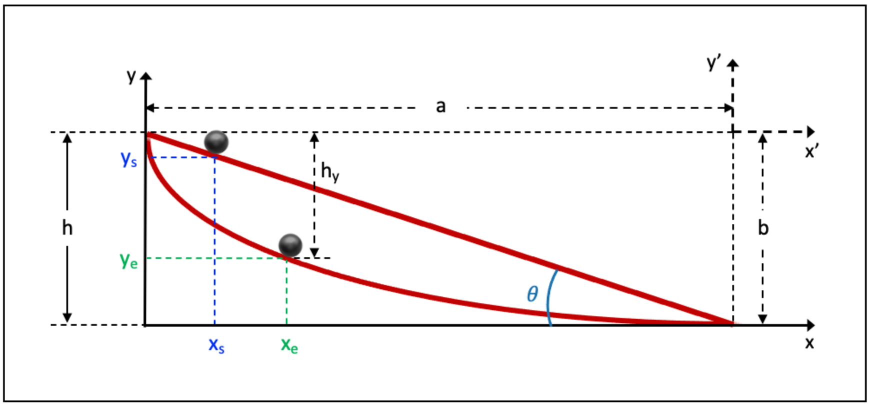


***Figure 12***

In the experimental classroom of the University of Costa Rica in the capital city of San José, Samuel is putting his knowledge on the laws of physics into practice. One of the experiments consists of two large, differently shaped, frictionless ramps put right next to each other, whereby two small steel bearing balls ($m = 0.354$ kg) are released simultaneously from the top of the slope (one ball for each ramp). The purpose of this particular experimental design is to demonstrate pratically how the law of energy conservation is at work. One of the ramps follows a straight path, whereas the other has an elliptical shape—in fact, it is the bottom left segment of an ellipse when dividing a full ellipse into four equal parts. Samuel wishes to figure out what the exact position is of each bearing ball when the speed $v_e$ of the ball on the elliptical trajectory is twice that of the ball on the straight path ($v_s$). If Samuel has already calculated that the ball on the straight trajectory needs $t = 1.91$ s to reach the bottom, at which moment it possesses a speed of $v_{s,f} = 5.83$ m/s, and if he has now installed one of the measuring devices next to the straight path at a height of $y_s = 1.44$ m, what coordinates—with respect to the coordinate system (x,y)—does Samuel find for both bearing balls?

## *Solution*

Given that only conservative forces (gravity) are acting on the bearing balls, we know that the total mechanical energy is conserved. If $y_s$ and $y_e$ are the respective heights for which the condition $v_e = 2 \cdot v_s$ is valid, we can apply the law of energy conservation to each of the bearing balls at those positions with respect to their starting position (at $t = 0$ s, we have that $y = h$, $x = 0$ m, and $v = 0$ m/s):

$$\begin{cases} \text{Straight path:} \quad m \cdot g \cdot h = \dfrac{m \cdot v_s^2}{2} + m \cdot g \cdot y_s \quad \Leftrightarrow \quad g \cdot h = \dfrac{v_s^2}{2} + g \cdot y_s \\ \\ \text{Elliptical path:} \quad m \cdot g \cdot h = \dfrac{m \cdot v_e^2}{2} + m \cdot g \cdot y_e \quad \Leftrightarrow \quad g \cdot h = \dfrac{v_e^2}{2} + g \cdot y_e \end{cases}$$

Setting the two equations equal to each other and making use of the condition that $v_e = 2 \cdot v_s$, we find the following expression for $v_s$:

$$\frac{v_s^2}{2} + g \cdot y_s = \frac{v_e^2}{2} + g \cdot y_e \quad \Leftrightarrow \quad \frac{v_s^2}{2} + g \cdot y_s = \frac{[2 \cdot v_s]^2}{2} + g \cdot y_e$$

$$\Leftrightarrow \quad v_s^2 = \frac{2}{3} \cdot g \cdot (y_s - y_e)$$

If we insert the above expression for $v_s^2$ into the equation of energy conservation for the straight path, we obtain the following expression for the height $y_e$:

$$g \cdot h = \frac{[\frac{2}{3} \cdot g \cdot (y_s - y_e)]}{2} + g \cdot y_s \quad \Leftrightarrow \quad h = \frac{1}{3} \cdot (y_s - y_e) + y_s$$

$$\Leftrightarrow \quad y_e = 4 \cdot y_s - 3 \cdot h$$

Before we can calculate the y-coordinate $y_e$, we need to find the height $h$. Given that the ball on the straight trajectory needs $t = 1.91$ s to reach a speed of $v_{s,f} = 5.83$ m/s at the bottom of the slope, we find that its acceleration over the length $L$ of this path is equal to:

$$v_{s,f} = v_0 + a_s \cdot t \quad \Leftrightarrow \quad a_s = \frac{v_{s,f} - v_0}{t} = \frac{5.83 - 0}{1.91} = 3.05 \text{ m/s}^2$$

As a result, the straight path has the following length $L$:

$$v_{s,f}^2 - v_0^2 = 2 \cdot a_s \cdot L \quad \Leftrightarrow \quad L = \frac{v_{s,f}^2 - v_0^2}{2 \cdot a_s} = \frac{5.83^2 - 0^2}{2 \cdot 3.05} = 5.57 \text{ m}$$

In a next step, applying Newton's second law to the ball on the straight path (in the direction of motion) allows us to determine the angle $\theta$ of the incline:

$$m \cdot g \cdot \sin\theta = m \cdot a_s \quad \Leftrightarrow \quad \theta = \sin^{-1}\left(\frac{a_s}{g}\right) = 18.1°$$

Finally, the height $h$ of both the ramps is then equal to:

$$h = L \cdot \sin\theta = 5.57 \cdot \sin(18.1°) = 1.73 \text{ m}$$

We can now calculate the y-coordinate of the bearing ball on the elliptical trajectory:

$$y_e = 4 \cdot y_s - 3 \cdot h = 4 \cdot 1.44 - 3 \cdot 1.73 = 56.3 \text{ cm}$$

Let us now focus on finding the x-coordinates of both the positions. With regard to the ball on the straight path, we find the x-coordinate with the assistance of trigonometry:

$$x_s = \frac{(h - y_s)}{\tan\theta} = \frac{(1.73 - 1.44)}{\tan(18.1°)} = 89.3 \text{ cm}$$

Regarding the bearing ball on the elliptical path, we have to switch for a moment to the coordinate system (x',y'), whose origin sits at the center of the ellipse. Given that the semi-major axis $a$ and the semi-minor axis $b$ are equal to $a = L \cdot \cos\theta = 5.57 \cdot \cos(18.1°) = 5.29$ m and $b = h = 1.73$ m, respectively, we can calculate the x-position of the bearing ball *with respect to the coordinate system (x',y')* as follows (remember that we are dealing with the bottom left segment of the ellipse):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Leftrightarrow \quad \frac{x_e'^2}{a^2} + \frac{(h_y)^2}{b^2} = 1 \quad \Leftrightarrow \quad \frac{x_e'^2}{a^2} + \frac{(b - y_e)^2}{b^2} = 1$$

$$x_e' = -a \cdot \sqrt{1 - \frac{(b - y_e)^2}{b^2}}$$

$$= -5.29 \cdot \sqrt{1 - \frac{(1.73 - 0.563)^2}{1.73^2}}$$

$$= -3.90 \text{ m}$$

With respect to the coordinate system (x,y), we find the following x-coordinate of the bearing ball on the elliptical path:

$$x_e = a - (0 - x_e') = 5.29 - [0 - (-3.90)] = 1.39 \text{ m}$$

The coordinates for both bearing balls are (0.893, 1.44) for the straight path and (1.39, 0.563) for the elliptical trajectory under the condition that $v_e = 2 \cdot v_s$ for the given height $y_s$.

# Exercise 12

## *Problem Statement*

According to one report of the Umeå University in Umeå, Sweden, the tenuous atmosphere of the planet Mercury is mainly composed of oxygen (42%), sodium (29%), hydrogen (22%), helium (6%), and minor traces of other elements, among which potassium (0.5%). Suppose now that a sodium atom ($m_{Na} = 22.9898$ amu) is whizzing through Mercury's atmosphere relatively close to its surface with a speed of $v_{Na,i} = 1,252$ m/s and under an angle of $\phi = 65.4°$ with the horizontal. A potassium atom ($m_K = 39.0983$ amu), traveling at a speed $v_{K,i}$, is right behind the sodium atom and collides with it, sending the sodium atom straight up. (1) What should be the minimum incoming speed of the potassium atom so that the sodium atom is able to exit Mercury's atmosphere? (2) After the collision, is the kinetic energy of the potassium atom still sufficient to make it out of Mercury's well of gravitational potential energy? Ignore any solar radiation pressure or drag forces for this problem and assume that the collision is perfectly elastic. Remember that 1 atomic mass unit (amu) is equal to 1 amu = $1.661 \times 10^{-24}$ g, that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$), and that the mass and radius of Mercury is equal to $M = 3.30 \times 10^{23}$ kg and $r = 2.44 \times 10^6$ m, respectively.

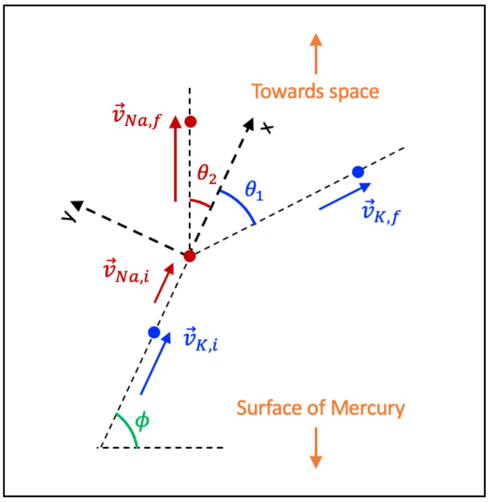


***Figure 13***

## *Solution*

(1) For our isolated system "sodium atom plus potassium atom" in which the collision occurs perfectly elastically we know that both the total linear momentum $\vec{p} = m \cdot \vec{v}$ (for such small masses, the gravitational force becomes negligible) and the total kinetic energy $E_{k,tot}$ are conserved quantities. Therefore, we can write the following three equations:

$$\begin{cases} x : (m_K \cdot v_{K,i}) + (m_{Na} \cdot v_{Na,i}) = (m_K \cdot v_{K,f} \cdot \cos\theta_1) + (m_{Na} \cdot v_{Na,f} \cdot \cos\theta_2) \\ \\ y : 0 = -(m_K \cdot v_{K,f} \cdot \sin\theta_1) + (m_{Na} \cdot v_{Na,f} \cdot \sin\theta_2) \\ \\ \quad (m_K \cdot v_{K,i}^2) + (m_{Na} \cdot v_{Na,i}^2) = (m_K \cdot v_{K,f}^2) + (m_{Na} \cdot v_{Na,f}^2) \end{cases}$$

Let us first have a look at what information we already have. The mass of the two atoms is equal to:

$$\begin{cases} m_K = 39.0983 \cdot 1.661 \times 10^{-27} = 6.49 \times 10^{-26} \text{ kg} \\ \\ m_{Na} = 22.9898 \cdot 1.661 \times 10^{-27} = 3.82 \times 10^{-26} \text{ kg} \end{cases}$$

We can also already determine the angle $\theta_2$, since the angles $\theta_2$ and $\phi$ are complementary angles. As a result, $\theta_2 = 90° - \phi = 90° - 65.4° = 24.6°$.

Next, there is the condition that the speed $v_{Na,f}$ of the sodium atom after the collision should be such that the atom is able to escape Mercury's gravitational pull and disappear into space. Objects that stay in an orbit around a planet are gravitationally bound to the planet. In that case, their total mechanical energy is negative, because their gravitational potential energy, which has a negative value, dominates. However, as soon as the mechanical energy becomes zero or greater (positive), it means that the object is no longer gravitationally bound and can follow a trajectory away from its original orbit around that planet. In other words, its kinetic energy is now the dominant term. The minimum speed $v_{Na,f}$ is then found when $E_{tot} = 0$ J (as the collision occurs relatively close to Mercury's surface, we can use the radius r in the expression for the gravitational potential energy):

$$E_{tot} = 0 \quad \Leftrightarrow \quad \frac{m_{Na} \cdot v_{Na,f}^2}{2} - \frac{G \cdot m_{Na} \cdot M}{r} = 0 \quad \Leftrightarrow \quad v_{Na,f} = \sqrt{\frac{2 \cdot G \cdot M}{r}}$$

$$= \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \cdot 3.30 \times 10^{23}}{2.44 \times 10^6}}$$

$$= 4.25 \times 10^3 \text{ m/s}$$

With three unknown variables left, we can now determine the initial speed $v_{K,i}$ of the potassium atom. In a first step, we rearrange and square the two equations related to the conservation of linear momentum and subsequently add them together (whereby we make use of the trigonometric identity "$\cos^2 \alpha + \sin^2 \alpha = 1$"):

$$\begin{cases} x: \quad [(m_K \cdot v_{K,i}) + (m_{Na} \cdot v_{Na,i}) - (m_{Na} \cdot v_{Na,f} \cdot \cos\theta_2)]^2 = m_K^2 \cdot v_{K,f}^2 \cdot \cos^2\theta_1 \\ y: \quad m_{Na}^2 \cdot v_{Na,f}^2 \cdot \sin^2\theta_2 = m_K^2 \cdot v_{K,f}^2 \cdot \sin^2\theta_1 \end{cases}$$

$$\Rightarrow \; m_K^2 \cdot v_{K,i}^2 + 2 \cdot m_K \cdot m_{Na} \cdot v_{Na,i} \cdot v_{K,i} + m_{Na}^2 \cdot v_{Na,i}^2 + m_{Na}^2 \cdot v_{Na,f}^2 - 2 \cdot m_K \cdot m_{Na} \cdot v_{Na,f} \cdot \cos\theta_2 \cdot v_{K,i}$$

$$- 2 \cdot m_{Na}^2 \cdot v_{Na,i} \cdot v_{Na,f} \cdot \cos\theta_2 = m_K^2 \cdot v_{K,f}^2$$

To write the above expression only in terms of the unknown variable $v_{K,i}$, we want to eliminate the unknown variable $v_{K,f}$. Using the equation of conservation of kinetic energy, we find the following expression for the term $m_K \cdot v_{K,f}^2$:

$$m_K \cdot v_{K,f}^2 = (m_K \cdot v_{K,i}^2) + (m_{Na} \cdot v_{Na,i}^2) - (m_{Na} \cdot v_{Na,f}^2)$$

Inserting the above expression into our first equation, we can calculate the value of $v_{K,i}$:

$$m_K^2 \cdot v_{K,i}^2 + 2 \cdot m_K \cdot m_{Na} \cdot v_{Na,i} \cdot v_{K,i} + m_{Na}^2 \cdot v_{Na,i}^2 + m_{Na}^2 \cdot v_{Na,f}^2 - 2 \cdot m_K \cdot m_{Na} \cdot v_{Na,f} \cdot \cos\theta_2 \cdot v_{K,i}$$

$$- 2 \cdot m_{Na}^2 \cdot v_{Na,i} \cdot v_{Na,f} \cdot \cos\theta_2 = m_K \cdot [(m_K \cdot v_{K,i}^2) + (m_{Na} \cdot v_{Na,i}^2) - (m_{Na} \cdot v_{Na,f}^2)]$$

$$\Leftrightarrow \; 2 \cdot m_K \cdot (v_{Na,i} - v_{Na,f} \cdot \cos\theta_2) \cdot v_{K,i} = (m_K - m_{Na}) \cdot v_{Na,i}^2 - (m_K + m_{Na}) \cdot v_{Na,f}^2 + 2 \cdot m_{Na} \cdot v_{Na,i} \cdot v_{Na,f} \cdot \cos\theta_2$$

$$\Leftrightarrow \; v_{K,i} = \frac{(m_K - m_{Na}) \cdot v_{Na,i}^2 - (m_K + m_{Na}) \cdot v_{Na,f}^2 + 2 \cdot m_{Na} \cdot v_{Na,i} \cdot v_{Na,f} \cdot \cos\theta_2}{2 \cdot m_K \cdot (v_{Na,i} - v_{Na,f} \cdot \cos\theta_2)}$$

$$= \frac{(6.49\times10^{-26} - 3.82\times10^{-26}) \cdot 1{,}252^2 - (6.49\times10^{-26} + 3.82\times10^{-26}) \cdot (4.25\times10^3)^2 + 2 \cdot 3.82\times10^{-26} \cdot 1{,}252 \cdot 4.25\times10^3 \cdot \cos(24.6°)}{2 \cdot 6.49\times10^{-26} \cdot [1{,}252 - 4.25\times10^3 \cdot \cos(24.6°)]}$$

$$= 4.28 \times 10^3 \text{ m/s}$$

(2) To find the kinetic energy $E_{k,K,f}$ of the potassium atom after the collision, we use the equation related to the conservation of kinetic energy:

$$E_{k,K,f} = \frac{m_K \cdot v_{K,f}^2}{2} = \frac{m_K \cdot v_{K,i}^2}{2} + \frac{m_{Na} \cdot v_{Na,i}^2}{2} - \frac{m_{Na} \cdot v_{Na,f}^2}{2}$$

$$= \frac{m_K \cdot v_{K,i}^2}{2} + \frac{m_{Na}}{2} \cdot (v_{Na,i}^2 - v_{Na,f}^2)$$

$$= \frac{6.49 \times 10^{-26} \cdot (4.28 \times 10^3)^2}{2} + \frac{3.82 \times 10^{-26}}{2} \cdot [1,252^2 - (4.25 \times 10^3)^2]$$

$$= 2.79 \times 10^{-19} \text{ J}$$

If we compare the kinetic energy against the gravitational potential energy $E_{p,K}$ of the potassium atom, which is equal to:

$$E_{p,K} = -\frac{G \cdot m_K \cdot M}{r} = -\frac{6.67 \times 10^{-11} \cdot 6.49 \times 10^{-26} \cdot 3.30 \times 10^{23}}{2.44 \times 10^6} = -5.86 \times 10^{-19} \text{ J}$$

it follows that the potential energy dominates, resulting in a negative value for the total mechanical energy. In other words, the potassium atom will not be able to escape Mercury's gravitational pull. Note, however, that if the potassium atom did not collide with the sodium atom, it would have escaped into space, since its initial speed ($v_{K,i} = 4.28 \times 10^3$ m/s) exceeded the magnitude of the escape velocity ($v_{esc} = v_{Na,f} = 4.25 \times 10^3$ m/s).

# Exercise 13

## *Problem Statement*

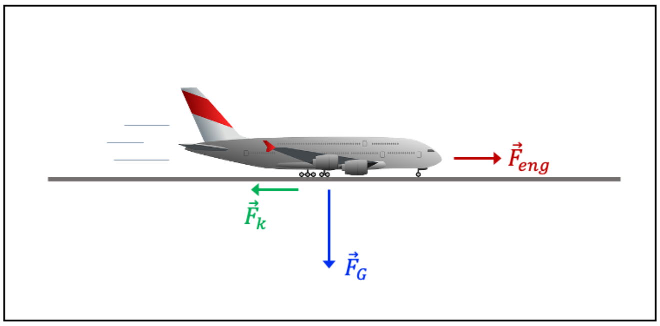


***Figure 14***

You are sitting at gate 15 of the Shenyang Taoxian International Airport, which is located at the capital city of Shenyang in the province Liaoning in China, waiting to board your flight CZ3602 to Guangzhou in the south of China. Being the astute engineer that you are, you immediately spot that the airplane model is the Airbus A320 Neo and since you have some free time on your hands, you decide to do some off-hand calculations. The flight attendant mentioned earlier that $n = 161$ passengers booked a seat, and you estimate that each person weighs about $m_{pas} = 75.0$ kg and that they carry $m_{hl} = 5.00$ kg of hand luggage and checked in a suitcase of $m_{sc} = 16.5$ kg. You further know that an empty A320 Neo model has a mass of $m_{pl} = 44.3$ t and that the fuel tanks contain approximately 27,500 L of jet fuel (with a density of $d = 692$ g/L). This specific model is furthermore equipped with two Pratt & Whitney PW1127G engines that each give a thrust of $T = 27,000$ lbf. As it is raining, you estimate that the tires create a slightly lower kinetic friction ($\mu_k = 0.135$) with the runway. (1) You're interested in finding the speed $v_h$ of the airplane halfway the runway, which has a total length of $L = 1,982$ m. What value for $v_h$ do you write down in your notebook? (2) If you estimate that the average power of the plane at that moment is equal to $P_h = 3.97$ MW and that the plane requires 70.7 % of the total takeoff time t to get to that point, how much time does it still need to accelerate before taking off? (3) What value do you find for the speed $v_f$ at lift-off? Remember that the pound-force is equal to 1 lbf = 1 lb × g, with $g$ the acceleration due to gravity, and you assume that the pilot needs the entire length of the runway to take off.

## *Solution*

(1) Let us in a first instance determine the total mass $m$ of the airplane at the moment when it is about to start accelerating at the beginning of the taxiway. The total mass of the passengers, including their luggage, is equal to:

$$m_{tot,p} = n \cdot (m_{pass} + m_{hl} + m_{sc}) = 161 \cdot (75.0 + 5.00 + 16.5) = 1.55 \times 10^4 \text{ kg}$$

Given that the mass of the empty airplane is equal to $m_{pl} = 4.43 \times 10^4$ kg and that of the jet fuel to $m_f = 27,500 \times 0.692 = 1.90 \times 10^4$ kg, we find the following total mass m:

$$m = m_{tot,p} + m_{pl} + m_f = 1.55 \times 10^4 + 4.43 \times 10^4 + 1.90 \times 10^4 = 7.89 \times 10^4 \text{ kg}$$

To determine the speed $v_h$, let us look at the work done on the system "Airbus A320 Neo". Given that the motion of the airplane is perpendicular to the gravitational force $\vec{F}_G$, we know that conservative forces are not performing any work on the system. Therefore, the total work $W_{tot,h}$ done is equal to the work $W_{ext}$ done by external forces, i.e., the thrust force $\vec{F}_{eng}$ generated by the engines and the kinetic friction force $\vec{F}_k$. The magnitude of the total thrust force $\vec{F}_{eng}$ delivered by the two engines is equal to (remember that 1 lb = 0.4536 kg):

$$F_{eng} = 2 \cdot (T \cdot 0.4536 \cdot g) = 2 \cdot (27,000 \cdot 0.4536 \cdot 9.81) = 2.40 \times 10^5 \text{ N}$$

The magnitude of the friction force $\vec{F}_k$ is equal to:

$$F_k = \mu_k \cdot F_N = \mu_k \cdot (m \cdot g) = 0.135 \cdot (7.89 \times 10^4 \cdot 9.81) = 1.04 \times 10^5 \text{ N}$$

The speed $v_h$ halfway the runway is then calculated as follows, whereby we make use of the work-energy relation:

$$\begin{aligned} W_{tot,h} = W_{ext} \quad \Leftrightarrow \quad \Delta E_k = W_k + W_{eng} \quad \Leftrightarrow \quad & \frac{m \cdot v_h^2}{2} - \frac{m \cdot v_0^2}{2} = (-F_k + F_{eng}) \cdot \frac{L}{2} \\ \Leftrightarrow \quad & v_h = \sqrt{v_0^2 + (-F_k + F_{eng}) \cdot \frac{L}{m}} \\ & \quad = \sqrt{0^2 + (-1.04 \times 10^5 + 2.40 \times 10^5) \cdot \frac{1,982}{7.89 \times 10^4}} \\ & \quad = 58.4 \text{ m/s or } 210 \text{ km/h} \end{aligned}$$

(2) Applying the definition of average power to the position halfway the runway allows us to find the respective amount of time $t_h$:

$$\begin{aligned} P_h = \frac{W_{tot,h}}{t_h} = \frac{(-F_k + F_{eng}) \cdot \frac{L}{2}}{t_h} \quad \Leftrightarrow \quad t_h &= \frac{(-F_k + F_{eng}) \cdot \frac{L}{2}}{P_h} \\ &= \frac{(-1.04 \times 10^5 + 2.40 \times 10^5) \cdot \frac{1,982}{2}}{3.97 \times 10^6} \\ &= 33.9 \text{ s} \end{aligned}$$

Given that the time $t_h$ required to reach the midpoint of the taxiway is equal to $t_h = 0.707 \cdot t$, we find the remaining amount of time $t_{rem}$ until the airplane reaches lift-off as follows:

$$t_{rem} = t - t_h = \frac{t_h}{0.707} - t_h = t_h \cdot \left(\frac{1 - 0.707}{0.707}\right) = 33.9 \cdot \left(\frac{1 - 0.707}{0.707}\right) = 14.1 \text{ s}$$

(3) Since the total work $W_{tot}$ done on the system "Airbus A320 Neo" by the end of the runway is twice that at the midpoint position—the external forces are performing work over twice the same distance $\frac{L}{2}$—we can find the lift-off speed $v_f$ as follows:

$$W_{tot} = 2 \cdot W_{tot,h} \quad \Leftrightarrow \quad \frac{m \cdot v_f^2}{2} = 2 \cdot W_{tot,h} \quad \Leftrightarrow \quad v_f = \sqrt{\frac{4 \cdot W_{tot,h}}{m}} = \sqrt{\frac{4 \cdot (-F_k + F_{eng}) \cdot \frac{L}{2}}{m}}$$

$$= \sqrt{2} \cdot \sqrt{(-F_k + F_{eng}) \cdot \frac{L}{m}}$$

$$= \sqrt{2} \cdot v_h$$

$$= \sqrt{2} \cdot 58.4$$

$$= 82.6 \text{ m/s or } 297 \text{ km/h}$$

# Exercise 14

## ***Problem Statement***

At high school, Hilde enjoyed studying mathematics and during her two final years she chose the advanced course option whereby she was taught 8 hours of mathematics per week. After high school, Hilde wanted to combine her interest in mathematics with her fascination for the natural laws that explain how the physical world works. As a result, she decided to pursue a master's degree in physics and astronomy at the Free University of Brussels, in Belgium. After a semester of hard work, Hilde is ready to tackle her first exam, which, according to her schedule, is that of the course "Classical Mechanics". The opening question consists of three parts and reads as follows. A particle is undergoing a force $\vec{F}(\vec{r})$, which is equal to $\vec{F}(\vec{r}) = \left[\frac{xy^2z^2}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_x + \left[\frac{x^2yz^2}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_y + \left[\frac{x^2y^2z}{2} \cdot \cos(kxyz)\right] \cdot \vec{i}_z$, with k a constant equal to $k = 0.453$. (1) Show that the force $\vec{F}(\vec{r})$ is conservative. (2) Determine the potential energy function $V(\vec{r})$, whereby $V(\vec{0}) = 0$. (3) Calculate the work done on the particle by this force as it moves from $\vec{r}_1 = (2, 2, 2)$ to $\vec{r}_2 = (1, -3, 5)$. How did Hilde answer this opening question?

## ***Solution***

(1) A conservative force is a force for which the net (total) work performed by this force to move a particle from one point in space to another is independent of the path taken between those two points. As a result, the net work done by a conservative force on a particle *with respect to a closed path* is always equal to zero. In mathematical language, this means that the line integral of that force over the closed path must be zero. The question of whether a force is conservative can be addressed relying on a mathematical theorem called Stokes' theorem, which relates this line integral to the amount of circular movement that this force sends through the surface enclosed by the closed path—in more technical terms, we say that the line integral is equal to the flux of the curl of this force. Mathematically, this reads as follows:

$$\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}(\vec{r})\right) \cdot \vec{n} \cdot dS$$

If the line integral of a conservative force over a closed path is zero, then the curl of that force, which is equal to the cross (vector) product between the nabla operator $\vec{\nabla}$ and the force $\vec{F}(\vec{r})$, must be equal to the null vector $\vec{0}$. In other words:

$$\vec{\nabla} \times \vec{F}(\vec{r}) = \vec{0} \quad \Leftrightarrow \quad \left[\frac{\partial}{\partial x} \cdot \vec{i}_x + \frac{\partial}{\partial y} \cdot \vec{i}_y + \frac{\partial}{\partial z} \cdot \vec{i}_z\right] \times \left[F_x \cdot \vec{i}_x + F_y \cdot \vec{i}_y + F_z \cdot \vec{i}_z\right] = \vec{0}$$

$$\Leftrightarrow \quad \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \cdot \vec{i}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \cdot \vec{i}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cdot \vec{i}_z = \vec{0}$$

At the component level, this means that a force is conservative if the following three conditions are satisfied *simultaneously*:

$$\begin{cases} x: \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} = 0 \quad \Leftrightarrow \quad \dfrac{\partial F_z}{\partial y} = \dfrac{\partial F_y}{\partial z} \\[2ex] y: \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x} = 0 \quad \Leftrightarrow \quad \dfrac{\partial F_x}{\partial z} = \dfrac{\partial F_z}{\partial x} \\[2ex] z: \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y} = 0 \quad \Leftrightarrow \quad \dfrac{\partial F_y}{\partial x} = \dfrac{\partial F_x}{\partial y} \end{cases}$$

We are now ready to apply this criterion to the force $\vec{F}(\vec{r})$ of Hilde's exam question. Let us take the left-hand side of the first equation:

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y^2z}{2} \cdot \cos(kxyz)\right] = x^2yz \cdot \cos(kxyz) - \frac{kx^3y^2z^2}{2} \cdot \sin(kxyz)$$

The right-hand side of the first equation is equal to:

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x^2yz^2}{2} \cdot \cos(kxyz)\right] = x^2yz \cdot \cos(kxyz) - \frac{kx^3y^2z^2}{2} \cdot \sin(kxyz)$$

The above two partial derivates are equal to each other so that the condition with respect to the x-component is satisfied. Similarly, for the y-and z-component, we find the following:

$$y: \begin{cases} \frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}\left[\frac{xy^2z^2}{2} \cdot \cos(kxyz)\right] = xy^2z \cdot \cos(kxyz) - \frac{kx^2y^3z^2}{2} \cdot \sin(kxyz) \\[2ex] \frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2y^2z}{2} \cdot \cos(kxyz)\right] = xy^2z \cdot \cos(kxyz) - \frac{kx^2y^3z^2}{2} \cdot \sin(kxyz) \end{cases}$$

$$z: \begin{cases} \frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2yz^2}{2} \cdot \cos(kxyz)\right] = xyz^2 \cdot \cos(kxyz) - \frac{kx^2y^2z^3}{2} \cdot \sin(kxyz) \\[2ex] \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left[\frac{xy^2z^2}{2} \cdot \cos(kxyz)\right] = xyz^2 \cdot \cos(kxyz) - \frac{kx^2y^2z^3}{2} \cdot \sin(kxyz) \end{cases}$$

Since the condition of each of the three components is satisfied, we can conclude that the force $\vec{F}(\vec{r})$ is conservative.

(2) Let us start with writing the definition of the work W done by the force $\vec{F}(\vec{r})$, whereby we

transform the initial integral with respect to the position into an integral expressed in terms of the time variable:

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r} \cdot \frac{dt}{dt} = \int_{t_1}^{t_2} \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt} \cdot dt$$

We also know that the work done by conservative forces between two points is equal to minus the difference of the potential energy at these two points. If we choose the origin ($\vec{0}$) as our first point and a random position vector $\vec{r}$ as our second point, and given our initial condition of $V(\vec{0}) = 0$, we can write the following:

$$W = -\Delta V = -\left[V(\vec{r}) - V(\vec{0})\right] = -\left[V(\vec{r}) - 0\right] = -V(\vec{r})$$

Combining the above two equations for the work W, we obtain the following expression for a conservative force $\vec{F}(\vec{r})$, whereby we make use of the product between two vectors, i.e., the dot product:

$$\int_{t_1}^{t_2} \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt} \cdot dt = -V(\vec{r}) \quad \Leftrightarrow \quad \frac{d}{dt}\left[\int_{t_1}^{t_2} \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt} \cdot dt\right] = \frac{d}{dt}\left[-V(\vec{r})\right]$$

$$\Leftrightarrow \quad \vec{F}(\vec{r}) \cdot \frac{d\vec{r}}{dt} = -\frac{dV(\vec{r})}{dt}$$

$$= -\left[\frac{\partial V(\vec{r})}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V(\vec{r})}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial V(\vec{r})}{\partial z} \cdot \frac{dz}{dt}\right]$$

$$= -\left[\vec{\nabla} V(\vec{r}) \cdot \frac{d\vec{r}}{dt}\right]$$

$$\Leftrightarrow \quad \vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

The three components of this expression are then equal to:

$$F_x = -\frac{\partial V(\vec{r})}{\partial x} \qquad F_y = -\frac{\partial V(\vec{r})}{\partial y} \qquad F_z = -\frac{\partial V(\vec{r})}{\partial z}$$

Returning now to Hilde's exam, if we take, for instance, the x-direction, the potential energy function $V(\vec{r})$ can be obtained as follows:

$$-\frac{\partial V(\vec{r})}{\partial x} = F_x \quad \Leftrightarrow \quad dV(\vec{r}) = -F_x \cdot dx \quad \Leftrightarrow \quad \int dV(\vec{r}) = \int -F_x \cdot dx \quad \Leftrightarrow \quad V(\vec{r}) = -\int F_x \cdot dx$$

$$\begin{aligned}
\Leftrightarrow \quad V(\vec{r}) &= -\int \left[\frac{xy^2z^2}{2} \cdot \cos(kxyz)\right] \cdot dx \\
&= -\frac{y^2z^2}{2} \cdot \int x \cdot \cos(kxyz) \cdot dx \\
&= -\frac{y^2z^2}{2} \cdot \left[\frac{1}{k^2y^2z^2} \cdot \cos(kxyz) + \frac{x}{kyz} \cdot \sin(kxyz)\right] + c \\
&= -\frac{1}{2k^2} \cdot [\cos(kxyz) + kxyz \cdot \sin(kxyz)] + c
\end{aligned}$$

The integration constant c can be found by implementing the initial condition $V(\vec{0}) = 0$:

$$V(\vec{0}) = 0 = -\frac{1}{2k^2} \cdot [\cos(0) + 0 \cdot \sin(0)] + c \quad \Leftrightarrow \quad c = \frac{1}{2k^2}$$

The potential energy function $V(\vec{r})$ then becomes:

$$\begin{aligned}
V(\vec{r}) &= -\frac{1}{2k^2} \cdot [\cos(kxyz) + kxyz \cdot \sin(kxyz)] + \frac{1}{2k^2} \\
&= -\frac{1}{2k^2} \cdot [\cos(kxyz) + kxyz \cdot \sin(kxyz) - 1]
\end{aligned}$$

(3) The work done by the force $\vec{F}(\vec{r})$ to move the particle from $\vec{r}_1 = (2,2,2)$ to $\vec{r}_2 = (1,-3,5)$ is calculated as follows:

$$\begin{aligned}
W &= -\left[V(\vec{r_2}) - V(\vec{r_1})\right] \\
&= -\left[-\tfrac{1}{2k^2} \cdot [\cos(kx_2y_2z_2) + kx_2y_2z_2 \cdot \sin(kx_2y_2z_2) - 1] + \tfrac{1}{2k^2} \cdot [\cos(kx_1y_1z_1) + kx_1y_1z_1 \cdot \sin(kx_1y_1z_1) - 1]\right] \\
&= -\left[-\frac{1}{2 \cdot 0.453^2} \cdot [\cos(0.453 \cdot 1 \cdot (-3) \cdot 5) + (0.453 \cdot 1 \cdot (-3) \cdot 5) \cdot \sin(0.453 \cdot 1 \cdot (-3) \cdot 5) - 1] + \right. \\
&\quad \left. \frac{1}{2 \cdot 0.453^2} \cdot [\cos(0.453 \cdot 2 \cdot 2 \cdot 2) + 0.453 \cdot 2 \cdot 2 \cdot 2 \cdot \sin(0.453 \cdot 2 \cdot 2 \cdot 2) - 1]\right] \\
&= -(-1.94 + 0.553) \\
&= 1.39 \text{ J}
\end{aligned}$$

# Exercise 15

## *Problem Statement*

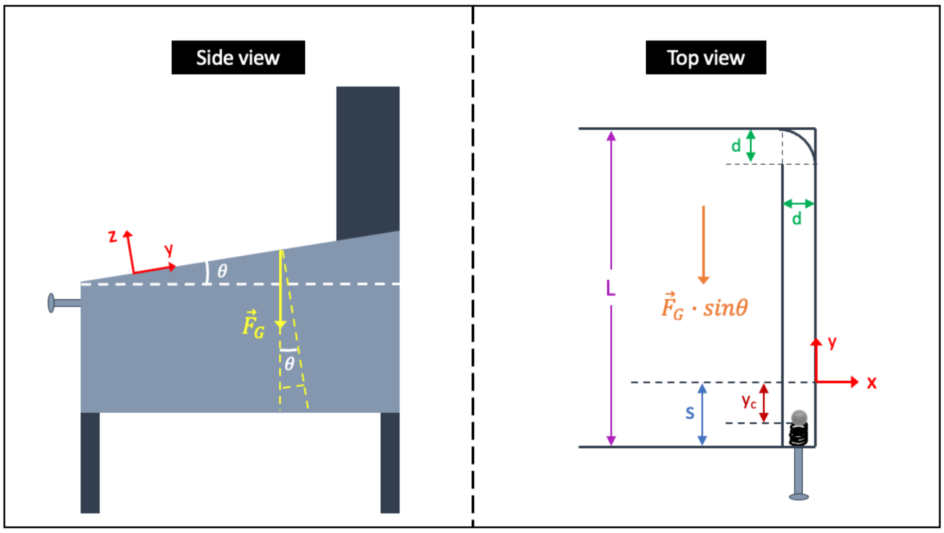


***Figure 15***

For the past 2 years, Toivo has held the maximum score on the Cyclone pinball machine in the local pub Kurva Kodu in Rakvere, Estonia. However, last night, his best friend Kaarli broke his record, and since then Toivo has been trying non-stop to regain his leader position on the Hall of Fame Scoreboard. The pinball machine has a length of $L = 1.25$ m and the playfield makes an angle of $\theta = 9.65°$ with the horizontal. The top left and top right corners of the playfield are round in shape and on the right-hand side, there is a long isolated compartment from where the metal ball (with a mass and radius equal to $m = 0.252$ kg and $r = 0.550$ cm) is launched. The compartment has a width of $d = 8.00$ cm and its left edge stops at a distance $d$ from the top edge of the pinball machine. The launch mechanism is a spring ($k = 155$ N/m), which compresses when being pulled backwards from outside of the machine. In resting mode, the equilibrium length of the spring is equal to $s = 14.0$ cm. Toivo has also figured out that it greatly benefits his game if the ball enters the playfield when it still "sticks" to the top edge of the pinball machine as it exits the top right rounded corner. If you know that the metal ball produces kinetic friction ($\mu_k = 0.228$) with the bottom surface of the playfield, how far back, at a minimum, should Toivo pull the external handle so that, upon release, the metal ball enters the playfield with the greatest odds of beating Kaarli's maximum score?

## *Solution*

Given that a kinetic friction force $\vec{F}_k$ acts on the metal ball (opposite to its direction of motion), the total mechanical energy of the ball is not conserved. What has to be taken into account is the work $W_{ext}$ done by $\vec{F}_k$. The work-energy relation is then written as follows, whereby $E_k$ and $E_p$ represent the kinetic and potential energy, respectively:

$$E_{k,1} + E_{p,1} + W_{ext,1} = E_{k,2} + E_{p,2}$$

Suppose that we consider the mechanical energy $E_1$ as the energy of the ball at its position whereby the spring is compressed by the distance $y_c$, whereas the mechanical energy $E_2$ is the energy of the ball right before it enters the rounded top right corner at a distance $L - s - d$. The work-energy relation then obtains the following form (note that $E_{k,1} = 0$ J):

$$E_{p,1} + W_{ext} = E_{k,2} + E_{p,2}$$

$$\Leftrightarrow \left[\tfrac{k \cdot y_c^2}{2} - m \cdot g \cdot \sin\theta \cdot y_c\right] + [-\mu_k \cdot m \cdot g \cdot \cos\theta \cdot (L - s - d + y_c)] = \left[\tfrac{m \cdot v^2}{2}\right] + [m \cdot g \cdot \sin\theta \cdot (L - s - d)]$$

$$\Leftrightarrow \frac{k \cdot y_c^2}{2} - [m \cdot g \cdot (\sin\theta + \mu_k \cdot \cos\theta)] \cdot y_c - m \cdot g \cdot (L - s - d) \cdot (\sin\theta + \mu_k \cdot \cos\theta) = \frac{m \cdot v^2}{2}$$

What we know want to do is the find an expression for the kinetic energy $E_{k,2} = \frac{m \cdot v^2}{2}$ which we can insert into the above work-energy equation. In a next step, let us go through the same exercise as above considering now the energy $E_2$ and the energy $E_3$, which is the energy of the ball when it exits the top right corner, whereby we demand that it follows a path maximally outwards when going through the corner. As the ball has a radius of $r = 0.550$ cm, the radius $r_c$ of the circular trajectory followed by the ball through the corner is then equal to $r_c = d - r = 8.00 - 0.550 = 7.45$ cm. The work-energy equation then states:

$$E_{k,2} + E_{p,2} + W_{ext,2} = E_{k,3} + E_{p,3}$$

$$\Leftrightarrow \left[\frac{m \cdot v^2}{2}\right] + [m \cdot g \cdot \sin\theta \cdot (L - s - d)] + \left[-\mu_k \cdot m \cdot g \cdot \cos\theta \cdot (r_c \cdot \frac{\pi}{2})\right] = \left[\frac{m \cdot v_f^2}{2}\right] +$$

$$[m \cdot g \cdot \sin\theta \cdot (L - s - r)]$$

The requirement that the metal ball sticks to the top edge of the pinball machine leads to another constraint, which we can use to replace the expression $v_f^2$ in the above work-energy equation. When the ball exits the top right corner, the centripetal force exerted on the ball must be, at a minimum, equal to the gravitational force. Therefore, we can write:

$$\frac{m \cdot v_f^2}{r_c} = m \cdot g \sin\theta \quad \Leftrightarrow \quad v_f^2 = r_c \cdot g \sin\theta$$

Plugging this expression for $v_f^2$ back into our second work-energy equation, we obtain the following expression for the kinetic energy $E_{k,2} = \frac{m \cdot v^2}{2}$:

$$\left[\frac{m \cdot v^2}{2}\right] + [m \cdot g \cdot \sin\theta \cdot (L - s - d)] + \left[-\mu_k \cdot m \cdot g \cdot \cos\theta \cdot (r_c \cdot \frac{\pi}{2})\right] = \left[\frac{m \cdot (r_c \cdot g \cdot \sin\theta)}{2}\right] +$$

$$[m \cdot g \cdot \sin\theta \cdot (L - s - r)]$$

$$\Leftrightarrow \frac{m \cdot v^2}{2} = \frac{m \cdot g}{2} \cdot [(2d - 2r + r_c) \cdot \sin\theta + \mu_k \cdot \pi \cdot r_c \cdot \cos\theta]$$

Inserting the above expression into our first work-energy equation, we obtain the following quadratic equation:

$$\frac{k \cdot y_c^2}{2} - [m \cdot g \cdot (\sin\theta + \mu_k \cdot \cos\theta)] \cdot y_c - m \cdot g \cdot (L - s - d) \cdot (\sin\theta + \mu_k \cdot \cos\theta) =$$

$$\frac{m \cdot g}{2} \cdot [(2d - 2r + r_c) \cdot \sin\theta + \mu_k \cdot \pi \cdot r_c \cdot \cos\theta]$$

$$\Leftrightarrow \ \frac{k \cdot y_c^2}{2} - [m \cdot g \cdot (\sin\theta + \mu_k \cdot \cos\theta)] \cdot y_c - m \cdot g \cdot \Big[(L - s - r + \frac{r_c}{2}) \cdot \sin\theta +$$

$$\mu_k \cdot (L - s - d + \frac{\pi \cdot r_c}{2}) \cdot \cos\theta\Big] = 0$$

$$\Leftrightarrow \ \tfrac{155}{2} \cdot y_c^2 - [0.252 \cdot 9.81 \cdot [\sin(9.65°) + 0.228 \cdot \cos(9.65°)]] \cdot y_c -$$

$$0.252 \cdot 9.81 \cdot \big[\big(1.25 - 0.14 - 0.00550 + \tfrac{0.0745}{2}\big) \cdot \sin(9.65°) + 0.228 \cdot \big(1.25 - 0.14 - 0.08 + \tfrac{\pi \cdot 0.0745}{2}\big) \cdot \cos(9.65°)\big] = 0$$

Solving the above quadratic equation gives a physically sensible solution ($y_c \geq 0$ since we treated it as a distance) equal to $y_c = 12.6$ cm. When Toivo pulls back the external handle by a distance of 12.6 cm, then the metal ball will "stick" to the upper edge of the pinball machine when exiting the top right corner, maximizing his chances of beating the top score of his best friend Kaarli.

Note furthermore that we took an extra step in the above solution, i.e., we considered the mechanical energy $E_2$ of the metal ball right before entering the top right corner. A more efficient approach would have been to only consider the mechanical energy $E_1$ and $E_3$, which should give the same result.

# Exercise 16

## ***Problem Statement***

Suppose that 30,000 years ago, at a distance of $d = 2.50$ light years away from the center of the Sun, two massive rocks collided. The first rock, with a mass of $m_1 = 5.95 \times 10^5$ kg, smashed with a speed of $v_{1,i} = 95,400$ km/h into a heavier second rock ($m_2 = 1.22 \times 10^6$ kg), which was traveling slower at $v_{2,i} = 10,200$ km/h. After the collision, which happened to be perfectly elastic, rock 1 deviated from its original path by an angle of $\alpha = 33.2°$ and headed straight towards our Solar System, which we consider, for practical purposes, to be equal to the system "the Sun plus planet Earth". Moreover, as soon as it followed its new course, rock 1 became sensitive to the gravitational influence of our Solar System (ignore the gravitational pull by rock 2). Today, rock 1 finally reached our Solar system and is about to hit the surface of the Sun. If you know that at that moment the Earth is in an orbital position at 90° with respect to the line of trajectory of rock 1, what is the rock's speed as it crashes into the Sun? Remember that the universal gravitational constant $G$ is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$), that the mass and radius of the Sun are equal to $M_s = 1.99 \times 10^{30}$ kg and $r_s = 6.96 \times 10^5$ km, respectively, that the mass of the Earth is equal to $M_E = 5.97 \times 10^{24}$ kg, and that 1 light year measures $9.46 \times 10^{12}$ km. Also take into account that at a distance $d$ the Earth-Sun distance ($r_{ES} = 1.496 \times 10^8$ km) becomes, relatively speaking, very small and can be ignored in the calculations.

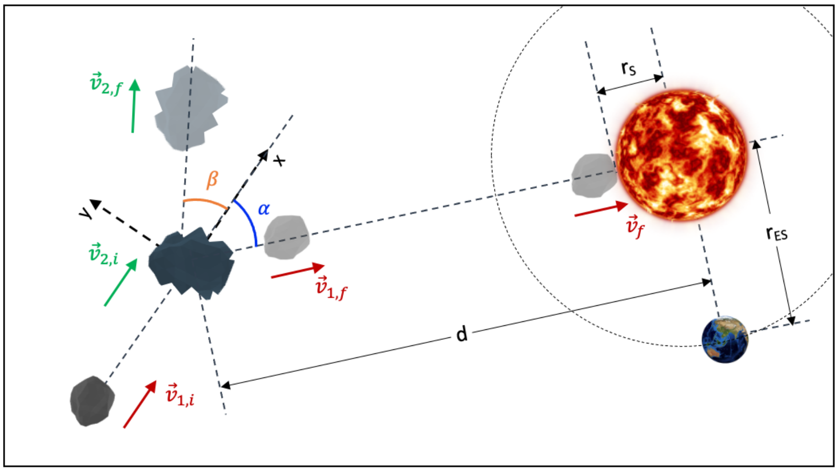


***Figure 16***

## ***Solution***

As the gravitational influence of our Solar System only becomes noticeable after the collision, we can assume that the collision occurred in an isolated system. Therefore, the total linear momentum $\vec{p} = m \cdot \vec{v}$ is conserved, and given that both rocks collided in a perfectly elastically fashion, the total kinetic energy $E_{k,tot}$ is equally constant. We can then write the following three equations:

$$\begin{cases} x : (m_1 \cdot v_{1,i}) + (m_2 \cdot v_{2,i}) = (m_1 \cdot v_{1,f} \cdot \cos\alpha) + (m_2 \cdot v_{2,f} \cdot \cos\beta) \\ \\ y : 0 = -(m_1 \cdot v_{1,f} \cdot \sin\alpha) + (m_2 \cdot v_{2,f} \cdot \sin\beta) \\ \\ \quad (m_1 \cdot v_{1,i}^2) + (m_2 \cdot v_{2,i}^2) = (m_1 \cdot v_{1,f}^2) + (m_2 \cdot v_{2,f}^2) \end{cases}$$

Rearranging and squaring both the x- and y-equation and subsequently adding them together and making use of the trigonometric identity "$\sin^2\theta + \cos^2\theta = 1$" as well as the equation related to the conservation of kinetic energy, we obtain the following quadratic equation:

$$\begin{cases} x: & [(m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i}) - (m_1 \cdot v_{1,f} \cdot \cos\alpha)]^2 = m_2^2 \cdot v_{2,f}^2 \cdot \cos^2\beta \\ \\ y: & m_1^2 \cdot v_{1,f}^2 \cdot \sin^2\alpha = m_2^2 \cdot v_{2,f}^2 \cdot \sin^2\beta \end{cases}$$

$$\Rightarrow \; (m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i})^2 - 2 \cdot m_1 \cdot v_{1,f} \cdot (m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i}) \cdot \cos\alpha + m_1^2 \cdot v_{1,f}^2 = m_2^2 \cdot v_{2,f}^2$$

$$\Leftrightarrow \; (m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i})^2 - 2 \cdot m_1 \cdot v_{1,f} \cdot (m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i}) \cdot \cos\alpha + m_1^2 \cdot v_{1,f}^2 =$$

$$m_2 \cdot [m_1 \cdot v_{1,i}^2 + m_2 \cdot v_{2,i}^2 - m_1 \cdot v_{1,f}^2]$$

$$\Leftrightarrow \; (m_1 + m_2) \cdot v_{1,f}^2 - 2 \cdot \cos\alpha \cdot (m_1 \cdot v_{1,i} + m_2 \cdot v_{2,i}) \cdot v_{1,f} + [(m_1 - m_2) \cdot v_{1,i} + 2 \cdot m_2 \cdot v_{2,i}] \cdot v_{1,i} = 0$$

$$\Leftrightarrow \; (5.95 \times 10^5 + 1.22 \times 10^6) \cdot v_{1,f}^2 - 2 \cdot \cos(33.2^\circ) \cdot (5.95 \times 10^5 \cdot 2.65 \times 10^4 + 1.22 \times 10^6 \cdot 2.83 \times 10^3) \cdot v_{1,f} +$$

$$[(5.95 \times 10^5 - 1.22 \times 10^6) \cdot 2.65 \times 10^4 + 2 \cdot 1.22 \times 10^6 \cdot 2.83 \times 10^3] \cdot 2.65 \times 10^4 = 0$$

The above quadratic equation provides two solutions, i.e., $v_{1,f} = 2.37 \times 10^4$ m/s and $v_{1,f} = -5.95 \times 10^3$ m/s. For the remainder of this exercise, we will work with the first solution.

Now that rock 1 has changed its course and is headed towards our Solar System with a speed $v_{1,f}$, we know that it becomes sensitive to the gravitational pull of our Solar System. With regard to the system "rock 1 plus our Solar System", the total mechanical energy of the rock is conserved given that only conservative forces are at play. We choose to compare its mechanical energy right after the collision ($E_{tot,i}$) with that as it is about to hit the Sun ($E_{tot,f}$). Regarding $E_{tot,i}$, although the gravitational force of both the Sun and the Earth is acting upon the rock, given that the Earth-Sun distance $r_{ES}$ becomes very small, relatively speaking, we can assume that there is just one mass, i.e., $M_s + M_E$, at a distance $d$ exerting its gravitational influence upon the rock.

When it comes to $E_{tot,f}$, since we know that the orbital position of the Earth around the Sun makes an angle of 90° with the trajectory line of rock 1, the distance $r_{r,E}$ between the rock and the Earth is equal to $r_{r,E} = \sqrt{r_s^2 + r_{ES}^2}$. The final speed $v_f$ with which rock 1 strikes against the Sun is then calculated as follows:

$$E_{tot,i} = E_{tot,f} \quad \Leftrightarrow \quad \frac{m_1 \cdot v_{1,f}^2}{2} - \frac{G \cdot m_1 \cdot (M_s + M_E)}{d} = \frac{m_1 \cdot v_f^2}{2} - \frac{G \cdot m_1 \cdot M_s}{r_s} - \frac{G \cdot m_1 \cdot M_E}{\sqrt{r_s^2 + r_{ES}^2}}$$

$$\Leftrightarrow \; v_f = \sqrt{v_{1,f}^2 + 2 \cdot G \cdot \left[\frac{M_s}{r_s} + \frac{M_E}{\sqrt{r_s^2 + r_{ES}^2}} - \frac{(M_s + M_E)}{d}\right]}$$

$$= \sqrt{(2.37 \times 10^4)^2 + 2 \cdot 6.67 \times 10^{-11} \cdot \left[\frac{1.99\times10^{30}}{6.96\times10^8} + \frac{5.97\times10^{24}}{\sqrt{(6.96\times10^8)^2+(1.496\times10^{11})^2}} - \frac{(1.99\times10^{30}+5.97\times10^{24})}{2.50\cdot9.46\times10^{15}}\right]}$$

$= 6.18 \times 10^5$ m/s or $2.22 \times 10^6$ km/h

# Exercise 17

## *Problem Statement*

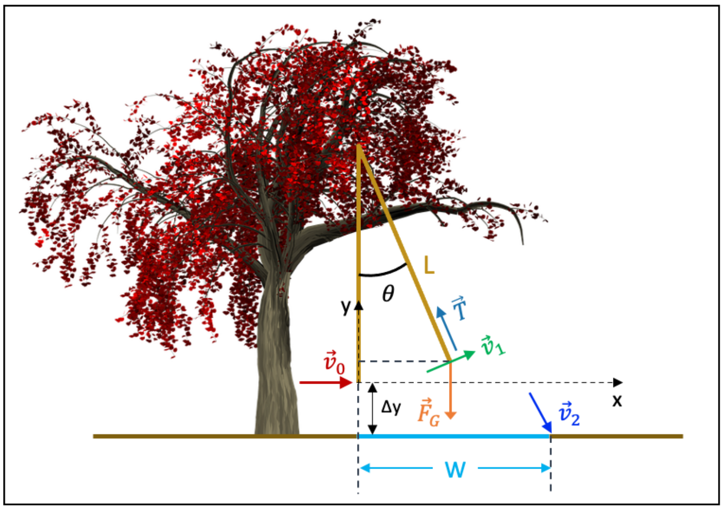


***Figure 17***

After spending their entire morning attending classes at the Norbuling Central School in Gelephu, Bhutan, Sangay ($m$ = 57.4 kg) and her friends Sherab and Kim rush to one of the nearby tributary streams of the Manas River. On one side of the river bank, a couple of indigenous trees called Ehretia acuminata are standing tall next to each other and Sangay has attached a rope of length $L$ = 8.55 m to one of their branches, whereby one end of the rope is a distance $\Delta y$ = 1.75 m short from touching the ground. Sangay runs up to the rope with an initial speed $v_0$, grabs it and subsequently swings on it until she briefly comes to a halt, at which moment the rope is making an angle of $\theta_{max}$ = 28.4° with the vertical. After a couple of swings, Sangay feels adventurous and she quickly estimates that when releasing the rope at an angle $\theta$, whereby $\theta < \theta_{max}$, she will make it to the other riverbank. Sangay is right in her calculations and she indeed just reaches the other side of the stream. (1) If you know that the tension in the rope right before the moment when Sangay releases it is equal to $T$ = 565 N, what is the value of the angle $\theta$? (2) How wide is the river? (3) With what speed does Sangay hit the ground on the other side?

## *Solution*

(1) As the upwards pointing tension force $\vec{T}$ is perpendicular to Sangay's direction of motion along an arc-shaped path, it is not performing any work on the system "Sangay", so that her total mechanical energy is conserved at all times. In a first instance, let us compare the mechanical energy $E_1$ at the moment Sangay grabs the rope with the energy $E_2$ when the rope makes an angle $\theta$ with the vertical:

$$E_1 = E_2 \quad \Leftrightarrow \quad \frac{m \cdot v_0^2}{2} = \frac{m \cdot v_1^2}{2} + m \cdot g \cdot L \cdot (1 - \cos\theta)$$

Since we do not know the values for either $v_0$ or $v_1$, we will find expressions for both of them. Regarding $v_0$, we can apply the law of energy conservation to Sangay's initial position ($E_1$) and the moment when she comes briefly to a stop at an angle $\theta_{max}$ ($E_{max}$):

$$E_1 = E_{max} \quad \Leftrightarrow \quad \frac{m \cdot v_0^2}{2} = m \cdot g \cdot L \cdot (1 - \cos\theta_{max})$$

With respect to $v_1$, when applying Newton's second law to Sangay, we obtain the following equation:

$$T - m \cdot g \cdot \cos\theta = \frac{m \cdot v_1^2}{L} \quad \Leftrightarrow \quad m \cdot v_1^2 = L \cdot T - L \cdot m \cdot g \cdot \cos\theta$$

Inserting the two above expression back into our first equation, we can calculate the angle $\theta$:

$$[m \cdot g \cdot L \cdot (1 - \cos\theta_{max})] = \frac{[L \cdot T - L \cdot m \cdot g \cdot \cos\theta]}{2} + m \cdot g \cdot L \cdot (1 - \cos\theta)$$

$$\Leftrightarrow \quad - m \cdot g \cdot \cos\theta_{max} = \frac{T}{2} - \frac{3}{2} \cdot m \cdot g \cos\theta$$

$$\Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{T}{3 \cdot m \cdot g} + \frac{2}{3} \cdot \cos\theta_{max}\right) = \cos^{-1}\left[\frac{565}{3 \cdot 57.4 \cdot 9.81} + \frac{2}{3} \cdot \cos(28.4°)\right] = 22.9°$$

(2) To find the width $W$ of the river, we first have to determine the speed $v_1$ as well as the time $t$ Sangay spends in the air during her free fall. The speed $v_1$ can be calculated by using the expression obtained through Newton's second law in part (1):

$$m \cdot v_1^2 = L \cdot T - L \cdot m \cdot g \cdot \cos\theta \quad \Leftrightarrow \quad v_1 = \sqrt{\frac{L}{m} \cdot (T - m \cdot g \cdot \cos\theta)}$$

$$= \sqrt{\frac{8.55}{57.4} \cdot [565 - 57.4 \cdot 9.81 \cdot \cos(22.9°)]}$$

$$= 2.63 \text{ m/s}$$

To determine the time $t$ during the free fall, we write the following equation of motion in the y-direction:

$$y = y_0 + v_0 \cdot t + \frac{a_y}{2} \cdot t^2 \quad \Leftrightarrow \quad - \Delta y = L \cdot (1 - \cos\theta) + v_1 \cdot \sin\theta \cdot t - \frac{g}{2} \cdot t^2$$

$$- 1.75 = 8.55 \cdot [1 - \cos(22.9°)] + 2.63 \cdot \sin(22.9°) \cdot t - \frac{9.81}{2} \cdot t^2$$

The above quadratic equation provides a physically sensible solution (i.e., $t > 0$) equal to $t = 0.816$ s. The width $W$ of the river can now be calculated as follow:

$$W = (L \cdot \sin\theta) + (v_1 \cdot \cos\theta \cdot t) = [8.55 \cdot \sin(22.9°)] + [2.63 \cdot \cos(22.9°) \cdot 0.816] = 5.31 \text{ m}$$

(3) We can obtain the speed $v_2$ by comparing, for instance, the energy of Sangay at the moment when she lets go of the rope ($E_2$) with that right before she lands on the opposite riverbank ($E_3$):

$$
\begin{aligned}
E_2 = E_3 \quad &\Leftrightarrow \quad \frac{m \cdot v_1^2}{2} + m \cdot g \cdot L \cdot (1 - \cos\theta) = \frac{m \cdot v_2^2}{2} + m \cdot g \cdot (-\Delta y) \\
&\Leftrightarrow \quad v_2 = \sqrt{v_1^2 + 2 \cdot g \cdot [L \cdot (1 - \cos\theta) + \Delta y]} \\
&\qquad\quad\; = \sqrt{2.63^2 + 2 \cdot 9.81 \cdot [8.55 \cdot [1 - \cos(22.9^\circ)] + 1.75]} \\
&\qquad\quad\; = 7.38 \text{ m/s}
\end{aligned}
$$

# Exercise 18

## *Problem Statement*

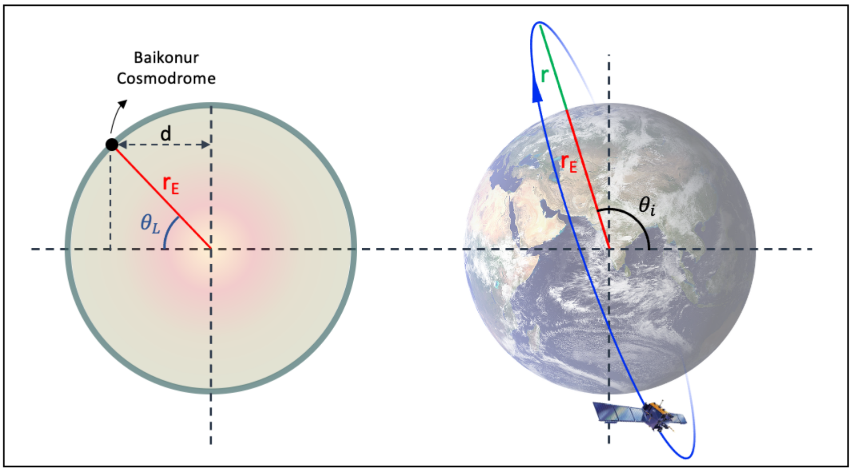


***Figure 18***

On 14 December 2007, the Earth surveillance satellite RADARSAT-2 ($m = 2,250$ kg) was launched with the assistance of a Soyuz launch vehicle from the Baikonur Cosmodrome in the south of Kazakhstan (45°58′42.3″N 63°17′31.9″E). The data gathered during its observation is used for research as well as for developing applications and services in a wide range of areas, including pollution monitoring, ice monitoring, agricultural crop monitoring, geological mapping, and disaster management. RADARSAT-2 has been put in a near polar heliosynchronous orbit with an orbital period roughly equal to $T_R = 101$ min at an inclination angle of $\theta_i = 98.6°$. In a heliosynchronous orbit, a satellite crosses the equator always at the same local time, which in the case of RADARSAT-2 is about 18:00 hrs (when moving from south to north). The inclination angle $\theta_i$ is the angle between the equator and the orbital plane of the satellite, whereby 0° corresponds to a satellite orbiting along the equator in the same direction as the Earth's spin. If you estimate that the average power of the engines combined was about $P = 10.6$ MW, how long did it take the Soyuz launch vehicle to place RADARSAT-2 into orbit? Remember that the universal gravitational constant $G$ is equal to $G = 6.67 \times 10^{-11}$ m$^3$/(kg$\cdot$ s$^2$) and that the mass and radius of the Earth are equal to $M_E = 5.97 \times 10^{24}$ kg and $r_E = 6.38 \times 10^3$ km, respectively. Assume a circular orbit for the satellite.

## *Solution*

Given that the thrust force of the engines is a non-conservative force, we know that the total mechanical energy of the system "RADARSAT-2" is not conserved, so that the total mechanical energy of the satellite within its near polar orbit ($E_2$) will be equal to the mechanical energy at the moment of its launch ($E_1$) plus the total energy of the engines needed to put the satellite into orbit ($E_{eng}$).

In Exercise 3, we have seen that the total mechanical energy of an object orbiting around a massive body at a distance $r$ from its center is equal to $E_{tot} = -\frac{G \cdot m \cdot M}{2 \cdot r}$. Therefore, in the case of the satellite RADARSAT-2, the total mechanical energy equation can be written in the following way:

$$E_1 + E_{eng} = E_2 \quad \Leftrightarrow \quad \left[\frac{m \cdot v_1^2}{2} - \frac{G \cdot m \cdot M_E}{r_E}\right] + E_{eng} = \left[-\frac{G \cdot m \cdot M_E}{2 \cdot (r_E + r)}\right]$$

The reason that the kinetic energy term at the moment of the satellite's launch is not zero is because, viewed from the center of the Earth, an object positioned on the Earth's surface at a non-zero angle with the Earth's rotation axis is undergoing rotational motion due to the Earth's spin. Let us now, in a first instance, determine the distance $d$ between the satellite's position at the Baikonur Cosmodrome and the Earth's rotation axis. Given an angle of latitude $\theta_L$ equal to $\theta_L = 45 + \frac{58}{60} + \frac{42.3}{60} = 46.0°$, the distance d becomes:

$$d = r_E \cdot \cos\theta_L = 6.38 \times 10^6 \cdot \cos(46.0°) = 4.43 \times 10^6 \text{ m}$$

The speed $v_1$ due to the rotation of the Earth at the location of the Baikonur Cosmodrome is then equal to (with $T_E$ the period of the Earth's rotation):

$$v_1 = \frac{2 \cdot \pi}{T_E} \cdot d = \frac{2 \cdot \pi}{86,400} \cdot 4.43 \times 10^6 = 322 \text{ m/s}$$

Before returning to our initial energy equation, we still need to determine the height $r$ above the Earth's surface of RADARSAT-2's near polar heliosynchronous (circular) orbit, which is found with the help of Kepler's third law:

$$r = \left(\sqrt[3]{\frac{T_R^2}{4 \cdot \pi^2} \cdot G \cdot M_E}\right) - r_E = \left(\sqrt[3]{\frac{(101 \cdot 60)^2}{4 \cdot \pi^2} \cdot 6.67 \times 10^{-11} \cdot 5.97 \times 10^{24}}\right) - 6.38 \times 10^6 = 802 \text{ km}$$

Going back to our initial energy equation, we now have all the information available to calculate the combined energy $E_{eng}$ of the engines:

$$E_{eng} = G \cdot m \cdot M_E \cdot \left[\frac{r_E + 2 \cdot r}{2 \cdot r_E \cdot (r_E + r)}\right] - \frac{m \cdot v_1^2}{2}$$

$$= 6.67 \times 10^{-11} \cdot 2,250 \cdot 5.97 \times 10^{24} \cdot \left[\frac{6.38 \times 10^6 + 2 \cdot 8.02 \times 10^5}{2 \cdot 6.38 \times 10^6 \cdot (6.38 \times 10^6 + 8.02 \times 10^5)}\right] - \frac{2,250 \cdot 322^2}{2}$$

$$= 7.79 \times 10^{10} \text{ J}$$

The time t it took the Soyuz launch vehicle to place RADARSAT-2 into its orbit is then found as follows:

$$P = \frac{E_{eng}}{t} \quad \Leftrightarrow \quad t = \frac{E_{eng}}{P} = \frac{7.79 \times 10^{10}}{10.6 \times 10^6} = 7.35 \times 10^3 \text{ s or } 2.04 \text{ hours}$$

# Exercise 19

## *Problem Statement*

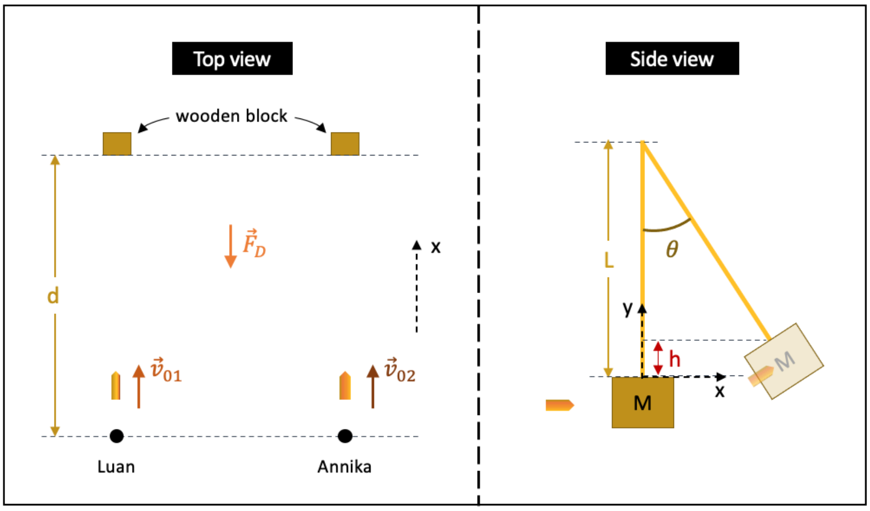


***Figure 19***

Luan and Annika are spending their Saturday afternoon improving their shooting skills at the Long Range Shooting Club in Leandra, South Africa. Luan owns a .380 ACP gun, while Annika brought her .40 Smith & Wesson to the shooting range. Luan's .380 ACP fires its bullets of 9.0 mm caliber with a muzzle velocity of $\vec{v}_{01} = v_{01} \cdot \vec{i}_x$, whereas the muzzle velocity of the 10.2 mm caliber bullets of Annika's .40 S&W is equal to $\vec{v}_{02} = v_{02} \cdot \vec{i}_x$. They are each standing at a distance of $d = 45.5$ m away from a small wooden block ($M = 2.55$ kg), suspended from a rope with length $L = 0.750$ m. When firing their gun, aimed at their respective wooden block, a drag force $\vec{F}_D = -b \cdot v^2 \cdot \vec{i}_x$ has slowed the bullet's muzzle velocity (in the x-direction) by 5% by the time the bullet hits the block. Upon impact, the block swings slightly backwards until it reaches a height h (with respect to the top edge of the block) and makes an angle $\theta$ with the vertical. If you know that the muzzle speed $v_{02}$, the mass $m_2$ of the .40 S&W's bullets, and the maximum swinging height $h_2$ of Annika' wooden block relative to Luan are equal to 1.122, 1.722, and 3.721, respectively, and that Luan's block makes a $\theta_1 = 13.95°$ angle when at its maximum swinging height $h_1$, (1) what is the mass $m_1$ and $m_2$ of the 9.0 mm and 10.2 mm caliber bullets, respectively, expressed in grains, whereby 1 grain = $6.48 \times 10^{-2}$ g? (2) What is the magnitude of the muzzle velocity $\vec{v}_{01}$ (.380 ACP) and $\vec{v}_{02}$ (.40 S&W)? (3) Which angle $\theta_2$ does Annika's block make with the vertical at its maximum height $h_2$? Assume that between the moment when the bullets enter the wooden block and until they come to a halt within the block, the block is not experiencing any major changes in its motion.

## *Solution*

(1) Given that the bullet does not cause significant motion of the block *during the deceleration* of the bullet as it penetrates the wooden block, the system "bullet plus wooden block" has a net force equal to zero, so that the linear momentum is conserved during this perfectly inelastic collision—if the block would start swinging during this moment of deceleration, the system would experience a net force due to gravity and the total linear momentum would not remain constant.

As a result of to the 5% loss in velocity due to the drag force $\vec{F}_D$, the incoming speeds $v_{11}$ and $v_{12}$ of the 9.0 mm caliber and 10.2 mm caliber bullet right before hitting the block are equal to $v_{11} = 0.95 \cdot v_{01}$ and $v_{12} = 0.95 \cdot v_{02}$, respectively. The conservation of the total linear momentum therefore provides us with the following two equations, with $v_{1f}$ and $v_{2f}$ the speed of the block and the bullet combined right after the collision:

$$\begin{cases} m_1 \cdot v_{11} = (m_1 + M) \cdot v_{1f} \\ \\ m_2 \cdot v_{12} = (m_2 + M) \cdot v_{2f} \end{cases}$$

For both Luan and Annika, the block will start swinging after the collision, and at that moment the only force doing work on the combined system "bullet and block" is the gravitational force. We therefore know that the system's total mechanical energy is conserved and we can write the following two equations, whereby $E_{1i}$ (with $i \in \{1, 2\}$) the mechanical energy at the block's lowest point and $E_{2i}$ that at the height $h_i$:

$$\begin{cases} E_{11} = E_{21} \quad \Leftrightarrow \quad \dfrac{(m_1 + M) \cdot v_{1f}^2}{2} = (m_1 + M) \cdot g \cdot h_1 \quad \Leftrightarrow \quad v_{1f} = \sqrt{2 \cdot g \cdot h_1} \\ \\ E_{12} = E_{22} \quad \Leftrightarrow \quad \dfrac{(m_2 + M) \cdot v_{2f}^2}{2} = (m_2 + M) \cdot g \cdot h_2 \quad \Leftrightarrow \quad v_{2f} = \sqrt{2 \cdot g \cdot h_2} \end{cases}$$

Plugging the above two expression for $v_{1f}$ and $v_{2f}$ into the two equations related to the conservation of linear momentum, we obtain the following two expressions:

$$\begin{cases} m_1 \cdot v_{11} = (m_1 + M) \cdot \sqrt{2 \cdot g \cdot h_1} \\ \\ m_2 \cdot v_{12} = (m_2 + M) \cdot \sqrt{2 \cdot g \cdot h_2} \end{cases}$$

Since we know that $v_{12} = 0.95 \cdot v_{02} = 0.95 \cdot (1.122 \cdot v_{01}) = 1.122 \cdot (0.95 \cdot v_{01}) = 1.122 \cdot v_{11}$, that $m_2 = 1.722 \cdot m_1$, and that $h_2 = 3.721 \cdot h_1$, we can reformulate the second equation as follows:

$$[1.722 \cdot m_1] \cdot [1.122 \cdot v_{11}] = ([1.722 \cdot m_1] + M) \cdot \sqrt{2 \cdot g \cdot [3.721 \cdot h_1]} \quad \Leftrightarrow \quad \sqrt{2 \cdot g \cdot h_1} = \frac{1.722 \cdot 1.122 \cdot m_1 \cdot v_{11}}{\sqrt{3.721} \cdot (1.722 \cdot m_1 + M)}$$

Inserting this final expression back into the first equation, we can calculate the mass $m_1$ of the .380 ACP's 9.0 mm caliber bullets:

$$m_1 \cdot v_{11} = (m_1 + M) \cdot \left[ \frac{1.722 \cdot 1.122 \cdot m_1 \cdot v_{11}}{\sqrt{3.721} \cdot (1.722 \cdot m_1 + M)} \right] \quad \Leftrightarrow \quad m_1 = \frac{\left(1.722 \cdot 1.122 - \sqrt{3.721}\right)}{1.722 \cdot \left(\sqrt{3.721} - 1.122\right)} \cdot M$$

$$= \frac{\left(1.722 \cdot 1.122 - \sqrt{3.721}\right)}{1.722 \cdot \left(\sqrt{3.721} - 1.122\right)} \cdot 2.55$$

$$= 5.68 \text{ g or } 87.6 \text{ grains}$$

The mass $m_2$ of the .40 S&W's 10.2 mm caliber bullets is then equal to $m_2 = 1.722 \cdot m_1 = 1.722 \cdot 5.68 = 9.78$ g or 151 grains.

(2) Given that the height $h_i$ of the wooden block at its maximum angle $\theta_i$ (when the block briefly comes to a halt) is equal to $h_i = L \cdot (1 - \cos\theta_i)$, the value of $h_1$ is found as follows:

$$h_1 = L \cdot (1 - \cos\theta_1) = 0.750 \cdot [1 - \cos(13.95°)] = 2.21 \text{ cm}$$

The height $h_2$ is then equal to $h_2 = 3.721 \cdot h_1 = 3.721 \cdot 2.21 = 8.23$ cm, and the magnitude of the velocities $v_{11}$ and $v_{12}$ is calculated in the following way:

$$\begin{cases} v_{11} = \left(1 + \dfrac{M}{m_1}\right) \cdot \sqrt{2 \cdot g \cdot h_1} = \left(1 + \dfrac{2.55}{0.00568}\right) \cdot \sqrt{2 \cdot 9.81 \cdot 0.0221} = 296 \text{ m/s} \\ \\ v_{12} = \left(1 + \dfrac{M}{m_2}\right) \cdot \sqrt{2 \cdot g \cdot h_2} = \left(1 + \dfrac{2.55}{0.00978}\right) \cdot \sqrt{2 \cdot 9.81 \cdot 0.0823} = 333 \text{ m/s} \end{cases}$$

Finally, the muzzle speeds $v_{01}$ and $v_{02}$ of the .380 ACP and .40 S&W, respectively, are then equal to $v_{01} = \frac{v_{11}}{0.95} = \frac{296}{0.95} = 312$ m/s and $v_{02} = \frac{v_{12}}{0.95} = \frac{333}{0.95} = 350$ m/s.

(3) The angle $\theta_2$ that Annika's wooden block makes with the vertical when the block is at its maximum height $h_2$ is found as follows:

$$h_2 = L \cdot (1 - \cos\theta_2) \quad \Leftrightarrow \quad \theta_2 = \cos^{-1}\left(1 - \frac{h_2}{L}\right) = \cos^{-1}\left(1 - \frac{0.0823}{0.750}\right) = 27.1°$$

Since the .40 S&W's muzzle speed is higher than that of the .380 ACP ($v_{02} = 350$ m/s $> v_{01} = 312$ m/s), it indeed makes sense that $\theta_2 = 27.1° > \theta_1 = 13.95°$ due to a greater impact at the moment of the collision between the bullet and the wooden block.

# Exercise 20

## ***Problem Statement***

Last week, Olivia and Adam have been watching a dozen of Youtube videos on how to make your own water bottle rocket. They even went the extra mile and figured out some of the basics of the underlying physics of their new project. Earlier this morning, they went to buy all the required equipment and with the assembled bottle rocket under their arm, Olivia and Adam are now headed to the nearby Blatherskite Park in Alice Springs, Australia, to try out their first design. Their rocket consists of three empty 2.00 L plastic soda water bottles ($m_b$ = 44.9 g per unit) firmly taped together and designed in such a way that combined they make one cylindrical container. The rocket is filled with heated soda water at a temperature of 42.5°C (for some extra kinetic energy) for a total volume of a little under one third per bottle ($V_b$ = 0.63 L per unit). Once Olivia and Adam start pumping air into the rocket, the growing pressure increasingly pushes on the water until at one point the water will come rushing out of the nozzle at the bottom of the rocket, providing the bottle rocket with upwards thrust and sending it flying through the air. Olivia and Adam estimate that about 459 g of soda water will shoot out of the rocket every second at a constant speed of $v_w$ = 36.5 m/s, relative to the rocket. They furthermore take into account an average drag force of $\vec{F}_D = -25.5 \cdot \vec{i}_y$ N during the rocket's ascent. If you know that the density of soda water is equal to $\rho$ = 1.01442 kg/L, (1) how high will the bottle rocket go? (2) How much time did the rocket spend in the air (ignore air friction during the rocket's descent)? (3) What is the total power supplied by the thrust force $\vec{F}_T$ of the bottle rocket, expressed in horsepower (hp)? Remember that 1 hp = 745.7 W.

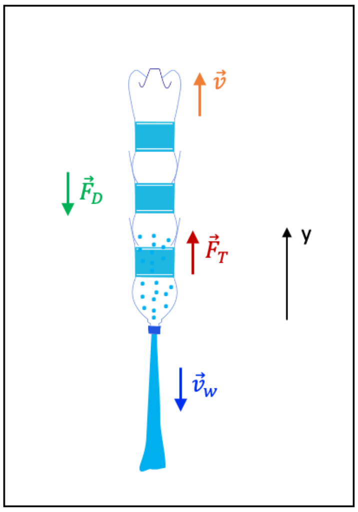


***Figure 20***

## ***Solution***

(1) Let us in a first instance determine the total mass $m_{tot}$ of the bottle rocket and the water together:

$$m_{tot} = m_b \cdot 3 + (V_b \cdot \rho) \cdot 3 = 0.0449 \cdot 3 + (0.63 \cdot 1.01442) \cdot 3 = 2.05 \text{ kg}$$

As the rocket shoots up, it is transferring its linear momentum $\vec{p}_r = m_{tot} \cdot \vec{v}$ to that of the water ($\vec{p}_w = m_{tot} \cdot \vec{v}_w'$) since the rocket's mass gradually reduces, whereby the velocity $\vec{v}_w'$ is the velocity of the water with respect to an inertial frame of reference, i.e., an observer standing on the ground, and is therefore equal to $\vec{v}_w' = \vec{v} + \vec{v}_w$.

Without any net force acting on the system "bottle rocket plus water", the change in the total linear momentum $\vec{p}_{tot} = \vec{p}_r + \vec{p}_w$ would be zero and we would write:

$$\frac{d\vec{P}_{tot}}{dt} = \frac{d\vec{P}_r}{dt} + \frac{d\vec{P}_w}{dt} = 0$$

However, as the combination of the gravitational force $\vec{F}_G$ and the drag force $\vec{F}_D$ exert a net force on the system during its ascent, the above equation becomes:

$$\frac{d\vec{P}_{tot}}{dt} = \frac{d\vec{P}_r}{dt} + \frac{d\vec{P}_w}{dt} = \vec{F}_G + \vec{F}_D$$

As we consider a one-dimensional system, we will omit the vector notation from now on. The two terms on the left-hand side of the equation are equal to:

$$\begin{cases} \dfrac{dP_r}{dt} = \dfrac{d(m_{tot} \cdot v)}{dt} = m_{tot} \cdot \dfrac{dv}{dt} + v \cdot \dfrac{dm_{tot}}{dt} \\ \\ \dfrac{dP_w}{dt} = -\dfrac{d(m_{tot} \cdot v_w^{'})}{dt} = -v_w^{'} \cdot \dfrac{dm_{tot}}{dt} \end{cases}$$

Regarding the term $\frac{dP_w}{dt}$, we introduced a minus sign in front of the fraction, because the momentum of the water is increasing while the rocket is losing mass, i.e., the term $\frac{d(m_{tot})}{dt}$ is negative. Also, the speed $v_w^{'}$ with which the water is being pushed out of the rocket is assumed constant, so that it can be brought outside of the differential. Putting the above two expressions back into our original equation of the change in total linear momentum, we obtain the following equation:

$$\begin{aligned} \frac{dP_r}{dt} + \frac{dP_w}{dt} = -F_G - F_D \quad &\Leftrightarrow \quad \left[m_{tot} \cdot \frac{dv}{dt} + v \cdot \frac{dm_{tot}}{dt}\right] + \left[-v_w^{'} \cdot \frac{dm_{tot}}{dt}\right] = -(m_{tot} \cdot g) - F_D \\ &\Leftrightarrow \quad \left[m_{tot} \cdot \frac{dv}{dt} + v \cdot \frac{dm_{tot}}{dt}\right] + \left[-(v - v_w) \cdot \frac{dm_{tot}}{dt}\right] = -(m_{tot} \cdot g) - F_D \\ &\Leftrightarrow \quad \frac{dv}{dt} = -\left(\frac{v_w}{m_{tot}} \cdot \frac{dm_{tot}}{dt}\right) - g - \frac{F_D}{m_{tot}} \end{aligned}$$

To express the height in terms of the time variable t, let us integrate the above equation:

$$\begin{aligned} \frac{dv}{dt} = -\left(\frac{v_w}{m_{tot}} \cdot \frac{dm_{tot}}{dt}\right) - g - \frac{F_D}{m_{tot}} \quad &\Leftrightarrow \quad dv = -\left(\frac{v_w}{m_{tot}} \cdot dm_{tot}\right) - \left(g + \frac{F_D}{m_{tot}}\right) \cdot dt \\ &\Leftrightarrow \quad \int_{v_0}^{v(t)} dv' = -v_w \cdot \int_{m_i}^{m_f} \frac{dm'_{tot}}{m'_{tot}} - \left(g + \frac{F_D}{m_{tot}}\right) \cdot \int_0^t dt' \end{aligned}$$

$$\Leftrightarrow \; v(t) - v_0 = -v_w \cdot \left[\ln(m_{tot})|_{m_i=m_{tot}}^{m_f=3\cdot m_b}\right] - \left(g + \frac{F_D}{m_{tot}}\right) \cdot t$$

$$\Leftrightarrow \; v(t) = v_0 - v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right) - \left(g + \frac{F_D}{m_{tot}}\right) \cdot t$$

$$\Leftrightarrow \; \frac{dy(t)}{dt} = v_0 - v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right) - \left(g + \frac{F_D}{m_{tot}}\right) \cdot t$$

$$\Leftrightarrow \; dy(t) = \left[v_0 - v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right] \cdot dt - \left(g + \frac{F_D}{m_{tot}}\right) \cdot t \cdot dt$$

$$\Leftrightarrow \; \int_{y_0}^{y(t)} dy'(t) = \left[v_0 - v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right] \cdot \int_0^t dt' - \left(g + \frac{F_D}{m_{tot}}\right) \cdot \int_0^t t' \cdot dt'$$

$$\Leftrightarrow \; y(t) = y_0 + \left[v_0 - v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right] \cdot t - \left(g + \frac{F_D}{m_{tot}}\right) \cdot \frac{t^2}{2}$$

As the bottle rocket leaves from the ground, we have that $y_0 = 0$ m and given that its initial speed is zero we can also write that $v_0 = 0$ m/s. The final equation of motion is then formulated as follows:

$$y(t) = \left[-v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right] \cdot t - \left(g + \frac{F_D}{m_{tot}}\right) \cdot \frac{t^2}{2}$$

Since the speed briefly becomes zero at the rocket's maximum height, we calculate the time $t_{max}$ for the rocket to reach that height in the following way:

$$v(t_{max}) = 0 \quad \Leftrightarrow \; v(t_{max}) = -v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right) - \left(g + \frac{F_D}{m_{tot}}\right) \cdot t_{max} = 0$$

$$\Leftrightarrow \; t_{max} = \frac{\left[-v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right]}{\left[g + \frac{F_D}{m_{tot}}\right]} = \frac{\left[-36.5 \cdot \ln\left(\frac{3 \cdot 0.0449}{2.05}\right)\right]}{\left[9.81 + \frac{25.5}{2.05}\right]} = 4.47 \text{ s}$$

Plugging this value for $t_{max}$ into our equation of motion gives us the maximum height $y_{max}$ of the bottle rocket:

$$y(t_{max}) = \left[-v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right] \cdot t_{max} - \left(g + \frac{F_D}{m_{tot}}\right) \cdot \frac{t_{max}^2}{2}$$

$$= \left[-36.5 \cdot \ln\left(\frac{3 \cdot 0.0449}{2.05}\right)\right] \cdot 4.47 - \left(9.81 + \frac{25.5}{2.05}\right) \cdot \frac{4.47^2}{2}$$

$$= 222 \text{ m}$$

(2) Since the bottle rocket loses its water fuel at a constant rate of 0.459 kg/s, it will have expelled all of its fuel in a time window of $t_{empty} = \frac{m_{tot} - 3 \cdot m_b}{\left[-\frac{dm_{tot}}{dt}\right]} = \frac{2.05 - 3 \cdot 0.0449}{[-(-0.459)]} = 4.18$ s, i.e., slightly less than the time it takes the rocket to reach its maximum height $y_{max}$. The altitude of the rocket at the moment of $t_{empty}$ is equal to $y_{empty} = 221$ m (calculated with the expression derived in part (1)). In other words, right after the rocket loses all of its water fuel, it climbs another $y_{max} - y_{empty} = 222 - 221 = 95.7$ cm before reaching its maximum height.

At the maximum height of $y_{max} = 222$ m, the three bottles combined are empty and now start their free fall. The respective time $t_{free}$ is equal to (remember that the speed at $y_{max}$ is equal to zero, so that $v_0 = 0$ m/s, and we ignore any drag force):

$$y(t_{free}) = y_0 + v_0 \cdot t_{free} + \frac{a_y}{2} \cdot t_{free}^2 \quad \Leftrightarrow \quad 0 = y_{max} + 0 \cdot t_{free} - \frac{g}{2} \cdot t_{free}^2$$

$$\Leftrightarrow \quad t_{free} = \sqrt{\frac{2 \cdot y_{max}}{g}} = \sqrt{\frac{2 \cdot 222}{9.81}} = 6.73 \text{ s}$$

The total time $t_{tot}$ that the bottle rocket spends in the air is then equal to $t_{tot} = t_{max} + t_{free} = 4.47 + 6.73 = 11.2$ s.

(3) From the equation of the change in the total linear momentum derived in part (1), we find that the term corresponding to the magnitude of the thrust force $\vec{F}_T$ is the term "$-v_w \cdot \frac{dm_{tot}}{dt}$". Similarly, from the expression obtained for the velocity $v(t)$, the term that contributes to the velocity due to the thrust force is equal to "$v_T = -v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)$".

The total power delivered by the thrust force then becomes:

$$P = F_T \cdot v_T = \left[-v_w \cdot \frac{dm_{tot}}{dt}\right] \cdot \left[-v_w \cdot \ln\left(\frac{3 \cdot m_b}{m_{tot}}\right)\right]$$

$$= [-36.5 \cdot (-0.459)] \cdot \left[-36.5 \cdot \ln\left(\frac{3 \cdot 0.0449}{2.05}\right)\right]$$

$$= 1.67 \times 10^3 \text{ J or } 2.23 \text{ hp}$$