Physics

Exercises on Rotational Motion

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Summary of Exercises

Exercise 1

When massive stars have exhausted their nuclear fuel, i.e., they have fused all the nuclei of lighter elements, such as hydrogen and helium, into heavier nuclei (a process referred to as stellar nucleosynthesis), an immense explosion (a supernova) follows and under the right circumstances the stellar core residue forms a very dense neutron star. Some of these stars are spinning around their rotation axis at very high velocities and are designated pulsars, whereby streams of electromagnetic radiation are blasted from their magnetic poles at velocities up to 70% of the speed of light. One such pulsar is the Vela Pulsar ($M_V = 2.26\ M_s$), which is located about 959 light years away from us and rotates around its axis about 11.195 times every second. Suppose now that a subatomic particle, let's say a proton (m_p), is sitting on the equator of Vela. (1) If you know that the proton has to be accelerated by a factor of 315 to gain the minimum speed to escape Vela's gravitational pull, what is the Vela Pulsar's radius and the proton's initial speed v_r (viewed from a stationary reference frame)? (2) Suppose that this same proton left Vela 400 years ago and that 100 years later a proton escapes from another pulsar called PSR J0437-4715 ($M_P = 1.44\ M_s$ and $r_P = 13.6$ km) whereby both protons are on a straight collision course. If you know that the distance between both pulsars is equal to $d = 449.2$ light years, in which year will/did the two protons collide? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2), that 1 light year measures 9.46×10^{12} km, and that one solar mass is equal to $M_s = 1.99 \times 10^{30}$ kg.

Exercise 2

You work at The Escape Game in New York City, the United States, and as you're designing a new escape room, you need three 2.50 m long bamboo poles. Luckily, your friend Akira, who works at the Japanese restaurant Sen Sekana only three blocks up, stores a lot of decoration material in the basement, including a couple of bamboo poles. You go on foot to pick them up and on your way back, 10.0 m before getting to the zebra crossing on East 42nd Street, the red hand of the pedestrian traffic lights starts flashing on and off and you start running. You make it on time to the other side of the street, but you've hit the street name sign "East 42nd St", which is attached to a metal pole on the sidewalk, with one of the bamboo poles with a force $\vec{F}$ of a magnitude equal to $F = 68.2$ N under an angle of $\phi = 26.6°$ at about one fourth of the sign's length from the sign's end. If you know that the street sign's mass, length, height, and width are equal to $m = 3.74$ kg, $L = 45.7$ cm, $h = 17.8$ cm, and $w = 4.35$ cm, respectively, that the collision lasted $t = 0.425$ s, and that during the impact the sign experienced a frictional torque of $\tau_f = 10.2\ N \cdot m$, by what angle θ (in degrees) did you cause the street name sign to rotate?

Exercise 3

The University of Sonora in Hermosillo, Mexico, organized a two weeks long science summer camp for young adolescents between 16 and 18 years to bring them closer into contact with several main topics in the field of biology, chemistry, and physics. At the end of the two weeks, the students are asked to present an own project in their field of interest, and Seina has designed an integrated system to showcase the physical concept of rotational motion. Her project consists of four systems, which are all attached onto a thin wooden board. The first system ("System 1") is a pendulum, whereby a ball ($m_1 = 158$ g) is attached to a basically massless thin rope of length $L_1 = 33.4$ cm. When pulling the ball to the right over an angle θ and subsequently releasing it, the ball hits at its lowest point a hard plastic cube ($m_2 = 215$ g) with an edge of $d = 1.00$ cm, which is fixed at the top end of a vertically placed iron rod ($M_2 = 439$ g) with length $L_2 = 25.6$ cm. This rod ("rod 2") makes up "System 2" and rotates about its axis located at the middle of the rod. Due to the collision between the ball and the cube, rod 2 rotates counterclockwise and the plastic cube hits in turn a horizontal aluminum rod—this "rod 3" constitutes System 3—with a mass of $M_3 = 312$ g and a length equal to that of rod 2. Upon collision, rod 3 swings counterclockwise about its axis at the left end and when it has turned 90°, its right end (now at the bottom) collides with a mini basketball ($m_4 = 194$ g), which lies unrestrained on a horizontal platform. Upon impact with rod 3, the basketball is kicked to the left and lands into a small basket, which is installed $h = 50.0$ cm below the horizontal platform and $\Delta x = 85.0$ cm to the left. If you assume that the kinetic energy at each collision is completely transferred from one system to the next, at what angle θ should Seina release the ball of System 1, so that the mini basketball of System 4 ends up right into the basket?

Exercise 4

Aminah is taking an advanced course in physics at the Rawalpindi Women University in Rawalpindi, Pakistan, and for her final project she wishes to determine with the help of Lagrangian mechanics the equation of motion of an elliptical cylinder (with mass M) rolling down an incline. Aminah's project is particularly challenging since the elliptical cylinder not only sporadically loses contact with the surface at higher speeds, but it simultaneously slips and rolls. As Aminah wants to prepare thoroughly for the first experimental test, whereby she will gather and map the measurements of the motion of both the center point and the two focus points of one of the sides of the cylinder, she decides to calculate beforehand the moment of inertia I of an elliptical cylinder. What is the outcome of Aminah's calculation?

Exercise 5

Martin and his two best friends, Cristobal and Catalina, are going bowling tonight in the local bowling center in Antofagasta, Chile, to celebrate the good news that Catalina obtained her official PADI certification for scuba diving. In the afternoon, they have some spare time, and to warm up for the evening event they play around at Cristobal's house with ramps, bowling balls, and bowling pins. They let a first bowling ball (with mass $M_1 = 2.54$ kg and radius $R_1 = 10.8$ cm) roll down a long ramp at an incline of $\theta = 25.2°$ from a starting height h, and when it reaches the bottom it hits (perfectly elastically) a second bowling ball (with mass $M_2 = 5.22$ kg and radius $R_2 = 11.6$ cm). Upon collision, the second ball receives an initial velocity with which it is sent up a $L = 5.00$ m long second ramp with a slope equal to $\phi = 14.6°$, whereby the ramp itself is placed loosely on a metal cylinder, which is fixed to the ground, and whereby the perpendicular bisector of the ramp intersects with the cylinder's rotation axis. As soon as the second bowling ball crosses the middle of the ramp, the ramp tilts and the ball rolls down from the second half of the ramp, eventually hitting a nicely assembled group of bowling pins. Martin did a too good a job of polishing the first bowling ball, so that it starts sliding ($\mu_k = 0.102$) on the first ramp when it reaches a speed equal to $v_s = \sqrt{\frac{5 \cdot g}{4}}$.

What should be the minimum height h_{min} from which the first bowling ball is released on the first ramp, so that the second bowling ball is able to tilt the second ramp and hit the pins? Assume that the second bowling ball does not slip, that the second ramp does not undergo translational motion when the second bowling ball starts rolling uphill, and that the rotation axis of both bowling balls does not change direction during their motion.

Exercise 6

In his village Qafmollë in Albania, Agim is known as someone who pretty much always goes against the grain. At this moment, he is welding a new weathercock to put on the roof of his 18^{th} century old house, but instead of placing the cock ($m_c = 4.67$ kg) in the middle of the horizontal iron rod ($m_r = 1.12$ kg), Agim feels that it belongs at one end of the rod, so that it can proudly gaze into the wind. At the other end of the $L = 85.0$ cm long rod, Agim attached a copper ball (with mass $m_b = 2.15$ kg and radius $r_b = 3.86$ cm) as counterweight. After trying out various positions along the rod, he finds that the weathercock is most agile and receptive to the wind when a second rod, which acts as the rotation axis, is placed vertically at the center of mass of the first rod. As Agim wants to fully comprehend why this is the case, he emails his son Spiro, who is pursuing a bachelor's degree in physics at the University of Tirana, asking him for a mathematical explanation. Later that day, Agim receives a response from Spiro. What did the email say?

Exercise 7

Emily has been living her whole life in Vallée de l'Ernz, Luxembourg, where she owns 45 hectares of farmland on which she has built her house and a large barn, used to store grain and straw bales ($m_b = 20.0$ kg per bale). The bales are kept in a separate section 6.00 m above the ground and Emily uses an iron platform ($M_{pl} = 66.3$ kg) attached to a large pulley system to lower the bales to the ground floor. When she puts two bales on the platform, it takes $t = 2.55$ s for the platform to reach the ground. Directly above the platform, a large wheel, which has a diameter of $d_{w1} = 91.8$ cm and a total mass of $m_{w1} = 21.4$ kg, which includes the mass of six steel spokes ($m_{s1} = 1.45$ kg per spoke), guides the cable that is holding the platform towards a disk-shaped pulley with a mass and radius of $m_d = 1.50$ kg and $r_d = 11.6$ cm, respectively. The cable then continues straight downwards, round a second wheel, which has a diameter of $d_{w2} = 63.4$ cm, a $w = 4.50$ cm wide edge, and a total mass of $m_{w2} = 13.6$ kg (this includes the mass m_{s2} of four iron spokes), and finally back up vertically where it is attached to the ceiling. Serving as a counterweight of this pulley system, a granite block ($M_{gb} = 129$ kg) is hanging from the center of the second wheel. Emily notices that one of the iron spokes shows a crack, and she replaces it straight away, otherwise the entire pulley system can come crashing down. If you know that the frictional torque caused by the bearings of the first wheel, the disk, and the second wheel are equal to $\tau_{f1} = 7.23$ N·m, $\tau_{f2} = 2.19$ N·m, and $\tau_{f3} = 5.41$ N·m, respectively, what is the mass of the iron spoke that Emily has just fixed? Assume that the cable does not slip when the pulley system is in motion and that counterclockwise is the positive direction of rotation.

Exercise 8

Gracjan finally put his two kids to bed and is now cleaning up. Earlier this afternoon, Gracjan threw a creative birthday party for his seven-year-old daughter Joanna and her friends whereby an art teacher was invited to show them all kinds of neat tricks with coloured cardboard paper. Gracjan promised Joanna to take her the next day to the beach in Ustka, Poland, which is a two-hour drive from their home in Szczecinek. When he sees during the cleanup a bunch of cut out isosceles triangles (with a mass of $m_t = 24.5$ g per piece) he gets the idea of creating a frisbee for Joanna that she can take with her to the beach tomorrow. Gracjan tapes eight triangles together to form an octagon (with a combined area of $A = 3,230$ cm^2) and to ensure stability during its flight, he glues a rubber band ($m_{rb} = 42.0$ g) onto the outer edge of each triangle. Gracjan then takes the octagonal frisbee for a test flight in the backyard and he is pleased with the result. When he grabs the frisbee at one edge (not at a vertex point), it takes $t = 0.620$ s to launch it, whereby it is given a frequency of 384 rpm. What is the magnitude of the perpendicular force $\vec{F}_{\perp}$ due to the friction between his hand and the frisbee that provides the frisbee with its initial spin?

Exercise 9

In the field of chemistry, organic compounds are defined as compounds that contain the chemical element carbon (C). Hydrocarbons are an example of organic compounds and only consist of the elements carbon and hydrogen (H). The hydrocarbons can be further subdivided into the homologous series alkanes and alkenes, whereby the latter are uniquely characterized by a double carbon bond (which the alkanes do not possess). An example of an alkane and an alkene is methane (CH_4) and ethene (CH_2=CH_2 or C_2H_4), respectively. The methane compound has a tetrahedral structure, which means that the angle $\theta = \angle HCH$ between two H atoms equals $\theta = 109.47°$. The length L_{CHm} of the C-H bond in methane is measured to be $L_{CHm} = 109.4$ pm. The six atoms of the compound ethene are coplanar and due to the presence of the double carbon bond, the length L_{CHe} of the C-H bond is slightly shorter relative to methane, i.e., $L_{CHe} = 108.7$ pm. The angle $\phi = \angle HCC$ between an H atom and the double bond is equal to $\phi = 121.7°$. If we let the compound ethene rotate about the axis that runs right through the middle of the double C-C bond and lies within the plane of the compound and if the methane compound rotates about the axis that connects an H atom with the central C atom, then the moment of inertia I_e of ethene is larger than that of methane (I_m) by a factor of 5.276. What is then the length L_{CC} of the double carbon bond in the compound ethene? Remember that 1 picometer is equal to 1 pm = 10^{-12} m, that the mass of an H and a C atom is equal to $m_H = 1.00797$ amu and $m_C = 12.011$ amu, respectively, and that 1 atomic mass unit is equivalent to 1 amu = 1.66054×10^{-27} kg.

Exercise 10

Roslyn is strolling about the different shops on Main Street in Ardara, Ireland, until her interest is suddenly drawn to a mysterious object in the display window of the shop All Kinds of Everything. In front of the object a label is placed with the words: "Perpetual Motion Machine". What Roslyn sees is a round platform with a diameter of $d = 9.50$ cm positioned at a certain height from the object's base, and from a hole in the middle of the platform a metal ball (with mass $m_b = 375$ g and radius $R_b = 9.50$ mm) falls down along a slide, which curls back up after touching the base, so that the ball eventually gets flung back onto the platform. Roslyn enters the shop, buys the intriguing device, and heads back home. It doesn't take Roslyn much time to figure out that a magnet hidden within the base is providing the required energy (and acceleration) to the ball to reach the platform—otherwise, the laws of thermodynamics would have been violated. When taking a closer look, Roslyn observes that the ball undergoes two types of motion on the slide: during the first part, the ball slides and slips ($\mu_k = 0.354$) along a $L = 20.2$ cm long straight segment tilted by $\phi = 27.4°$ and then follows a circular-shaped path (with a radius of $R_c = 6.50$ cm) whereby it now rolls without slipping, until the ball leaves the slide at the point where the tangent is making an angle of $\theta = 75.0°$ with the horizontal. If Roslyn switches off the magnet, how far, in terms of vertical distance, is the ball now removed from the platform while being at its highest point mid-air, if at all?

Exercise 11

Liam and Robert are having fun with their kids at the Greenview Playground in Edmonton, Canada, on a chilly, yet sunny day. After three hours of entertainment on the hopping stools, the monkey bars, the various slides, the chain ladders, the rope webs, the climbing forest, the swings, the spinners, and the climbing dome, the kids are now resting on the merry-go-round. That is, until their fathers suggest they go and get an ice-cream. With unprecedented excitement and loud cheering, they dash off to collect their well-deserved afternoon snack. As they jump off the merry-go-round, which has a radius of $R = 2.55$ m, they leave it spinning counterclockwise at a constant angular velocity of $\omega = 0.455$ rad/s. Because of all the screaming, the squirrel, who was enjoying her own snack up in the tree, is startled and drops her acorn ($m_a = 105$ g), which lands on the merry-go-round $d = 55.0$ cm from the edge and it rolls with an initial speed of $v_0 = 1.24$ m/s at an angle of $\theta = 156°$ with the radial line segment that intersects with the landing spot of the acorn. (1) How long does the acorn stay on the merry-go-round before flying off of it? (2) What are the coordinates of the acorn's point of exit (as seen from the rotating reference frame)? Ignore any kind of kinetic friction for this problem.

Exercise 12

Two large asteroids (with mass $m_1 = 2.62 \times 10^{12}$ kg and m_2) are tumbling through the vast emptiness of space with a speed of $v_1 = 88,394$ km/h and $v_2 = 52,872$ km/h, respectively, whereby asteroid 2 is following a straight path at an angle of $\theta = 76.5°$ with respect to the straight path of asteroid 1 and both move in the same plane. The shape of both asteroids can be approximated by a cuboid, i.e., a rectangular prism, with length $l_1 = 38.1$ km, width $w_1 = 20.6$ km, and height $h_1 = 90.3$ km for asteroid 1 and length $l_2 = 155$ km, width $w_2 = 53.8$ km, and height $h_2 = 72.4$ km for asteroid 2. Asteroid 1 rotates every 6.50 hours clockwise around the z-axis through its center of mass, while asteroid 2, which spins counterclockwise around an axis that runs parallel to the z-axis and along one of the four edges, requires 22.3 hours to complete one revolution. Moreover, asteroid 2 has a hole of cuboidal shape at its center that stretches across the entire height h_2 and whereby its length and width are about one fifth of that of the asteroid. At one point, the two asteroids collide and merge their mass into one spherical-like object, which has a radius of $R = 53,750$ m and rotates about an axis running through its center of mass. If you know that the consolidated asteroid is headed into the direction that makes an angle of $\alpha = 58.9°$ relative to the original path of the first asteroid, (1) what was the mass m_2 of asteroid 2? (2) At which speed is the newly assembled asteroid hurtling through space? (3) How long does it take the spherical asteroid to spin just once around its rotation axis? Assume that no mass is lost during the collision and transformation of the asteroids.

Exercise 13

Nina is pursuing her PhD in Theoretical Astrophysics at the Lorentz Institute in Leiden, The Netherlands, and she is specifically interested in studying binary systems of spinning black holes, a.k.a. Kerr black holes. A binary system consists of two massive bodies orbiting each other around their common center of mass called the barycenter. When a spinning object is orbiting around a certain point in space, its total angular moment $\vec{L}_{tot}$ is composed of two terms, i.e., the orbital angular momentum $\vec{L}_{orb} = \vec{r} \times \vec{p}$ and the spin angular momentum $\vec{J} = I \cdot \vec{\Omega}$. For her doctoral thesis, Nina is currently investigating the galaxy NGC 7674, which is located within the Pegasus constellation about 400 million light years away and houses a binary system of supermassive black holes. Nina finds that the magnitude of the orbital angular momentum $\vec{L}_{orb}$ is equal to $L_{orb} = M_1 \cdot \sqrt{G \cdot d_1 \cdot M_2}$, with d_1 the distance of black hole 1 from the barycenter and M_1 and M_2 the mass of the black hole 1 and 2, respectively. How did Nina obtain this result? Make the assumption that the orbits are circular in nature.

Exercise 14

The sub-Antarctic island of South Georgia, which belongs to the British Overseas Territories, is home to the world's largest colony of King Penguins. During the winter, many are often found more southwards along the coastal regions of the Antarctic continent. On one of the Antarctic islands called Spert Island, three King Penguins feel particularly playful today and they suddenly notice a floating piece of ice near the shore. Two of them ($m_{p1} = 12.3$ kg and $m_{p2} = 17.6$ kg) are quick to react, make their way towards the ice platform and jump onto it. Right before the third penguin ($m_{p3} = 23.2$ kg) also jumps onto the platform, the ice shelf is rotating slowly in the clockwise direction around its center of mass at a rate of 1 revolution every 1.22 minutes. The shape of the platform is a square prism (with length $l = 2.00$ m and height $h = 5.00$ cm) onto which an isosceles right-angled triangular prism of corresponding dimensions is attached to one of its sides. When visualizing the triangular prism at the right-hand side of the square prism, then the two penguins are standing $d_1 = 35.5$ cm and $d_2 = 82.7$ cm from the top left and bottom left corner under an angle of $\theta_1 = 50.6°$ and $\theta_2 = 38.2°$ with the horizontal, respectively. If you know that the density of ice is equal to $\rho = 917$ kg/m^3 and that the third penguin lands right at the center of mass, at what rate is the ice platform now rotating? Treat the ice shelf as a thin plate and the penguins as solid cylinders with an internal radius equal to $r_1 = 16.9$ cm, $r_2 = 19.1$ cm, and $r_3 = 22.3$ cm, respectively.

Exercise 15

Laniyan is one of the Cameroonian artists who are invited to exhibit their work at the temporary exposition "l'Asymétrie et la Rotation" at the contemporary art center doul'art at Douala, Cameroon. Laniyan designed his own creative version of a newly discovered, young planetary system of six planets. Instead of orbiting within a fixed plane, the planets are spatially arranged in a stepwise fashion whereby their total angular momentum $\vec{L}_{tot}$ is tilted by an angle of $\theta = 23.2°$ relative to the axis of rotation, i.e., the y-axis. Planet 1 ($m_1 = 6.55$ kg), which is the planet in the highest orbit, is located at a distance of $D_x = 92.2$ cm horizontally and $D_y = 66.6$ cm vertically from the planet in the lowest orbit, i.e., Planet 6 ($m_6 = 8.21$ kg), at the other end of the planetary system. Planet 2 ($m_2 = 7.60$ kg) is positioned $d_1 = 42.5$ cm to the south of Planet 1, and Planet 3 ($m_3 = 4.35$ kg), which finds itself $d_2 = 24.4$ cm to the east of Planet 2, is the planet closest to the origin of the coordinate system at a distance of $d_3 = 15.0$ cm, making thereby an angle α with the rotation axis. Planet 4 (m_4) is orbiting at $d_4 = 26.4$ cm to the east of the origin, and $d_5 = 13.9$ cm away from Planet 4 in the direction east of south at an angle β is the location of the orbit of Planet 5 ($m_5 = 3.67$ kg). Finally, Planet 6 is positioned $d_6 = 26.7$ cm further to the east relative to Planet 5. If you know that in Laniyan's planetary system the planets are not spinning and rotate counterclockwise, what is the mass m_4 of Planet 4? Neglect the mass of the connecting rods between the planets and take clockwise as the positive direction of rotation.

Exercise 16

Neylan is trekking with her friend Eldar through the national park Bozdağ Milli Parkı, which is located east of Konya, Turkey, and they just made a stop since Neylan wishes to practice her Robin Hood archery skills. Eldar finds a thin wooden board of length $L = 82.5$ cm and mass $m_b = 855$ g, balances it upright on two fingers, and throws it up in the air. The board spins in a counterclockwise direction around the axis perpendicular to its length and parallel to its width at 210 rpm, whereby the angular velocity vector points southwards. From a distance of $\Delta x = 55.0$ m away, Neylan shoots an arrow of length $d = 61.6$ cm and mass $m_a = 40.6$ g with an initial speed of $v = 89.3$ m/s eastwards towards the spinning board. When the arrow is at its highest point during its trajectory, it hits the uppermost end of the board, which is at that precise moment vertically oriented, right in the middle. If you know that the arrow remains stuck after hitting the wooden board, at what rate does the combined object now spin and in which direction? Treat the arrow as a long rod.

Exercise 17

When stars exhaust their nuclear fuel at the end of their lifetime, they shed off their outer layers, often accompanied by a supernova explosion, and the stellar core remnant converts, broadly speaking, into a white dwarf, a neutron star, or a black hole. About 2.6 billion years ago, this process created a rapidly spinning neutron star—called a pulsar—which goes by the name PSR J0348+0432. (1) If we suppose that the original star had a mass, diameter, and rotational period of $M_i = 4.68 \cdot M_s$, $d_i = 2.56 \times 10^4$ km, and $T_i = 1.05$ days, respectively, that during the formation of the pulsar a total of 57% of its mass was lost (without dissipating any angular momentum) and that the star's diameter shrunk by 99.9%, at what rate is the pulsar now spinning? (2) Suppose that a rock ($m_r = 1.87 \times 10^4$ kg) is following a synchronous, circular orbit around the pulsar PSR J0348+0432 and is suddenly hit by an asteroid from outer space along the radial direction of the rock's orbit. As a result of the collision, the rock is sent straight down towards the pulsar's surface at a velocity of $\vec{v}_0 = -15.5 \cdot \vec{i}_y$ km/s. If you know that the rock at the moment of impact is positioned above the pulsar's southern hemisphere at a latitude of $51°18'4.04''$S, by how much is the rock deflected due to the Coriolis effect when it hits the pulsar's surface? Use the average value of the gravitational field strength g between the orbital height and the pulsar's surface, and remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2) and that one solar mass measures $M_s = 1.99 \times 10^{30}$ kg.

Exercise 18

Tarif is living in the one house that stands somewhat isolated from the rest of the houses in his village Al-Mani'a in Algeria. Everywhere in and around his house Tarif has spinning objects hanging from the ceiling or from the roof of his veranda, since he believes that rotating objects produce powerful vortices of spiritual energy and attract benevolent spirits. In the Saharan cypress tree in his backyard, Tarif has tied two ropes to a branch, whereby two rods of length $D = 62.0$ cm are each attached to the free end of a rope. Towards the free end of each rod a spherical-like object is mounted that is able to spin around the rod. The object consists of two copper rings welded together in such a way so that the rod passes through the center of one ring and makes up the central diameter of the other ring. The rings of object 1 are $w_1 = 9.30$ cm wide and the left side of the object is located at a distance of $d_1 = 7.50$ cm from the rod's free end. Object 2 is $d_2 = 15.2$ cm away from its rod's free end and its rings have a width of $w_2 = 8.60$ cm. When holding the free end of the rod, Tarif gives object 1 a clockwise initial spin of 12 rps, whereas object 2 receives an anti-clockwise initial spin of 960 rpm. When he subsequently lets go of the respective rod, both objects start to precess. As Tarif is fascinated by periodic relations, he wants the first object to precess at half the rate of the second object. He achieves this configuration by gently touching a metal ring with a stick for about $t = 3.50$ s, producing thereby an angular deceleration of $\alpha = 16.7$ rad/s^2. (1) When looking from above, what is the direction of precession of both objects? (2) If you know that object 1 precesses initially at a higher rate, to which object does Tarif has to apply his angular deceleration technique? (3) If you know that the radius R_1 of the two metal rings of object 1 measures $R_1 = 14.2$ cm, what is the radius R_2 of the two metal rings of object 2?

Exercise 19

Yulissa decided to take her five-year-old nieces Camila ($m_1 = 20.6$ kg) and Sofia ($m_2 = 18.3$ kg) to the playground in the Parque Central Juan Pablo Duarte in Nagua, The Dominican Republic, so that their parents could celebrate their eighth wedding anniversary during a lunch at the seafood restaurant Junior Natura. The first attraction to which Camila and Sofia run off when they arrive at the Parque Central is the seesaw, which has a length and mass of $L = 4.50$ m and $M = 12.5$ kg, respectively, and makes an angle of $\theta = 9.50°$ with the horizontal when one side touches the ground. After only 10 minutes, Camila, who is the cheekiest of the two sisters, gets bored and climbs on the nearby Disney tower. From up there, she sees that Sofia is still sitting on the seesaw and without hesitation Camila jumps from a height of $h = 2.25$ m onto her empty seat. To Camila's great delight (and also Sofia's), Sofia is being ejected out of her seat for just a brief moment in time. (1) Where exactly does Sofia land? (2) How high did she go? Assume that the average force of impact that Camila exerts upon her empty seat is about twice her kinetic energy (per unit length) right before landing on the seesaw.

Exercise 20

The local council of the city of Bruges in Belgium is calling tenders for the construction of a new pedestrian drawbridge over one of the city's many canals. The bridge has to be $L_b = 12.0$ m long and two galvanized steel cables will have to do the important job of safely drawing up the bridge. Two large iron disk-shaped pulleys with a radius and mass of $R = 57.5$ cm and $M_p = 165$ kg, respectively, are rolling up the cables as the bridge is being lifted. Bart is one of the engineers responsible for listing the technical details for one of the tenders. According to his conservative calculations, the cables combined could lift a bridge of no more than $M = 4,250$ kg. When the bridge is down the cables make an angle of $\theta = 42.0°$ with the horizontal, and Bart estimates that the small boats could easily pass under the bridge when it is drawn up at an angle of $\phi_{up} = 65.0°$ with the horizontal. Bart furthermore assesses that the motor that rotates the pulleys can comfortably provide a torque of $\vec{\tau}_0 = 8,985 \cdot \vec{i}_z$ N·m at the initial moment when the bridge is being lifted (position 0), whereas at an angle of $\phi = 45.0°$ (position 1)—that is, when the process of drawing up the bridge is slowing down—a minimum torque of 54.5% of its initial value is required. (1) What angle do the cables make with the bridge when it is in position 1? (2) What is the length of the visible part of the cables at that moment? (3) What is the magnitude of the tension forces $\vec{T}_0$ and $\vec{T}_1$ in the cables when the bridge is in position 0 and 1, respectively? (4) What is the average amount of time needed for the bridge to reach position 1?

Exercise 1

Problem Statement

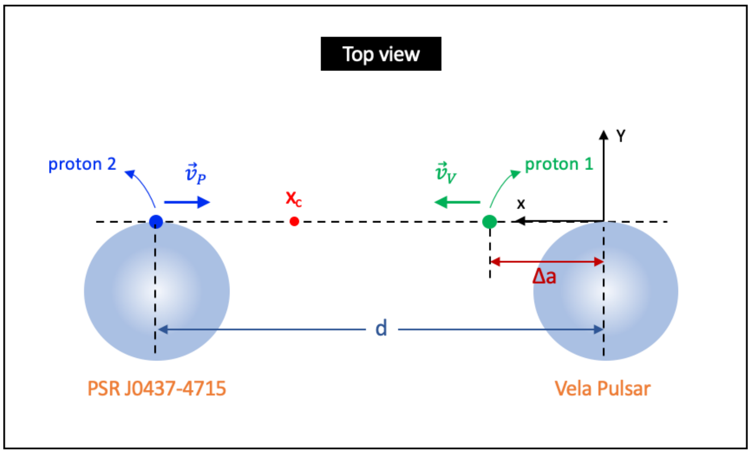

Figure 1

When massive stars have exhausted their nuclear fuel, i.e., they have fused all the nuclei of lighter elements, such as hydrogen and helium, into heavier nuclei (a process referred to as stellar nucleosynthesis), an immense explosion (a supernova) follows and under the right circumstances the stellar core residue forms a very dense neutron star. Some of these stars are spinning around their rotation axis at very high velocities and are designated pulsars, whereby streams of electromagnetic radiation are blasted from their magnetic poles at velocities up to 70% of the speed of light. One such pulsar is the Vela Pulsar ($M_V = 2.26\ M_s$), which is located about 959 light years away from us and rotates around its axis about 11.195 times every second. Suppose now that a subatomic particle, let's say a proton (m_p), is sitting on the equator of Vela. (1) If you know that the proton has to be accelerated by a factor of 315 to gain the minimum speed to escape Vela's gravitational pull, what is the Vela Pulsar's radius and the proton's initial speed v_r (viewed from a stationary reference frame)? (2) Suppose that this same proton left Vela 400 years ago and that 100 years later a proton escapes from another pulsar called PSR J0437-4715 ($M_P = 1.44\ M_s$ and $r_P = 13.6$ km) whereby both protons are on a straight collision course. If you know that the distance between both pulsars is equal to $d = 449.2$ light years, in which year will/did the two protons collide? Remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}\ \mathrm{m^3/(kg \cdot s^2)}$, that 1 light year measures 9.46×10^{12} km, and that one solar mass is equal to $M_s = 1.99 \times 10^{30}$ kg.

Solution

(1) For a particle that wants to become gravitationally *unbound* by a massive body, its total mechanical energy E_{tot} has to be at least zero (or greater). For the proton trying to get away from the Vela Pulsar's gravitational pull, we can write (with r_V the pulsar's radius and v_V the magnitude of the escape velocity $\vec{v}_V$):

$$E_{tot} = E_k + E_p = 0 \quad \Leftrightarrow \quad \frac{m_p \cdot v_V^2}{2} - \frac{G \cdot m_p \cdot M_V}{r_V} = 0 \quad \Leftrightarrow \quad v_V^2 = \frac{2 \cdot G \cdot M_V}{r_V}$$

When the proton is located at the equator of Vela, it is rotating at a tangential speed v_r (viewed from a stationary reference frame) due to the pulsar's rotation. We know that the proton has to be accelerated (by some unknown mechanism) by a factor of 315 so that its speed v_r is boosted to the value of the escape speed v_V. In other words, $v_V = 315 \cdot v_r$. Moreover, as Vela's frequency is equal to $f = 11.195$, we know that its angular velocity $\vec{\omega}$ has a value of $\omega = 2\pi \cdot f = 2\pi \cdot 11.195 = 70.3$ rad/s. Finally, given that $v_r = \omega \cdot r_V$ and inserting this expression into our above equation, we can find the radius of the Vela Pulsar as follows:

$$[315 \cdot (\omega \cdot r_V)]^2 = \frac{2 \cdot G \cdot M_V}{r_V} \quad \Leftrightarrow \quad r_V = \sqrt[3]{\frac{2 \cdot G \cdot M_V}{(315 \cdot \omega)^2}}$$

$$= \sqrt[3]{\frac{2 \cdot 6.67 \times 10^{-11} \cdot (2.26 \cdot 1.99 \times 10^{30})}{(315 \cdot 70.3)^2}}$$

$$= 10.7 \text{ km}$$

The proton's initial (tangential) speed v_r is then equal to $v_r = \omega \cdot r_V = 70.3 \cdot 10.7 \times 10^3 = 752$ km/s.

(2) The magnitude of the proton's escape velocity $\vec{v}_V$ is equal to $v_V = 315 \cdot v_r = 315 \cdot 7.52 \times 10^5 = 237,000$ km/s. Over a time period of $t_0 = 100$ years at a speed v_V, the proton will have covered a distance Δa of:

$$\Delta a = v_V \cdot t_0 = 2.37 \times 10^8 \cdot (100 \cdot 365 \cdot 86,400) = 7.48 \times 10^{17} \text{ m or } 79.0 \text{ light years}$$

At that moment, proton number 2 escapes from the pulsar PSR J0437-4715 at a speed equal to:

$$v_P = \sqrt{\frac{2 \cdot G \cdot M_V}{r_V}} = \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \cdot (1.44 \cdot 1.99 \times 10^{30})}{13.6 \times 10^3}} = 168,000 \text{ km/s}$$

Both protons will collide when the time t_1 needed by proton 1 to cover the displacement $x_c - \Delta a$ is equal to the time t_2 proton 2 needs to travel across the displacement $x_c - d$. The value of the position x_c at which the collision occurs is calculated as follows (bear in mind that proton 2 moves into the opposite direction of the x-axis):

$$t_1 = t_2 \quad \Leftrightarrow \quad \frac{x_c - \Delta a}{v_V} = \frac{x_c - d}{-v_P} \quad \Leftrightarrow \quad x_c = \frac{\Delta a \cdot v_P + d \cdot v_V}{v_P + v_V} = \frac{79.0 \cdot 1.68 \times 10^8 + 449.2 \cdot 2.37 \times 10^8}{1.68 \times 10^8 + 2.37 \times 10^8}$$

$$= 296 \text{ light years}$$

Starting from the position Δa, the time for proton 1 to get to the location of collision is equal to $t_1 = \frac{x_c - \Delta a}{v_V} = \frac{(296-79.0) \cdot 9.46 \times 10^{15}}{2.37 \times 10^8} = 274$ years. As proton 1 left the Vela Pulsar 400 years ago and given that the collision occurred $100 + 274 = 374$ years later, we know that the collision took place $400 - 374 = 25.6$ years ago, i.e., in the year 1996 (given that we are currently living in the year 2022).

Exercise 2

Problem Statement

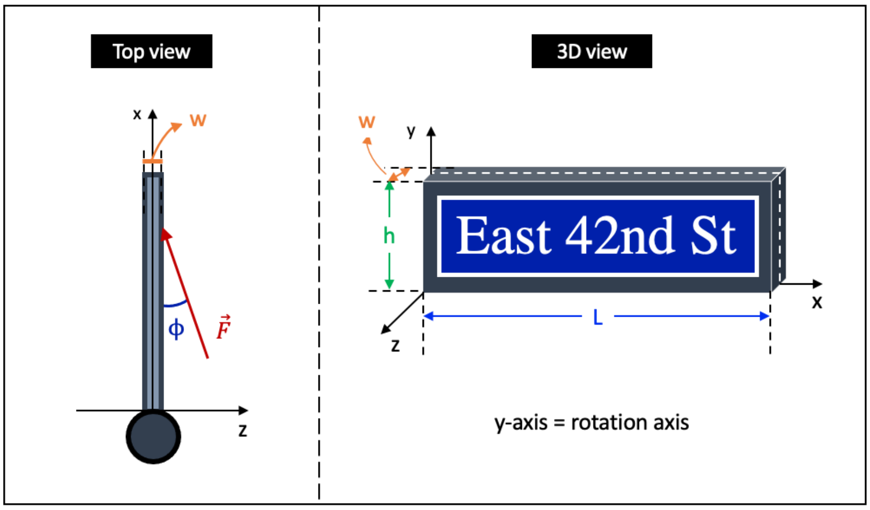

Figure 2

You work at The Escape Game in New York City, the United States, and as you're designing a new escape room, you need three 2.50 m long bamboo poles. Luckily, your friend Akira, who works at the Japanese restaurant Sen Sekana only three blocks up, stores a lot of decoration material in the basement, including a couple of bamboo poles. You go on foot to pick them up and on your way back, 10.0 m before getting to the zebra crossing on East 42nd Street, the red hand of the pedestrian traffic lights starts flashing on and off and you start running. You make it on time to the other side of the street, but you've hit the street name sign "East 42nd St", which is attached to a metal pole on the sidewalk, with one of the bamboo poles with a force $\vec{F}$ of a magnitude equal to $F = 68.2$ N under an angle of $\phi = 26.6°$ at about one fourth of the sign's length from the sign's end. If you know that the street sign's mass, length, height, and width are equal to $m = 3.74$ kg, $L = 45.7$ cm, $h = 17.8$ cm, and $w = 4.35$ cm, respectively, that the collision lasted $t = 0.425$ s, and that during the impact the sign experienced a frictional torque of $\tau_f = 10.2\ N \cdot m$, by what angle θ (in degrees) did you cause the street name sign to rotate?

Solution

In order to determine the angular displacement θ, we need to find the value of the angular acceleration α during the collision, which we can calculate as soon as we have figured out the value of both the net torque τ_n and the sign's moment of inertia I. Let us start with calculating the latter, for which we have to solve the following integral (with R the magnitude of the position vector $\vec{R}$ starting in the coordinate system's origin to a random point on the road sign):

$$I = \int R^2 \cdot dm$$

As the street name sign rotates about the y-axis, the dimensions that influence the moment of inertia are the x- and z-directions (note that the x-axis runs right through the middle of the width of the sign). Therefore, the sign's surface density ρ_s is equal to $\rho_s = \frac{dm}{dA}$, whereby $dA = dx \cdot dz$, and the value of R is equal to $R = \sqrt{x^2 + z^2}$. Plugging these expressions into the above integral allows us to find the sign's moment of inertia:

$$I = \int R^2 \cdot dm = \int_{x_1}^{x_2} \int_{z_1}^{z_2} [\sqrt{x'^2 + z'^2}]^2 \cdot [\rho_s \cdot dx' \cdot dz'] = \rho_s \cdot \int_{x_1}^{x_2} \int_{z_1}^{z_2} (x'^2 + z'^2) \cdot dx' \cdot dz'$$

$$= \rho_s \cdot \int_{x_1}^{x_2} \left[\left(x'^2 \cdot z + \frac{z^3}{3} \right) \Big|_{z_1=-\frac{w}{2}}^{z_2=\frac{w}{2}} \right] \cdot dx'$$

$$= \rho_s \cdot \int_{x_1}^{x_2} \left(x'^2 \cdot w + \frac{w^3}{12} \right) \cdot dx'$$

$$= \rho_s \cdot \left[\left(\frac{x^3}{3} \cdot w + \frac{w^3}{12} \cdot x \right) \Big|_{x_1=0}^{x_2=L} \right]$$

$$= \rho_s \cdot \left(\frac{L^3}{3} \cdot w + \frac{w^3}{12} \cdot L \right)$$

$$= \left[\frac{m}{L \cdot w} \right] \cdot \left(\frac{L^3}{3} \cdot w + \frac{w^3}{12} \cdot L \right)$$

$$= \frac{m}{12} \cdot \left(4 \cdot L^2 + w^2 \right)$$

$$= \frac{3.74}{12} \cdot \left(4 \cdot 0.457^2 + 0.0435^2 \right)$$

$$= 0.261 \text{ kg·m}^2$$

Next, we need to determine the value of the net torque τ_n. The torque τ_b facilitated by your bamboo pole hitting the street name sign at a distance $R_0 = 0.75 \cdot L$ from the rotation axis is equal to $\tau_b = (F \cdot \sin \phi) \cdot (0.75 \cdot L) = [68.2 \cdot \sin(26.6°)] \cdot (0.75 \cdot 0.457) = 10.5$ N·m. Given that there is a large frictional torque $\tau_f = 10.2$ N·m, the net torque is then $\tau_n = \tau_b - \tau_f = 10.5 - 10.2 = 0.267$ N·m.

Based on Newton's second law for rotation, we can now calculate the angular acceleration α:

$$\alpha = \frac{\tau_n}{I} = \frac{0.267}{0.261} = 1.02 \text{ rad/s}^2$$

In a final step, we determine the value of the street name sign's angular displacement θ:

$$\theta = \frac{\alpha}{2} \cdot t^2 = \frac{1.02}{2} \cdot 0.425^2 = 9.23 \times 10^{-2} \text{ rad or } 5.29°$$

Exercise 3

Problem Statement

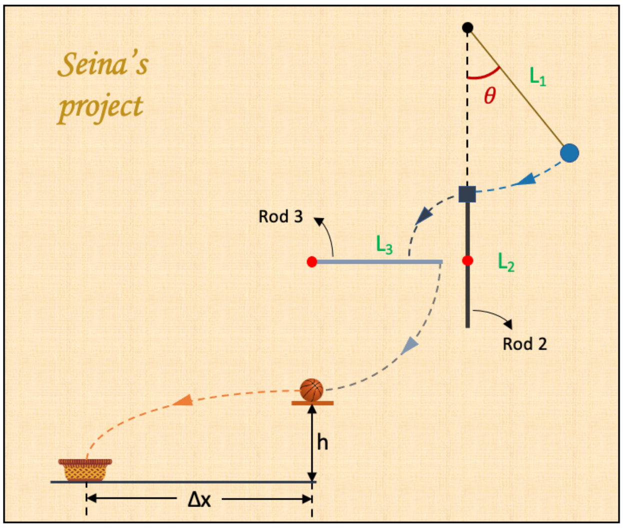

Figure 3

The University of Sonora in Hermosillo, Mexico, organized a two weeks long science summer camp for young adolescents between 16 and 18 years to bring them closer into contact with several main topics in the field of biology, chemistry, and physics. At the end of the two weeks, the students are asked to present an own project in their field of interest, and Seina has designed an integrated system to showcase the physical concept of rotational motion. Her project consists of four systems, which are all attached onto a thin wooden board. The first system ("System 1") is a pendulum, whereby a ball (m_1 = 158 g) is attached to a basically massless thin rope of length L_1 = 33.4 cm. When pulling the ball to the right over an angle θ and subsequently releasing it, the ball hits at its lowest point a hard plastic cube (m_2 = 215 g) with an edge of d = 1.00 cm, which is fixed at the top end of a vertically placed iron rod (M_2 = 439 g) with length L_2 = 25.6 cm. This rod ("rod 2") makes up "System 2" and rotates about its axis located at the middle of the rod. Due to the collision between the ball and the cube, rod 2 rotates counterclockwise and the plastic cube hits in turn a horizontal aluminum rod—this "rod 3" constitutes System 3—with a mass of M_3 = 312 g and a length equal to that of rod 2. Upon collision, rod 3 swings counterclockwise about its axis at the left end and when it has turned 90°, its right end (now at the bottom) collides with a mini basketball (m_4 = 194 g), which lies unrestrained on a horizontal platform. Upon impact with rod 3, the basketball is kicked to the left and lands into a small basket, which is installed h = 50.0 cm below the horizontal platform and Δx = 85.0 cm to the left. If you assume that the kinetic energy at each collision is completely transferred from one system to the next, at what angle θ should Seina release the ball of System 1, so that the mini basketball of System 4 ends up right into the basket?

Solution

Given that only conservative forces, i.e., gravity, are at work across the four systems, we know that the total mechanical energy within each of them is conserved. Regarding System 1, we can therefore write that the total work $W_{tot,1}$ done on the ball as it swings from its initial position at an angle θ to its lowest point is equal to the work $W_{c,1}$ performed by the conservative forces. As a result, we find the following expression for the ball's speed v_1 at its lowest point:

$$W_{tot,1} = W_{c,1} \quad \Leftrightarrow \quad \Delta E_{k,1} = -\Delta E_{p,1} \quad \Leftrightarrow \quad \frac{m_1 \cdot v_1^2}{2} = m_1 \cdot g \cdot L_1 \cdot (1 - \cos\theta)$$

$$\Leftrightarrow \quad v_1 = \sqrt{2 \cdot g \cdot L_1 \cdot (1 - \cos\theta)}$$

Upon hitting the cube on top of rod 2, the ball entirely transfers its kinetic energy to System 2, which consists of a rotating system, so that the initial rotational kinetic energy $E_{k,2,i}$ of System 2 equals the final kinetic energy $E_{k,1}$ of System 1:

$$E_{k,2,i} = E_{k,1} \quad \Leftrightarrow \quad \frac{I_2 \cdot \omega_{2,i}^2}{2} = \frac{m_1 \cdot v_1^2}{2}$$

Let us in a first instance determine the moment of inertia I_2 of rod 2. The moment of inertia of rod 2 without the cube is equal to $I_r = \frac{M_2 \cdot L_2^2}{12}$ and that of a cube that rotates around an axis at its center of mass to $I_{c,c} = \frac{m \cdot r^2}{6}$ (with r the length of one edge). Given that the cube in System 2 is located at a distance $R_c = \frac{L_2}{2} + \frac{d}{2}$ from the axis of rotation, we can apply the parallel-axis theorem to find the total moment of inertia I_2 of rod 2:

$$I_2 = I_r + (I_{c,c} + m_2 \cdot R_c^2) = \frac{M_2 \cdot L_2^2}{12} + \left[\frac{m_2 \cdot d^2}{6} + m_2 \cdot \left(\frac{L_2}{2} + \frac{d}{2}\right)^2\right]$$

$$= \frac{L_2^2}{12} \cdot (M_2 + 3 \cdot m_2) + \frac{m_2 \cdot d}{12} \cdot (5 \cdot d + 6 \cdot L_2)$$

Also, since there is only one cube attached to rod 2, the rod's center of mass x_c does not correspond to the position of the axis of rotation. In fact, x_c should be located a little bit above the rod's midpoint. Based on the definition of the center of mass, we can find the distance h_* between x_c and the rod's midpoint as follows:

$$h_* = \frac{\frac{(L_2+d)}{2} \cdot m_2}{M_2 + m_2} = \frac{\frac{(0.256+0.01)}{2} \cdot 0.215}{0.439 + 0.215} = 4.37 \text{ cm}$$

Therefore, when the ball of System 1 hits the cube of System 2, also gravity will act upon rod 2 as it rotates counterclockwise towards System 3. Using all the above information, when comparing the initial upright position of rod 2 to its final horizontal position, the work-energy relation gives us the following expression for the final angular velocity $\omega_{2,f}$ of rod 2 right before colliding with rod 3 :

$$W_{tot,2} = W_{c,2} \quad \Leftrightarrow \quad \Delta E_{k,2} = -\Delta E_{p,2} \quad \Leftrightarrow \quad \frac{I_2 \cdot \omega_{2,f}^2}{2} - \frac{I_2 \cdot \omega_{2,i}^2}{2} = (m_2 + M_2) \cdot g \cdot h_*$$

$$\Leftrightarrow \quad \frac{I_2 \cdot \omega_{2,f}^2}{2} - \frac{m_1 \cdot v_1^2}{2} = (m_2 + M_2) \cdot g \cdot h_*$$

$$\Leftrightarrow \quad \omega_{2,f} = \sqrt{\frac{2}{I_2} \cdot \left[(m_2 + M_2) \cdot g \cdot h_* + \frac{m_1 \cdot v_1^2}{2}\right]}$$

Applying the work-energy relation to System 3, whereby the moment of inertia I_3 of rod 3 equals $I_3 = \frac{M_3 \cdot L_3^2}{3}$ and whereby gravity exerts a force on the rod's center of mass, i.e., at its midpoint, and furthermore keeping in mind that the kinetic energy is being completely transferred, we find the following expression for the final angular velocity $\omega_{3,f}$ of rod 3 when it is positioned vertically and right before hitting the mini basketball:

$$W_{tot,3} = W_{c,3} \quad \Leftrightarrow \quad \Delta E_{k,3} = -\Delta E_{p,3} \quad \Leftrightarrow \quad \frac{I_3 \cdot \omega_{3,f}^2}{2} - \frac{I_3 \cdot \omega_{3,i}^2}{2} = M_3 \cdot g \cdot \frac{L_3}{2}$$

$$\Leftrightarrow \quad \frac{I_3 \cdot \omega_{3,f}^2}{2} - \frac{I_2 \cdot \omega_{2,f}^2}{2} = M_3 \cdot g \cdot \frac{L_3}{2}$$

$$\Leftrightarrow \quad \omega_{3,f} = \sqrt{\frac{M_3 \cdot g \cdot L_3 + I_2 \cdot \omega_{2,f}^2}{I_3}}$$

With respect to System 4, the equation related to the transfer of kinetic energy provides us with the speed v_4 with which the mini basketball is being kicked to the left:

$$\frac{I_3 \cdot \omega_{3,f}^2}{2} = \frac{m_4 \cdot v_4^2}{2} \quad \Leftrightarrow \quad v_4 = \sqrt{\frac{I_3}{m_4}} \cdot \omega_{3,f}$$

The time t the mini basketball spends in the air is calculated as follows:

$$y = y_0 - \frac{g}{2} \cdot t^2 \quad \Leftrightarrow \quad 0 = 0.5 - \frac{g}{2} \cdot t^2 \quad \Leftrightarrow \quad t = \sqrt{\frac{1}{g}}$$

The requirement that the mini basketball end up in the basket is mathematically translated as:

$$\Delta x = v_4 \cdot t = v_4 \cdot \sqrt{\frac{1}{g}}$$

To find the angle θ at which Seina should release the ball of System 1, we take the above requirement as starting position and insert step by step all of the above expressions into the equation until we, as in a sort of Matryoshka doll approach, come to the point where we rediscover the angle θ:

$$\Delta x = v_4 \cdot \sqrt{\frac{1}{g}} = \left[\sqrt{\frac{I_3}{m_4}} \cdot \omega_{3,f}\right] \cdot \sqrt{\frac{1}{g}} = \sqrt{\frac{I_3}{m_4 \cdot g}} \cdot \omega_{3,f}$$

$$= \sqrt{\frac{I_3}{m_4 \cdot g}} \cdot \left[\sqrt{\frac{M_3 \cdot g \cdot L_3 + I_2 \cdot \omega_{2,f}^2}{I_3}} \right] = \sqrt{\frac{M_3 \cdot g \cdot L_3 + I_2 \cdot \omega_{2,f}^2}{m_4 \cdot g}}$$

$$= \sqrt{\frac{M_3 \cdot g \cdot L_3 + I_2 \cdot \left[\frac{2}{I_2} \cdot \left[(m_2 + M_2) \cdot g \cdot h_* + \frac{m_1 \cdot v_1^2}{2}\right]\right]}{m_4 \cdot g}}$$

$$= \sqrt{\frac{M_3 \cdot g \cdot L_3 + 2 \cdot (m_2 + M_2) \cdot g \cdot h_* + m_1 \cdot v_1^2}{m_4 \cdot g}}$$

$$= \sqrt{\frac{M_3 \cdot g \cdot L_3 + 2 \cdot (m_2 + M_2) \cdot g \cdot h_* + m_1 \cdot [2 \cdot g \cdot L_1 \cdot (1 - \cos\theta)]}{m_4 \cdot g}}$$

$$\Leftrightarrow \quad \cos\theta = 1 - \frac{[(\Delta x)^2 \cdot m_4 - M_3 \cdot L_3 - 2 \cdot (m_2 + M_2) \cdot h_*]}{2 \cdot m_1 \cdot L_1}$$

$$\Leftrightarrow \quad \theta = \cos^{-1}\left(1 - \frac{[(\Delta x)^2 \cdot m_4 - M_3 \cdot L_3 - 2 \cdot (m_2 + M_2) \cdot h_*]}{2 \cdot m_1 \cdot L_1}\right)$$

$$= \cos^{-1}\left(1 - \frac{[0.850^2 \cdot 0.194 - 0.312 \cdot 0.256 - 2 \cdot (0.215 + 0.439) \cdot 0.0437]}{2 \cdot 0.158 \cdot 0.334}\right)$$

$$= 13.9^\circ$$

If Seina releases the ball of the pendulum at the moment when the rope makes an angle of $\theta = 13.9^\circ$ with the vertical, then the mini basketball will eventually end up within the basket, under the assumption, however, that the kinetic energy of the final state of a system is completely transferred to the next system.

Exercise 4

Problem Statement

Aminah is taking an advanced course in physics at the Rawalpindi Women University in Rawalpindi, Pakistan, and for her final project she wishes to determine with the help of Lagrangian mechanics the equation of motion of an elliptical cylinder (with mass M) rolling down an incline. Aminah's project is particularly challenging since the elliptical cylinder not only sporadically loses contact with the surface at higher speeds, but it simultaneously slips and rolls. As Aminah wants to prepare thoroughly for the first experimental test, whereby she will gather and map the measurements of the motion of both the center point and the two focus points of one of the sides of the cylinder, she decides to calculate beforehand the moment of inertia I of an elliptical cylinder. What is the outcome of Aminah's calculation?

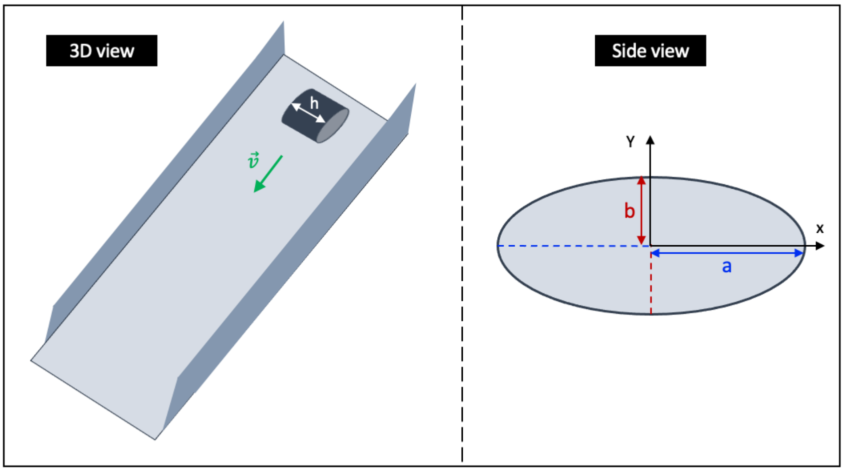

Figure 4

Solution

When rolling down the incline, the elliptical cylinder rotates about the z-axis, which is coming out of your screen with respect to the chosen coordinate system as depicted in Fig. 4 (right-hand side). The general integral for the moment of inertia I is equal to:

$$I = \int R^2 \cdot dm$$

The mass element dm can be reformulated as $dm = \rho \cdot dV$, with ρ the volume density of the element and dV the volume element, which is equal to $dV = dx \cdot dy \cdot dz$. The distance R represents the magnitude of the position vector $\vec{R}$, which indicates the position of the mass element dm and makes an angle of 90° with the axis of rotation, i.e., the z-axis. In other words, the vector $\vec{R}$ lies in the xy-plane and its magnitude is equal to $R = \sqrt{x^2 + y^2}$. Inserting this information back into our above equation, we obtain the following triple integral:

$$I = \iiint \left(x'^2 + y'^2\right) \cdot \rho \cdot dx' \cdot dy' \cdot dz'$$

Regarding the limits of this integral, if we let the limits of the x- and y-direction run from -a to a and from -b to b, respectively, and integrate this double integral, we would obtain the area of a rectangle with length $2a$ and height $2b$, which is not what we want as the area of an ellipse is equal to $\pi \cdot a \cdot b$ and not $4 \cdot a \cdot b$. Therefore, if we let, for instance, the limit of the x-direction run from -a to a, we then have to let the limit of the y-direction run from -y to y, whereby we obtain an expression for y from the equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Leftrightarrow \quad y = \pm\, b \cdot \sqrt{1 - \frac{x^2}{a^2}}$$

Given that the limit of the z-direction runs from 0 to the cylinder's height h, we can write the triple integral as follows:

$$I = \rho \cdot \int_0^h dz' \cdot \int_{-a}^{a} dx' \cdot \int_{-b\cdot\sqrt{1-\frac{x^2}{a^2}}}^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} \left(x'^2 + y'^2\right) \cdot dy'$$

We can further simplify this integral by observing in Fig. 4 (right-hand side) that the area of an ellipse can be divided in four equal parts, whereby the upper right part is constrained by the line segment [0,a] in the x-direction and [0,b] in the y-direction. The area of the ellipse can then be found by calculating the area of the upper right part and multiplying that by a factor of 4. The integral then becomes:

$$I = 4 \cdot \rho \cdot \int_0^h dz' \cdot \int_0^a dx' \cdot \int_0^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} \left(x'^2 + y'^2\right) \cdot dy'$$

We can now start solving the integral:

$$I = 4 \cdot \rho \cdot \left[z\big|_0^h \right] \cdot \int_0^a dx' \cdot \int_0^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} (x'^2 + y'^2) \cdot dy'$$

$$= 4 \cdot \rho \cdot h \cdot \int_0^a dx' \cdot \int_0^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} (x'^2 + y'^2) \cdot dy'$$

$$= 4 \cdot \rho \cdot h \cdot \int_0^a dx' \cdot \left[\left(x'^2 \cdot y + \frac{y^3}{3} \right) \Bigg|_0^{b\cdot\sqrt{1-\frac{x^2}{a^2}}} \right]$$

$$= 4 \cdot \rho \cdot h \cdot \int_0^a dx' \cdot \left[x'^2 \cdot b \cdot \sqrt{1 - \frac{x'^2}{a^2}} + \frac{b^3}{3} \cdot \left(1 - \frac{x'^2}{a^2}\right)^{\frac{3}{2}} \right]$$

$$= 4 \cdot \rho \cdot h \cdot b \cdot \int_0^a \left[x'^2 \cdot \sqrt{1 - \frac{x'^2}{a^2}} + \frac{b^2}{3} \cdot \left(1 - \frac{x'^2}{a^2}\right)^{\frac{3}{2}} \right] \cdot dx'$$

At this point, we make the substitution $x' = a \cdot \sin\theta'$, whereby $dx' = a \cdot \cos\theta' \cdot d\theta'$. The limits of integration thereby change from $0 \to a$ to $0 \to \frac{\pi}{2}$. In addition, we will apply the trigonometric identity "$\cos^2\alpha + \sin^2\alpha = 1$". The integral then becomes:

$$I = 4 \cdot \rho \cdot h \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot \sin^2\theta' \cdot \sqrt{1 - \frac{a^2 \cdot \sin^2\theta'}{a^2}} + \frac{b^2}{3} \cdot \left(1 - \frac{a^2 \cdot \sin^2\theta'}{a^2}\right)^{\frac{3}{2}} \right] \cdot a \cdot \cos\theta' \cdot d\theta'$$

$$= 4 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot \sin^2\theta' \cdot \cos\theta' + \frac{b^2}{3} \cdot \cos^3\theta' \right] \cdot \cos\theta' \cdot d\theta'$$

$$= 4 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot \sin^2\theta' \cdot \cos^2\theta' + \frac{b^2}{3} \cdot \cos^4\theta' \right] \cdot d\theta'$$

In the following lines, we make use of the trigonometric identity mentioned earlier, as well as two more identity relations, i.e., $\cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$ and $\cos^4\alpha = \frac{3+4\cdot\cos(2\alpha)+\cos(4\alpha)}{8}$, so that we eventually find the expression for the moment of inertia I of an elliptical cylinder:

$$I = 4 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot (1 - \cos^2\theta') \cdot \cos^2\theta' + \frac{b^2}{3} \cdot \cos^4\theta' \right] \cdot d\theta'$$

$$= 4 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot \cos^2\theta' + \left(\frac{b^2}{3} - a^2\right) \cdot \cos^4\theta' \right] \cdot d\theta'$$

$$= 4 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[a^2 \cdot \left(\frac{1 + \cos(2\theta')}{2}\right) + \left(\frac{b^2}{3} - a^2\right) \cdot \left(\frac{3 + 4 \cdot \cos(2\theta') + \cos(4\theta')}{8}\right) \right] \cdot d\theta'$$

$$= 2 \cdot \rho \cdot h \cdot a \cdot b \cdot \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} \cdot (a^2 + b^2) + \frac{b^2}{3} \cdot \cos(2\theta') + \frac{1}{4} \cdot \left(\frac{b^2}{3} - a^2\right) \cdot \cos(4\theta') \right] \cdot d\theta'$$

$$= 2 \cdot \rho \cdot h \cdot a \cdot b \cdot \left[\frac{1}{4} \cdot (a^2 + b^2) \cdot \left[\theta|_0^{\frac{\pi}{2}}\right] + \frac{b^2}{6} \cdot \left[\sin(2\theta)|_0^{\frac{\pi}{2}}\right] + \frac{1}{16} \cdot \left(\frac{b^2}{3} - a^2\right) \cdot \left[\sin(4\theta)|_0^{\frac{\pi}{2}}\right] \right]$$

$$= 2 \cdot \rho \cdot h \cdot a \cdot b \cdot \left[\frac{1}{4} \cdot (a^2 + b^2) \cdot \frac{\pi}{2} + 0 + 0 \right]$$

$$= 2 \cdot \left[\frac{M}{\pi \cdot a \cdot b \cdot h}\right] \cdot h \cdot a \cdot b \cdot \left[\frac{1}{4} \cdot (a^2 + b^2) \cdot \frac{\pi}{2} \right]$$

$$= \frac{M}{4} \cdot (a^2 + b^2)$$

Exercise 5

Problem Statement

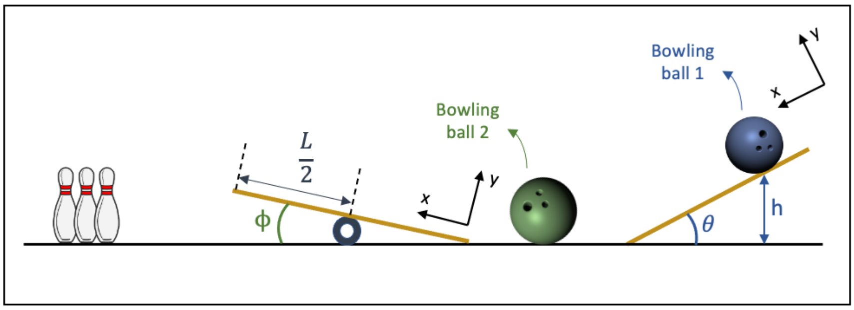

Figure 5

Martin and his two best friends, Cristobal and Catalina, are going bowling tonight in the local bowling center in Antofagasta, Chile, to celebrate the good news that Catalina obtained her official PADI certification for scuba diving. In the afternoon, they have some spare time, and to warm up for the evening event they play around at Cristobal's house with ramps, bowling balls, and bowling pins. They let a first bowling ball (with mass $M_1 = 2.54$ kg and radius $R_1 = 10.8$ cm) roll down a long ramp at an incline of $\theta = 25.2°$ from a starting height h, and when it reaches the bottom it hits (perfectly elastically) a second bowling ball (with mass $M_2 = 5.22$ kg and radius $R_2 = 11.6$ cm). Upon collision, the second ball receives an initial velocity with which it is sent up a $L = 5.00$ m long second ramp with a slope equal to $\phi = 14.6°$, whereby the ramp itself is placed loosely on a metal cylinder, which is fixed to the ground, and whereby the perpendicular bisector of the ramp intersects with the cylinder's rotation axis. As soon as the second bowling ball crosses the middle of the ramp, the ramp tilts and the ball rolls down from the second half of the ramp, eventually hitting a nicely assembled group of bowling pins. Martin did a too good a job of polishing the first bowling ball, so that it starts sliding ($\mu_k = 0.102$) on the first ramp when it reaches a speed equal to $v_s = \sqrt{\frac{5 \cdot g}{4}}$. What should be the minimum height h_{min} from which the first bowling ball is released on the first ramp, so that the second bowling ball is able to tilt the second ramp and hit the pins? Assume that the second bowling ball does not slip, that the second ramp does not undergo translational motion when the second bowling ball starts rolling uphill, and that the rotation axis of both bowling balls does not change direction during their motion.

Solution

Until the first bowling ball reaches the speed v_s, it rolls down ramp 1 without slipping. This means that the tangential acceleration of the rotating ball is equal to its translational acceleration down the slope. To determine the value of the acceleration a, we apply in a first instance Newton's second law to bowling ball 1 (whereby $\vec{F}_f$ represents a friction force smaller or equal to the static friction force and which is responsible for the torque exerted upon the ball):

$$-F_f + M_1 \cdot g \cdot \sin\theta = M_1 \cdot a$$

As we assumed that the direction of the ball's rotation axis does not change while rolling down the slope, Newton's second law for rotation is valid, so that we obtain the following expression for the magnitude of the friction force $\vec{F}_f$ (remember that the moment of inertia I of a ball is equal to $I = \frac{2}{5} \cdot M \cdot R^2$):

$$\tau = I \cdot \alpha \quad \Leftrightarrow \quad [F_f \cdot R_1] = \left[\frac{2}{5} \cdot M_1 \cdot R_1^2\right] \cdot \left[\frac{a}{R_1}\right] \quad \Leftrightarrow \quad F_f = \frac{2}{5} \cdot M_1 \cdot a$$

Plugging this expression into the equation of Newton's second law for linear motion provides us with an expression for the acceleration a:

$$-\left[\frac{2}{5} \cdot M_1 \cdot a\right] + M_1 \cdot g \cdot \sin\theta = M_1 \cdot a \quad \Leftrightarrow \quad a = \frac{5}{7} \cdot g \cdot \sin\theta$$

The distance x_s across which bowling ball 1 rolls on ramp 1 without slipping is equal to:

$$v_s^2 - v_0^2 = 2 \cdot a \cdot x_s \quad \Leftrightarrow \quad x_s = \frac{v_s^2 - v_0^2}{2 \cdot a} = \frac{\left[\sqrt{\frac{5 \cdot g}{4}}\,\right]^2 - 0^2}{2 \cdot \left[\frac{5}{7} \cdot g \cdot \sin\theta\right]} = \frac{7}{8} \cdot \frac{1}{\sin\theta}$$

Since the total distance d that bowling ball 1 travels when releasing it from a height h is equal to $d = \frac{h}{\sin\theta}$, the remaining distance x_r during which the ball slips until it reaches the ground is then equal to:

$$x_r = d - x_s = \frac{h}{\sin\theta} - \frac{7}{8} \cdot \frac{1}{\sin\theta} = \frac{1}{\sin\theta} \cdot \left(h - \frac{7}{8}\right)$$

The acceleration a_r during the distance x_r is found by applying Newton's second law to ball 1 (whereby $\vec{F}_k$ represents the kinetic friction force with magnitude $F_k = \mu_k \cdot F_N$):

$$-F_k + F_G = M_1 \cdot a_r \quad \Leftrightarrow \quad -\mu_k \cdot (M_1 \cdot g \cdot \cos\theta) + M_1 \cdot g \cdot \sin\theta = M_1 \cdot a_r \quad \Leftrightarrow \quad a_r = g \cdot (\sin\theta - \mu_k \cdot \cos\theta)$$

The expression for the final speed v_1 with which ball 1 reaches the bottom of the ramp is then found as follows:

$$\begin{aligned} v^2 - v_0^2 = 2 \cdot a \cdot \Delta x \quad &\Leftrightarrow \quad v_1^2 - v_s^2 = 2 \cdot a_r \cdot x_r \\ &\Leftrightarrow \quad v_1 = \sqrt{v_s^2 + 2 \cdot g \cdot (\sin\theta - \mu_k \cdot \cos\theta) \cdot \frac{1}{\sin\theta} \cdot \left(h - \frac{7}{8}\right)} \\ &\qquad\quad = \sqrt{v_s^2 + 2 \cdot g \cdot (1 - \mu_k \cdot \cot\theta) \cdot \left(h - \frac{7}{8}\right)} \end{aligned}$$

At this point, ball 1 collides at a speed v_1 with bowling ball 2, which is at rest ($v_2 = 0$ m/s). If we consider the isolated system of "bowling ball 1 plus bowling ball 2", then the total linear momentum $\vec{p}_{tot}$ is conserved as well as the total kinetic energy $E_{k,tot}$, given that their collision occurs perfectly elastically. We can therefore write:

$$\begin{cases} M_1 \cdot v_1 = M_1 \cdot v_1' + M_2 \cdot v_2' \\ \\ M_1 \cdot v_1^2 = M_1 \cdot v_1'^2 + M_2 \cdot v_2'^2 \end{cases}$$

Rearranging the equations and subsequently dividing one by the other, we find a relation between the three speeds v_1, v_1', and v_2'. If we then insert the newly found expression for the speed v_1' into the equation related to the conservation of linear momentum, we obtain an expression for the speed v_2':

$$\begin{cases} M_1 \cdot (v_1 - v_1') = M_2 \cdot v_2' \\ \\ M_1 \cdot (v_1 - v_1') \cdot (v_1 + v_1') = M_2 \cdot v_2'^2 \end{cases}$$

$$\Rightarrow \quad v_1 + v_1' = v_2'$$

$$\Rightarrow \quad M_1 \cdot v_1 = M_1 \cdot [v_2' - v_1] + M_2 \cdot v_2'$$

$$\Leftrightarrow \quad v_2' = \frac{2 \cdot M_1}{M_1 + M_2} \cdot v_1$$

The speed v_2' is the speed with which ball 2 starts rolling uphill (without slipping) on the second ramp. The minimum requirement of ball 2 tilting the ramp is that its speed is greater than zero when the ball is at the middle of the ramp. If the speed were equal to zero, the ball would come to a halt and roll back down. If the speed v_m represents the speed of ball 2 at the middle of the ramp, and with the acceleration a equal to $a = -\frac{5}{7} \cdot g \cdot \sin\phi$, the requirement in mathematical language reads as follows:

$$v_m^2 - v_2'^2 = 2 \cdot a \cdot \frac{L}{2} \quad \Leftrightarrow \quad v_m^2 - v_2'^2 = a \cdot L \quad \Leftrightarrow \quad v_m^2 - v_2'^2 = \left[-\frac{5}{7} \cdot g \cdot \sin\phi\right] \cdot L$$

$$\Leftrightarrow \quad v_2'^2 = v_m^2 + \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad v_2'^2 > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

If we now insert the previously obtained expressions for v_2' and v_1 into the above requirement, we find the minimum height h_{min} from which bowling ball 1 must be released if Martin and his friends want bowling ball 2 to tilt the second ramp and strike the pins:

$$v_2'^2 > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad \left[\frac{2 \cdot M_1}{M_1 + M_2} \cdot v_1\right]^2 > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad \left(\frac{2 \cdot M_1}{M_1 + M_2}\right)^2 \cdot v_1^2 > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad \left(\frac{2 \cdot M_1}{M_1 + M_2}\right)^2 \cdot \left[\sqrt{v_s^2 + 2 \cdot g \cdot (1 - \mu_k \cdot \cot\theta) \cdot \left(h - \frac{7}{8}\right)}\,\right]^2 > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad \left(\frac{2 \cdot M_1}{M_1 + M_2}\right)^2 \cdot \left[v_s^2 + 2 \cdot g \cdot (1 - \mu_k \cdot \cot\theta) \cdot \left(h - \frac{7}{8}\right)\right] > \frac{5}{7} \cdot g \cdot \sin\phi \cdot L$$

$$\Leftrightarrow \quad h > \frac{7}{8} + \frac{5 \cdot g \cdot \sin\phi \cdot L \cdot (M_1 + M_2)^2 - 28 \cdot v_s^2 \cdot M_1^2}{56 \cdot g \cdot M_1^2 \cdot (1 - \mu_k \cdot \cot\theta)}$$

$$\Leftrightarrow \quad h > \frac{7}{8} + \frac{5 \cdot 9.81 \cdot \sin(14.6^\circ) \cdot 5.00 \cdot (2.54 + 5.22)^2 - 28 \cdot \left[\sqrt{\frac{5 \cdot g}{4}}\,\right]^2 \cdot 2.54^2}{56 \cdot 9.81 \cdot 2.54^2 \cdot [1 - 0.102 \cdot \cot(25.2^\circ)]}$$

$$\Leftrightarrow \quad h > 1.42 \text{ m}$$

Exercise 6

Problem Statement

In his village Qafmollë in Albania, Agim is known as someone who pretty much always goes against the grain. At this moment, he is welding a new weathercock to put on the roof of his 18th century old house, but instead of placing the cock ($m_c = 4.67$ kg) in the middle of the horizontal iron rod ($m_r = 1.12$ kg), Agim feels that it belongs at one end of the rod, so that it can proudly gaze into the wind. At the other end of the $L = 85.0$ cm long rod, Agim attached a copper ball (with mass $m_b = 2.15$ kg and radius $r_b = 3.86$ cm) as counterweight. After trying out various positions along the rod, he finds that the weathercock is most agile and receptive to the wind when a second rod, which acts as the rotation axis, is placed vertically at the center of mass of the first rod. As Agim wants to fully comprehend why this is the case, he emails his son Spiro, who is pursuing a bachelor's degree in physics at the University of Tirana, asking him for a mathematical explanation. Later that day, Agim receives a response from Spiro. What did the email say?

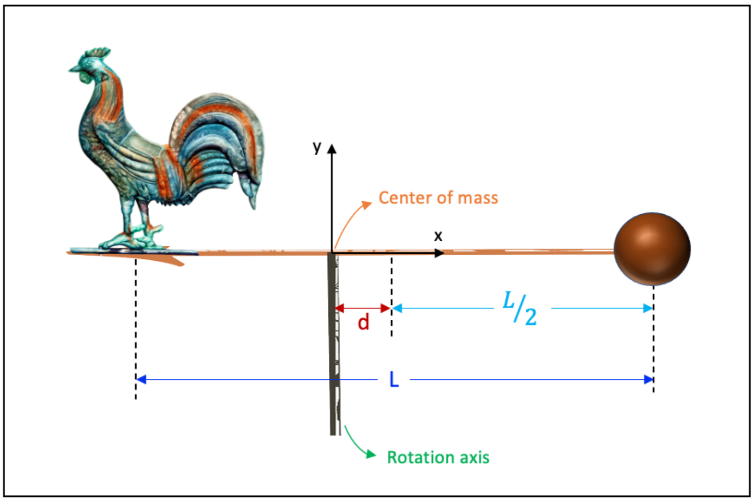

Figure 6

Solution

Hi Dad,

I see you've took on a new project? Cannot wait to see the final result this weekend!

As to your question, it has everything to do with the (mass) moment of inertia I of the entire weathercock. To be most agile and receptive to the wind means that the weathercock should experience the least amount of rotational inertia when rotating about its axis of rotation for a given wind force—you could interpret inertia as a kind of resistance to motion due to the object's total mass as well as the spatial distribution of that mass with respect to the rotation axis.

According to Newton's second law for rotation, i.e., $\tau_{net} = I \cdot \alpha$, and keeping in mind the definition of the torque τ, i.e., $\tau = F_\perp \cdot \Delta x$, you can see that for a constant wind force F a smaller rotational inertia I results in a larger rotational acceleration α, which translates into a greater agility of your weathercock.

Before diving into the math, let me also add that the weathercock's agility, and stability, is greatest when the rotation axis is placed at its center of mass, because then the net torque in the xy-plane (i.e., the rotation around the z-axis, which comes out of your screen) will be zero. Otherwise, it would start to wobble when rotating in the xz-plane (i.e., around the y-axis) instead of rotating smoothly.

Time for some math. You can determine the ideal position of the rotation axis if you minimize the total moment of inertia I_{tot} of the weathercock. So, let us first find the expression for I_{tot}, which is equal to the sum of the three individual moments of inertia, i.e., that of the cock (I_c), the rod (I_r), and the copper ball (I_b). If we assume that the ideal position is located at some distance *d to the left* of the middle of the iron rod (after all, the cock is heavier than the rod and the ball combined), and applying the parallel-axis theorem, we find the following expressions for I_c, I_r, and I_b, respectively:

$$\begin{cases} I_c = I_{c,CM} + m_c \cdot \left(\dfrac{L}{2} - d\right)^2 \\ \\ I_r = \dfrac{m_r \cdot L^2}{12} + m_r \cdot d^2 \\ \\ I_b = \dfrac{2}{5} \cdot m_b \cdot r_b^2 + m_b \cdot \left(\dfrac{L}{2} + d\right)^2 \end{cases}$$

As the shape of the cock is quite irregular, we don't know the exact expression of its moment of inertia $I_{c,CM}$ whereby the axis is located at the cock's center of mass. Anyway, for the purpose of our exercise, this term is irrelevant since it will vanish when we take the derivative of I_{tot}, because $I_{c,CM}$ does not depend on the distance d. Adding them all up, we find the following expression for I_{tot}:

$$I_{tot} = I_c + I_r + I_b = \left[I_{c,CM} + m_c \cdot \left(\frac{L}{2} - d\right)^2\right] + \left[\frac{m_r \cdot L^2}{12} + m_r \cdot d^2\right] + \left[\frac{2}{5} \cdot m_b \cdot r_b^2 + m_b \cdot \left(\frac{L}{2} + d\right)^2\right]$$

Taking the derivative of I_{tot} with respect to the distance d and setting it equal to zero allows us to find the distance d from the axis for which I_{tot} is minimal:

$$\frac{dI_{tot}}{d(d)} = 0 \quad \Leftrightarrow \quad -2 \cdot m_c \cdot \left(\frac{L}{2} - d\right) + 2 \cdot m_r \cdot d + 2 \cdot m_b \cdot \left(\frac{L}{2} + d\right) = 0$$

$$\Leftrightarrow \quad d = \frac{(m_c - m_b) \cdot L}{2 \cdot (m_c + m_r + m_b)} = \frac{(4.67 - 2.15) \cdot 0.85}{2 \cdot (4.67 + 1.12 + 2.15)} = 13.5 \text{ cm}$$

If we now calculate the position of the center of mass x_c according to the definition of the center of mass, we should obtain the same result! With the origin of our coordinate system located at the center of mass, we indeed find the same expression for the distance d:

$$x_c = 0 = \frac{-m_c \cdot \left(\frac{L}{2} - d\right) + m_r \cdot d + m_b \cdot \left(\frac{L}{2} + d\right)}{m_c + m_r + m_b} \quad \Leftrightarrow \quad d = \frac{(m_c - m_b) \cdot L}{2 \cdot (m_c + m_r + m_b)}$$

As a final check, we can verify whether the net torque in the xy-plane is indeed zero when you put the axis at the weathercock's center of mass:

$$\begin{aligned}\tau_{net} &= m_c \cdot g \cdot \left(\frac{L}{2} - d\right) - m_r \cdot g \cdot d - m_b \cdot g \cdot \left(\frac{L}{2} + d\right)\\ &= 4.67 \cdot 9.81 \cdot \left(\frac{0.85}{2} - 0.135\right) - 1.12 \cdot 9.81 \cdot 0.135 - 2.15 \cdot 9.81 \cdot \left(\frac{0.85}{2} + 0.135\right)\\ &= 0 \text{ N·m}\end{aligned}$$

So, Dad, if you put the rotation axis 13.5 cm to the left of the middle of the iron rod, it should hopefully all work out.

See you this weekend!
Spiro

Exercise 7

Problem Statement

Emily has been living her whole life in Vallée de l'Ernz, Luxembourg, where she owns 45 hectares of farmland on which she has built her house and a large barn, used to store grain and straw bales (m_b = 20.0 kg per bale). The bales are kept in a separate section 6.00 m above the ground and Emily uses an iron platform (M_{pl} = 66.3 kg) attached to a large pulley system to lower the bales to the ground floor. When she puts two bales on the platform, it takes t = 2.55 s for the platform to reach the ground. Directly above the platform, a large wheel, which has a diameter of d_{w1} = 91.8 cm and a total mass of m_{w1} = 21.4 kg, which includes the mass of six steel spokes (m_{s1} = 1.45 kg per spoke), guides the cable that is holding the platform towards a disk-shaped pulley with a mass and radius of m_d = 1.50 kg and r_d = 11.6 cm, respectively. The cable then continues straight downwards, round a second wheel, which has a diameter of d_{w2} = 63.4 cm, a w = 4.50 cm wide edge, and a total mass of m_{w2} = 13.6 kg (this includes the mass m_{s2} of four iron spokes), and finally back up vertically where it is attached to the ceiling. Serving as a counterweight of this pulley system, a granite block (M_{gb} = 129 kg) is hanging from the center of the second wheel. Emily notices that one of the iron spokes shows a crack, and she replaces it straight away, otherwise the entire pulley system can come crashing down. If you know that the frictional torque caused by the bearings of the first wheel, the disk, and the second wheel are equal to τ_{f1} = 7.23 N·m, τ_{f2} = 2.19 N·m, and τ_{f3} = 5.41 N·m, respectively, what is the mass of the iron spoke that Emily has just fixed? Assume that the cable does not slip when the pulley system is in motion and that counterclockwise is the positive direction of rotation.

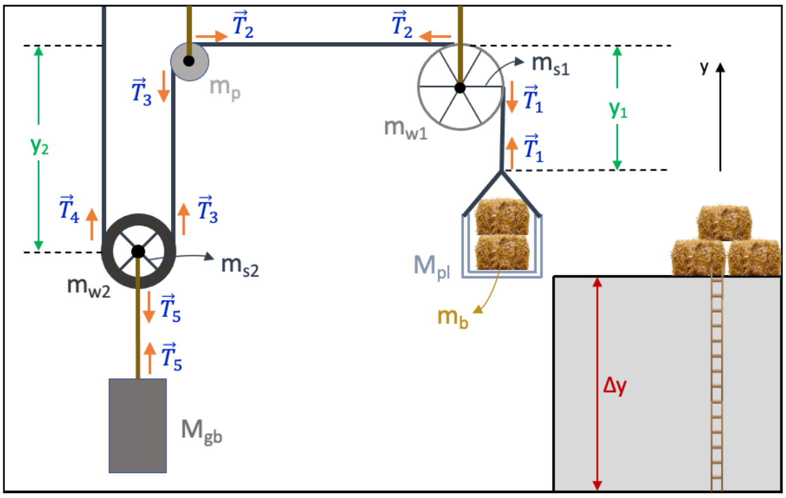

Figure 7

Solution

If we choose upwards as the positive y-direction and if the ground level is equal to $y = 0$ m, then the following equation of motion provides the acceleration a_{pl} of the platform carrying two bales of straw:

$$y = y_0 + \frac{a_{pl}}{2} \cdot t^2 \quad \Leftrightarrow \quad a_{pl} = \frac{2 \cdot (y - y_0)}{t^2} = \frac{2 \cdot (0 - 6.00)}{2.55^2} = -1.85 \text{ m/s}^2$$

As the acceleration a_{pl} is negative, the platform descends, and the granite block goes upwards. We can find the acceleration a_{gb} of the block by considering a constraint related to the length of the cable. Given that the total length of the cable does not change, Fig. 7 then tells us that the segment "$y_1 + 2 \cdot y_2$" also has to remain constant. Put another way, $y_1 + 2 \cdot y_2 = c$, with c some constant. Taking the second derivative with respect to time of this constraint gives us the following relation between the acceleration a_{pl} and a_{gb}:

$$\frac{d^2 y_1}{dt^2} + 2 \cdot \frac{d^2 y_2}{dt^2} = \frac{d^2 c}{dt^2} \quad \Leftrightarrow \quad a_{pl} + 2 \cdot a_{gb} = 0 \quad \Leftrightarrow \quad a_{gb} = -\frac{a_{pl}}{2} = -\frac{(-1.85)}{2} = 0.923 \text{ m/s}^2$$

In a next step, we apply Newton's second law to both the granite block and the loaded platform (regarding the platform, keep in mind that the total mass M_P is equal to $M_P = M_{pl} + 2 \cdot m_b = 66.3 + 2 \cdot 20.0 = 106$ kg), which gives us the following two equations:

$$\begin{cases} \text{Platform:} \qquad -M_P \cdot g + T_1 = M_P \cdot a_{pl} \\ \\ \text{Block:} \qquad -M_{gb} \cdot g + T_5 = M_{gb} \cdot a_{gb} \end{cases}$$

We now want to apply Newton's second law for rotation to the two wheels and the disk. We first write the moment of inertia of each of the rotating objects. Regarding the first wheel, the outer thin ring has a mass of $m_{otr} = m_{w1} - 6 \cdot m_{s1} = 21.4 - 6 \cdot 1.45 = 12.7$ kg and radius equal to $r_{w1} = \frac{d_{w1}}{2} = \frac{91.8}{2} = 45.9$ cm. With respect to the second wheel, the outer thick edge has a mass of $m_{ote} = m_{w2} - 4 \cdot m_{s2}$ and an outer and inner radius equal to $r_{w2o} = \frac{d_{w2}}{2} = \frac{63.4}{2} = 31.7$ cm and $r_{w2i} = r_{w2o} - w = 31.7 - 4.50 = 27.2$ cm, respectively. The moments of inertia then become:

$$\begin{cases} \text{First wheel:} \quad I_{w1} = m_{otr} \cdot r_{w1}^2 + 6 \cdot \dfrac{m_{s1} \cdot r_{w1}^2}{3} = 12.7 \cdot 0.459^2 + 6 \cdot \dfrac{1.45 \cdot 0.459^2}{3} = 3.29 \text{ kg·m}^2 \\ \\ \text{Disk:} \quad I_d = \dfrac{m_d \cdot r_d^2}{2} = \dfrac{1.50 \cdot 0.116^2}{2} = 1.01 \times 10^{-2} \text{ kg·m}^2 \\ \\ \text{Second wheel:} \quad I_{w2} = \dfrac{(m_{w2} - 4 \cdot m_{s2})}{2} \cdot (r_{w2o}^2 + r_{w2i}^2) + 4 \cdot \dfrac{m_{s2} \cdot r_{w2i}^2}{3} \end{cases}$$

Given that the cable does not slip when passing over the pulleys, we know that the tangential acceleration is equal to the acceleration of the cable, which in turn is equal to the acceleration a_{pl} of the loaded platform. As the platform moves downwards, both the first wheel and the disk rotate clockwise, whereas the second wheel rotates counterclockwise. Keeping in mind that a_{pl} is negative and that counterclockwise is the positive direction of rotation, applying Newton's second law for rotation ($\tau_{net} = I \cdot \alpha$) to each of the pulleys, we obtain the following three relations (also remember that the tangential acceleration a is equal to $a = R \cdot \alpha$):

$$\begin{cases} \text{First wheel:} & -T_1 \cdot r_{w1} + T_2 \cdot r_{w1} + \tau_{f1} = I_{w1} \cdot \dfrac{a_{pl}}{r_{w1}} \\[2ex] \text{Disk:} & -T_2 \cdot r_d + T_3 \cdot r_d + \tau_{f2} = I_d \cdot \dfrac{a_{pl}}{r_d} \\[2ex] \text{Second wheel:} & T_3 \cdot r_{w2o} - T_4 \cdot r_{w2o} - \tau_{f3} = -I_{w2} \cdot \dfrac{a_{pl}}{r_{w2o}} \end{cases}$$

To find the mass m_{s2} of the spoke of the second wheel, we need to calculate the magnitude of the tension forces $\vec{T}_3$ and $\vec{T}_4$, so that we can use the above equation for the second wheel to obtain m_{s2}. For that, we must first determine the magnitude of the tension forces $\vec{T}_1$ and $\vec{T}_2$. The value of the former is calculated via the equation for the platform:

$$T_1 = M_P \cdot (a_{pl} + g) = 106 \cdot (-1.85 + 9.81) = 847 \text{ N}$$

Using the equation for the first wheel, we find the value for T_2:

$$T_2 = T_1 + I_{w1} \cdot \frac{a_{pl}}{r_{w1}^2} - \frac{\tau_{f1}}{r_{w1}} = 847 + 3.29 \cdot \frac{(-1.85)}{0.459^2} - \frac{7.23}{0.459} = 802 \text{ N}$$

The equation for the disk provides us with the value of T_3:

$$T_3 = T_2 + I_d \cdot \frac{a_{pl}}{r_d^2} - \frac{\tau_{f2}}{r_d} = 802 + 1.01 \times 10^{-2} \cdot \frac{(-1.85)}{0.116^2} - \frac{2.19}{0.116} = 782 \text{ N}$$

Before we can calculate the value of T_4, we determine the value of T_5 through the equation of the granite block:

$$-M_{gb} \cdot g + T_5 = M_{gb} \cdot a_{gb} \quad \Leftrightarrow \quad T_5 = M_{gb} \cdot (g + a_{gb}) = 129 \cdot (9.81 + 0.923) = 1,385 \text{ N}$$

The value of T_4 is then found when applying Newton's second law to the second wheel:

$$\begin{aligned} T_3 + T_4 - m_{w2} \cdot g - T_5 = m_{w2} \cdot a_{gb} \quad \Leftrightarrow \quad T_4 &= m_{w2} \cdot (g + a_{gb}) + T_5 - T_3 \\ &= 13.6 \cdot (9.81 + 0.923) + 1,385 - 782 \\ &= 749 \text{ N} \end{aligned}$$

The mass m_{s2} is then found through the equation of Newton's second law for rotation with respect to the second wheel:

$$
\begin{aligned}
& T_3 \cdot r_{w2o} - T_4 \cdot r_{w2o} - \tau_{f3} = -I_{w2} \cdot \frac{a_{pl}}{r_{w2o}} \\
\Leftrightarrow \quad & T_3 \cdot r_{w2o} - T_4 \cdot r_{w2o} - \tau_{f3} = -\left[\frac{(m_{w2} - 4 \cdot m_{s2})}{2} \cdot (r_{w2o}^2 + r_{w2i}^2) + 4 \cdot \frac{m_{s2} \cdot r_{w2i}^2}{3}\right] \cdot \frac{a_{pl}}{r_{w2o}} \\
\Leftrightarrow \quad & m_{s2} = \frac{\left([(T_3 - T_4) \cdot r_{w2o} - \tau_{f3}] \cdot \frac{r_{w2o}}{a_{pl}} + \frac{m_{w2}}{2} \cdot [r_{w2o}^2 + r_{w2i}^2]\right)}{2 \cdot \left(r_{w2o}^2 + \frac{r_{w2i}^2}{3}\right)} \\
& \quad\;\; = \frac{\left([(782 - 749) \cdot 0.317 - 5.41] \cdot \frac{0.317}{(-1.85)} + \frac{13.6}{2} \cdot [0.317^2 + 0.272^2]\right)}{2 \cdot \left(0.317^2 + \frac{0.272^2}{3}\right)} \\
& \quad\;\; = 1.24 \text{ kg}
\end{aligned}
$$

If Emily replaces the cracked iron spoke with a spoke of the same mass m_{s2} as the other three spokes, then the pulley system will continue to operate safely.

Exercise 8

Problem Statement

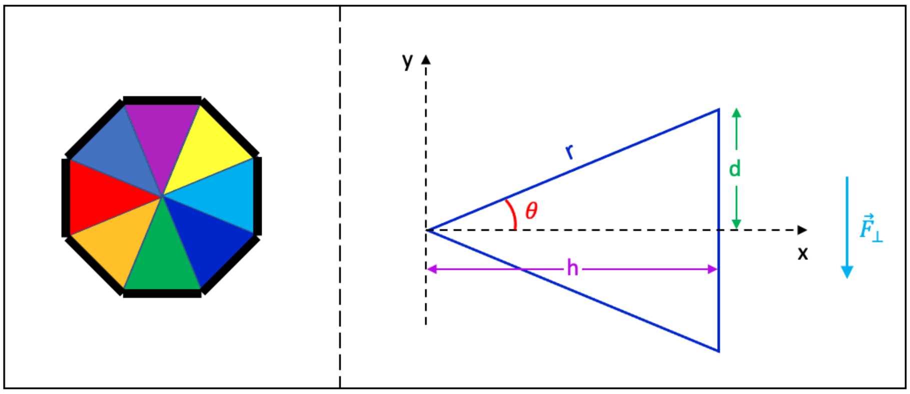

Figure 8

Gracjan finally put his two kids to bed and is now cleaning up. Earlier this afternoon, Gracjan threw a creative birthday party for his seven-year-old daughter Joanna and her friends whereby an art teacher was invited to show them all kinds of neat tricks with coloured cardboard paper. Gracjan promised Joanna to take her the next day to the beach in Ustka, Poland, which is a two-hour drive from their home in Szczecinek. When he sees during the cleanup a bunch of cut out isosceles triangles (with a mass of $m_t = 24.5$ g per piece) he gets the idea of creating a frisbee for Joanna that she can take with her to the beach tomorrow. Gracjan tapes eight triangles together to form an octagon (with a combined area of $A = 3,230$ cm^2) and to ensure stability during its flight, he glues a rubber band ($m_{rb} = 42.0$ g) onto the outer edge of each triangle. Gracjan then takes the octagonal frisbee for a test flight in the backyard and he is pleased with the result. When he grabs the frisbee at one edge (not at a vertex point), it takes $t = 0.620$ s to launch it, whereby it is given a frequency of 384 rpm. What is the magnitude of the perpendicular force $\vec{F}_\perp$ due to the friction between his hand and the frisbee that provides the frisbee with its initial spin?

Solution

Since an octagon consists of eight isosceles triangles, each triangle carves out an angle of $\frac{360°}{8} = 45.0°$ within the octagon. Therefore, the angle θ in Fig. 8 is equal to $\theta = \frac{45.0°}{2} = 22.5°$. Given that the area A of the octagon is equal to $A = 3,230$ cm^2, we find the radius r as follows:

$$A = 8 \cdot (h \cdot d) = 8 \cdot [(r \cdot \cos\theta) \cdot (r \cdot \sin\theta)] \quad \Leftrightarrow \quad r = \sqrt{\frac{A}{8 \cdot \sin\theta \cdot \cos\theta}}$$

$$= \sqrt{\frac{3,230}{8 \cdot \sin(22.5°) \cdot \cos(22.5°)}}$$

$$= 33.8 \text{ cm}$$

It follows then that $h = r \cdot \cos\theta = 33.8 \cdot \cos(22.5°) = 31.2$ cm and $d = r \cdot \sin\theta = 33.8 \cdot \sin(22.5°) = 12.9$

cm. Next, let us find an expression for the moment of inertia I_{ht} of a half triangle (with a mass equal to $m_{ht} = \frac{m_t}{2} = \frac{24.5}{2} = 12.3$ g) rotating around the origin of our coordinate system. The limits of integration run from 0 to h (for the x-direction) and from 0 to y (for the y-direction), whereby y represents the equation of the straight line $y = \frac{d}{h} \cdot x$. The moment of inertia I_{ht} is then found as follows (remember that the area density ρ is equal to $\rho = \frac{dm}{dx \cdot dy}$):

$$
\begin{aligned}
I_{ht} = \int R^2 \cdot dm = \rho \cdot \int_0^h dx' \cdot \int_0^{\frac{d}{h}\cdot x} (x'^2 + y'^2) \cdot dy' &= \rho \cdot \int_0^h dx' \cdot \left[\left(x'^2 \cdot y + \frac{y^3}{3} \right) \Big|_0^{\frac{d}{h}\cdot x} \right] \\
&= \rho \cdot \int_0^h \left(x'^2 \cdot \frac{d}{h} \cdot x' + \frac{d^3}{3 \cdot h^3} \cdot x'^3 \right) \cdot dx' \\
&= \rho \cdot \frac{d}{h^3} \cdot \left(h^2 + \frac{d^2}{3} \right) \cdot \int_0^h x'^3 \cdot dx' \\
&= \rho \cdot \frac{d}{h^3} \cdot \left(h^2 + \frac{d^2}{3} \right) \cdot \left[\left(\frac{x^4}{4} \Big|_0^h \right) \right] \\
&= \rho \cdot \frac{d \cdot h}{4} \cdot \left(h^2 + \frac{d^2}{3} \right) \\
&= \left[\frac{2 \cdot m_{ht}}{h \cdot d} \right] \cdot \frac{d \cdot h}{4} \cdot \left(h^2 + \frac{d^2}{3} \right) \\
&= \frac{m_{ht}}{2} \cdot \left(h^2 + \frac{d^2}{3} \right)
\end{aligned}
$$

In a next step, we determine an expression for the moment of inertia I_{hrb} of one half of a rubber band that rotates around the origin of the coordinate system at a distance h from that origin. With the help of the parallel-axis theorem, we obtain:

$$
I_{hrb} = \frac{\left(\frac{m_{rb}}{2}\right) \cdot d^2}{3} + \left(\frac{m_{rb}}{2}\right) \cdot h^2 = \frac{m_{rb}}{2} \cdot \left(\frac{d^2}{3} + h^2 \right)
$$

The total moment of inertia I_f of the octagonal frisbee is then equal to equal to:

$$
\begin{aligned}
I_f = 16 \cdot (I_{ht} + I_{hrb}) &= 16 \cdot \left[\frac{m_{ht}}{2} \cdot \left(h^2 + \frac{d^2}{3} \right) + \frac{m_{rb}}{2} \cdot \left(\frac{d^2}{3} + h^2 \right) \right] \\
&= 8 \cdot (m_{ht} + m_{rb}) \cdot \left(h^2 + \frac{d^2}{3} \right)
\end{aligned}
$$

$$= 8 \cdot (0.0123 + 0.0420) \cdot \left(0.312^2 + \frac{0.129^2}{3}\right)$$

$$= 4.47 \times 10^{-2}\ \text{kg·m}^2$$

Given that it takes $t = 0.620$ s to provide the frisbee with its initial spin, we calculate its angular acceleration α as follows (whereby the initial angular velocity ω_0 is equal to $\omega_0 = 0$ rad/s):

$$\omega = \omega_0 + \alpha \cdot t \quad \Leftrightarrow \quad \alpha = \frac{\omega - \omega_0}{t} = \frac{\left[2 \cdot \pi \frac{384}{60}\right] - 0}{0.620} = 64.9\ \text{rad/s}^2$$

In a final step, bearing in mind that Gracjan holds the frisbee at an outer edge, which is located at a distance h from the axis of rotation, the magnitude of the perpendicular force $\vec{F}_\perp$ that Gracjan gives to the frisbee is then found with the assistance of Newton's second law for rotation (if we assume that Gracjan throws the frisbee with his right hand and that anticlockwise is the positive direction of rotation, then the net torque τ_{net} will be negative):

$$\vec{\tau}_{net} = I \cdot \vec{\alpha} \quad \Leftrightarrow \quad -F_\perp \cdot h \cdot \vec{i}_z = -I_f \cdot \alpha \cdot \vec{i}_z \quad \Leftrightarrow \quad F_\perp = \frac{I_f \cdot \alpha}{h} = \frac{4.47 \times 10^{-2} \cdot 64.9}{0.312} = 9.29\ \text{N}$$

Exercise 9

Problem Statement

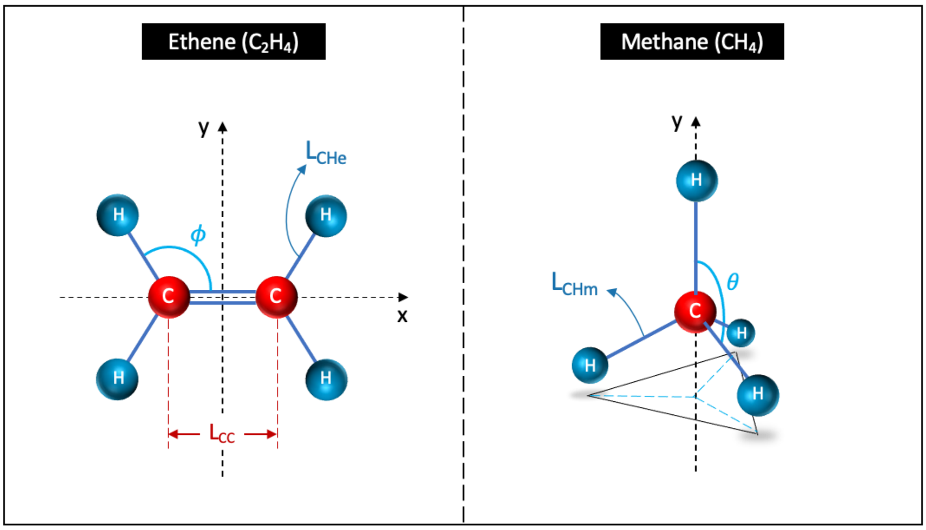

Figure 9

In the field of chemistry, organic compounds are defined as compounds that contain the chemical element carbon (C). Hydrocarbons are an example of organic compounds and only consist of the elements carbon and hydrogen (H). The hydrocarbons can be further subdivided into the homologous series alkanes and alkenes, whereby the latter are uniquely characterized by a double carbon bond (which the alkanes do not possess). An example of an alkane and an alkene is methane (CH_4) and ethene (CH_2=CH_2 or C_2H_4), respectively. The methane compound has a tetrahedral structure, which means that the angle $\theta = \angle HCH$ between two H atoms equals $\theta = 109.47°$. The length L_{CHm} of the C-H bond in methane is measured to be $L_{CHm} = 109.4$ pm. The six atoms of the compound ethene are coplanar and due to the presence of the double carbon bond, the length L_{CHe} of the C-H bond is slightly shorter relative to methane, i.e., $L_{CHe} = 108.7$ pm. The angle $\phi = \angle HCC$ between an H atom and the double bond is equal to $\phi = 121.7°$. If we let the compound ethene rotate about the axis that runs right through the middle of the double C-C bond and lies within the plane of the compound and if the methane compound rotates about the axis that connects an H atom with the central C atom, then the moment of inertia I_e of ethene is larger than that of methane (I_m) by a factor of 5.276. What is then the length L_{CC} of the double carbon bond in the compound ethene? Remember that 1 picometer is equal to 1 pm $= 10^{-12}$ m, that the mass of an H and a C atom is equal to $m_H = 1.00797$ amu and $m_C = 12.011$ amu, respectively, and that 1 atomic mass unit is equivalent to 1 amu $= 1.66054 \times 10^{-27}$ kg.

Solution

Let us in a first instance express the masses of the atoms in terms of kilograms:

$$\begin{cases} m_H = 1.00797 \cdot 1.66054 \times 10^{-27} = 1.674 \times 10^{-27} \text{ kg} \\ \\ m_C = 12.011 \cdot 1.66054 \times 10^{-27} = 1.994 \times 10^{-26} \text{ kg} \end{cases}$$

In order to find the expression for the moment of inertia of the compounds ethene and methane, we can see from Fig. 9 that both C_2H_4 and CH_4 rotate about the y-axis. With respect to the moment of inertia I_m of CH_4, the perpendicular distance from an H atom to the rotation axis is equal to $L_{CHm} \cdot \sin\theta$, given that the sine of the angle θ is equal to the sine of its supplementary angle $180° - \theta$. Since neither the H atom along the y-axis or the C atom contribute to I_m, we can write:

$$I_m = 3 \cdot m_H \cdot (L_{CHm} \cdot \sin\theta)^2 = 3 \cdot 1.674 \times 10^{-27} \cdot [1.094 \times 10^{-10} \cdot \sin(109.47°)]^2 = 5.342 \times 10^{-47}\ \text{kg}{\cdot}\text{m}^2$$

Given that $I_e = 5.276 \cdot I_m$, we can calculate the value of ethene's moment of inertia I_e:

$$I_e = 5.276 \cdot I_m = 5.276 \cdot 5.342 \times 10^{-47} = 2.818 \times 10^{-46}\ \text{kg}{\cdot}\text{m}^2$$

In a next step, we write down the expression for I_e. The C atom is at a distance $\frac{L_{CC}}{2}$ away from the rotation axis, whereas the four H atoms find themselves at a distance $\frac{L_{CC}}{2} - L_{CHe} \cdot \cos\phi$ from the axis. Since the angle ϕ is greater than 90°, its cosine is negative, which explains the minus sign in the latter expression for the distance. We then obtain the following expression:

$$I_e = 2 \cdot m_C \cdot \left(\frac{L_{CC}}{2}\right)^2 + 4 \cdot m_H \cdot \left(\frac{L_{CC}}{2} - L_{CHe} \cdot \cos\phi\right)^2$$

$$\Leftrightarrow \left[1 + \frac{m_C}{2 \cdot m_H}\right] \cdot L_{CC}^2 - [4 \cdot L_{CHe} \cdot \cos\phi] \cdot L_{CC} + \left[4 \cdot L_{CHe}^2 \cdot \cos^2\phi - \frac{I_e}{m_H}\right] = 0$$

$$\Leftrightarrow \left[1 + \frac{1.994 \times 10^{-26}}{2 \cdot 1.674 \times 10^{-27}}\right] \cdot L_{CC}^2 - \left[4 \cdot 1.087 \times 10^{-10} \cdot \cos(121.7°)\right] \cdot L_{CC} +$$

$$\left[4 \cdot (1.087 \times 10^{-10})^2 \cdot \cos^2(121.7°) - \frac{2.818 \times 10^{-46}}{1.674 \times 10^{-27}}\right] = 0$$

The physically sensible (i.e., the value of the distance must be greater than or equal to zero) solution to the above quadratic equation is equal to $L_{CC} = 1.339 \times 10^{-10}$ m or 133.9 pm.

Exercise 10

Problem Statement

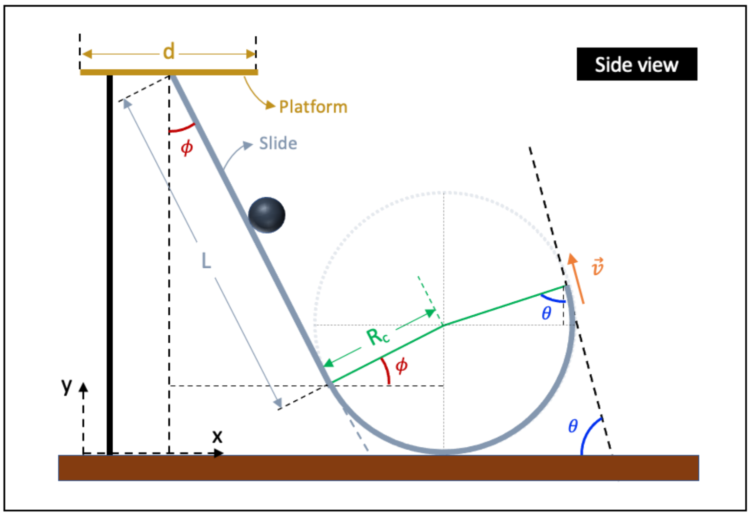

Figure 10

Roslyn is strolling about the different shops on Main Street in Ardara, Ireland, until her interest is suddenly drawn to a mysterious object in the display window of the shop All Kinds of Everything. In front of the object a label is placed with the words: "Perpetual Motion Machine". What Roslyn sees is a round platform with a diameter of $d = 9.50$ cm positioned at a certain height from the object's base, and from a hole in the middle of the platform a metal ball (with mass $m_b = 375$ g and radius $R_b = 9.50$ mm) falls down along a slide, which curls back up after touching the base, so that the ball eventually gets flung back onto the platform. Roslyn enters the shop, buys the intriguing device, and heads back home. It doesn't take Roslyn much time to figure out that a magnet hidden within the base is providing the required energy (and acceleration) to the ball to reach the platform—otherwise, the laws of thermodynamics would have been violated. When taking a closer look, Roslyn observes that the ball undergoes two types of motion on the slide: during the first part, the ball slides and slips ($\mu_k = 0.354$) along a $L = 20.2$ cm long straight segment tilted by $\phi = 27.4°$ and then follows a circular-shaped path (with a radius of $R_c = 6.50$ cm) whereby it now rolls without slipping, until the ball leaves the slide at the point where the tangent is making an angle of $\theta = 75.0°$ with the horizontal. If Roslyn switches off the magnet, how far, in terms of vertical distance, is the ball now removed from the platform while being at its highest point mid-air, if at all?

Solution

In a first step, we determine the speed v_1 of the metal ball as it reaches the end of the straight segment L. As the metal ball experiences kinetic friction while slipping along this first part of the slide, the magnitude of the kinetic friction force $\vec{F}_k$ is equal to $F_k = \mu_k \cdot (m_b \cdot g \cdot \sin\phi)$. Applying the work-energy theorem, we find the following speed v_1 (keep in mind that at its initial position on the slope, the ball's center of mass is positioned at a distance $R_b \cdot \sin\phi$ above the platform):

$$W_{tot,1} = W_{ext} + W_{c,1} \quad \Leftrightarrow \quad \Delta E_{k,1} = -F_k \cdot L - \Delta E_{p,1}$$

$$\Leftrightarrow \quad \frac{m_b \cdot v_1^2}{2} - \frac{m_b \cdot v_0^2}{2} = -\mu_k \cdot (m_b \cdot g \cdot \sin\phi) \cdot L -$$

$$[m_b \cdot g \cdot (R_c - (R_c - R_b) \cdot \sin\phi) - m_b \cdot g \cdot (R_b \cdot \sin\phi + L \cdot \cos\phi + R_c \cdot (1 - \sin\phi))]$$

$$\Leftrightarrow \frac{m_b \cdot v_1^2}{2} - \frac{m_b \cdot 0^2}{2} = -\mu_k \cdot (m_b \cdot g \cdot \sin\phi) \cdot L + m_b \cdot g \cdot L \cdot \cos\phi$$

$$\Leftrightarrow v_1 = \sqrt{2 \cdot g \cdot L \cdot (\cos\phi - \mu_k \cdot \sin\phi)}$$

$$= \sqrt{2 \cdot 9.81 \cdot 0.202 \cdot [\cos(27.4^\circ) - 0.354 \cdot \sin(27.4^\circ)]}$$

$$= 1.69 \text{ m/s}$$

Next, we find the speed v at which the metal ball leaves the slide. As we assume that the ball does not experience any friction during the circular-shaped second segment of the slide, we obtain the speed v with the help of the work-energy theorem (remember that the moment of inertia I_b of the metal ball and the angular velocity ω are equal to $I_b = \frac{2}{5} \cdot m_b \cdot R_b^2$ and $\omega = \frac{v}{R_b}$, respectively):

$$W_{tot,2} = W_{c,2} \quad \Leftrightarrow \quad \Delta E_{k,2} = -\Delta E_{p,2}$$

$$\Leftrightarrow \left(\frac{m_b \cdot v^2}{2} + \frac{I_b \cdot \omega^2}{2}\right) - \left(\frac{m_b \cdot v_1^2}{2} + \frac{I_b \cdot \omega_1^2}{2}\right) =$$

$$-\left[m_b \cdot g \cdot (R_c + (R_c - R_b) \cdot \cos\theta) - m_b \cdot g \cdot (R_c - (R_c - R_b) \cdot \sin\phi)\right]$$

$$\Leftrightarrow \left(\frac{m_b \cdot v^2}{2} + \frac{1}{5} \cdot m_b \cdot R_b^2 \cdot \frac{v^2}{R_b^2}\right) - \left(\frac{m_b \cdot v_1^2}{2} + \frac{1}{5} \cdot m_b \cdot R_b^2 \cdot \frac{v_1^2}{R_b^2}\right) =$$

$$m_b \cdot g \cdot (R_b - R_c) \cdot (\cos\theta + \sin\phi)$$

$$\Leftrightarrow \frac{7}{10} \cdot m_b \cdot v^2 = \frac{7}{10} \cdot m_b \cdot v_1^2 + m_b \cdot g \cdot (R_b - R_c) \cdot (\cos\theta + \sin\phi)$$

$$\Leftrightarrow v = \sqrt{v_1^2 + \frac{10}{7} \cdot g \cdot (R_b - R_c) \cdot (\cos\theta + \sin\phi)}$$

$$= \sqrt{1.69^2 + \frac{10}{7} \cdot 9.81 \cdot (9.50 \times 10^{-3} - 6.50 \times 10^{-2}) \cdot [\cos(75.0^\circ) + \sin(27.4^\circ)]}$$

$$= 1.52 \text{ m/s}$$

The maximum height h_{max} that the metal ball attains after it has left the slide is calculated as follows (remember that at h_{max} the corresponding speed v_{max} is zero):

$$v_{max}^2 - (v \cdot \sin\theta)^2 = 2 \cdot (-g) \cdot h_{max} \quad \Leftrightarrow \quad h_{max} = \frac{(v \cdot \sin\theta)^2}{2 \cdot g} = \frac{[1.52 \cdot \sin(75.0^\circ)]^2}{2 \cdot 9.81} = 11.0 \text{ cm}$$

The minimum vertical distance h_{min} that the metal ball should travel through the air if it wishes to end up back on the platform (which is only possible if Roslyn switches on the magnet) is expressed as follows:

$$
\begin{aligned}
h_{min} &= R_b + L \cdot \cos\phi - [R_c \cdot \sin\phi + (R_c - R_b) \cdot \cos\theta] \\
&= L \cdot \cos\phi + R_b \cdot (1 + \cos\theta) - R_c \cdot (\cos\theta + \sin\phi) \\
&= 0.202 \cdot \cos(27.4°) + 9.50 \times 10^{-3} \cdot [1 + \cos(75.0°)] - 6.50 \times 10^{-2} \cdot [\cos(75.0°) + \sin(27.4°)] \\
&= 14.5 \text{ cm}
\end{aligned}
$$

The metal ball would need another $h_{min} - h_{max} = 14.5 - 11.0 = 3.45$ cm to reach the platform. Bear in mind that in this exercise we assumed no friction during the circular-shaped path. In a real-life example, however, the metal ball experiences friction with the slide during the entire path and would most likely not even have sufficient energy to leave the slide.

Exercise 11

Problem Statement

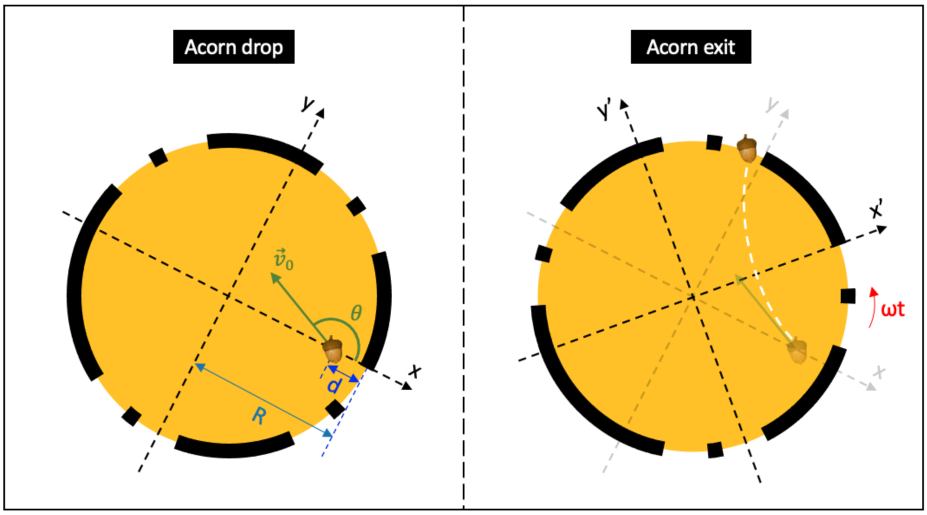

Figure 11

Liam and Robert are having fun with their kids at the Greenview Playground in Edmonton, Canada, on a chilly, yet sunny day. After three hours of entertainment on the hopping stools, the monkey bars, the various slides, the chain ladders, the rope webs, the climbing forest, the swings, the spinners, and the climbing dome, the kids are now resting on the merry-go-round. That is, until their fathers suggest they go and get an ice-cream. With unprecedented excitement and loud cheering, they dash off to collect their well-deserved afternoon snack. As they jump off the merry-go-round, which has a radius of $R = 2.55$ m, they leave it spinning counterclockwise at a constant angular velocity of $\omega = 0.455$ rad/s. Because of all the screaming, the squirrel, who was enjoying her own snack up in the tree, is startled and drops her acorn ($m_a = 105$ g), which lands on the merry-go-round $d = 55.0$ cm from the edge and it rolls with an initial speed of $v_0 = 1.24$ m/s at an angle of $\theta = 156°$ with the radial line segment that intersects with the landing spot of the acorn. (1) How long does the acorn stay on the merry-go-round before flying off of it? (2) What are the coordinates of the acorn's point of exit (as seen from the rotating reference frame)? Ignore any kind of kinetic friction for this problem.

Solution

(1) Since the acorn is moving within a non-inertial, rotating frame of reference, the forces acting upon the acorn are the normal force $\vec{F}_N$, the gravitational force $\vec{F}_G$, the centrifugal force $\vec{F}_{CF}$, and the Coriolis force $\vec{F}_C$. As we will consider the acorn's motion in the xy-plane, $\vec{F}_N$ and $\vec{F}_G$ are irrelevant to our problem. The acorn's horizontal motion is thus affected by the following net force $\vec{F}_{net}$ (with $\vec{r} = x \cdot \vec{i}_{x'} + y \cdot \vec{i}_{y'}$ representing the acorn's position vector, $\vec{v} = v_x \cdot \vec{i}_{x'} + v_y \cdot \vec{i}_{y'}$ its velocity vector, $\vec{a} = a_x \cdot \vec{i}_{x'} + a_y \cdot \vec{i}_{y'}$ its acceleration vector, and $\vec{\omega} = \omega \cdot \vec{i}_{z'}$ its angular velocity vector, respectively):

$$\vec{F}_{net} = \vec{F}_{CF} + \vec{F}_C \quad \Leftrightarrow \quad m_a \cdot \vec{a} = [m_a \cdot \vec{\omega} \times (\vec{r} \times \vec{\omega})] + [2 \cdot m_a \cdot (\vec{v} \times \vec{\omega})]$$

As the angular velocity vector $\vec{\omega}$ points upwards (out of your screen) along the z-axis, we can deduce from the cross products that the centrifugal force $\vec{F}_{CF}$ acts radially outwards and the Coriolis force

$\vec{F}_C$, when looking into the direction of the velocity vector $\vec{v}$, to the right. If we work out the cross products, we obtain the following equation:

$$\vec{a} = \left[\omega^2 \cdot x \cdot \vec{i}_{x'} + \omega^2 \cdot y \cdot \vec{i}_{y'}\right] + 2 \cdot \left[\omega \cdot v_y \cdot \vec{i}_{x'} - \omega \cdot v_x \cdot \vec{i}_{y'}\right]$$

In component form, this becomes:

$$\begin{cases} a_x = \omega^2 \cdot x + 2 \cdot \omega \cdot v_y \\ \\ a_y = \omega^2 \cdot y - 2 \cdot \omega \cdot v_x \end{cases}$$

In order to obtain an expression for the time variable t, we have to solve these two differential equations. To simplify our calculations, let us introduce a new, complex variable $r_c = x + i \cdot y$, which represents the position vector in the complex field, whereby $i^2 = -1$ (do not confuse i with the unit vectors $\vec{i}_{x'}$ and $\vec{i}_{y'}$). Taking the first and second derivative of r_c, we then obtain the respective complex equivalents of the velocity and acceleration vectors, i.e., $v_c = v_x + i \cdot v_y$ and $a_c = a_x + i \cdot a_y$. Using the above two equations, we now write:

$$\begin{aligned} a_c = a_x + i \cdot a_y &= \left[\omega^2 \cdot x + 2 \cdot \omega \cdot v_y\right] + i \cdot \left[\omega^2 \cdot y - 2 \cdot \omega \cdot v_x\right] \\ &= \omega^2 \cdot (x + i \cdot y) - 2 \cdot i \cdot \omega \cdot (v_x + i \cdot v_y) \\ &= \omega^2 \cdot r_c - 2 \cdot i \cdot \omega \cdot v_c \end{aligned}$$

$$\Leftrightarrow \quad \frac{d^2 r_c}{dt^2} + 2 \cdot i \cdot \omega \cdot \frac{dr_c}{dt} - \omega^2 \cdot r_c = 0$$

To solve this differential equation, let us try the solution $r_c = e^{\lambda \cdot t}$, which we insert into the above equation:

$$\frac{d^2}{dt^2}\left(e^{\lambda \cdot t}\right) + 2 \cdot i \cdot \omega \cdot \frac{d}{dt}\left(e^{\lambda \cdot t}\right) - \omega^2 \cdot \left(e^{\lambda \cdot t}\right) = 0$$

$$\Leftrightarrow \quad \lambda^2 + 2 \cdot i \cdot \omega \cdot \lambda - \omega^2 = 0$$

$$\Leftrightarrow \quad \lambda = \frac{1}{2} \cdot \left[\, -2 \cdot i \cdot \omega \pm \sqrt{(2 \cdot i \cdot \omega)^2 - 4 \cdot 1 \cdot (-\omega^2)}\, \right] = -i \cdot \omega$$

Therefore, our two solutions have the same form: $r_c = e^{-i \cdot \omega \cdot t}$. A general solution to our differential equation consists of a linear combination of these two solutions, which we write as $r_c =$

$c_1 \cdot e^{-i\cdot\omega\cdot t} + c_2 \cdot t \cdot e^{-i\cdot\omega\cdot t}$, with c_1 and c_2 two constants. Notice that we multiplied the second term by "t", which we have to do if we wish to avoid non-sensical answers when implementing the two initial conditions of our problem, i.e., the acorn's (complex) initial position $r_{c,0} = L + i \cdot 0 = L$ (with L the initial position of the acorn equal to $L = R - d = 2.55 - 0.550 = 2.00$ m) and its (complex) initial velocity $v_{c,0} = (v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta)$. To find the value of the two constants c_1 and c_2, we apply the two initial conditions to our general solution at $t = 0$ s:

$$\begin{cases} r_{c,0} = L \quad \Leftrightarrow \quad c_1 \cdot e^{-i\cdot\omega\cdot 0} + c_2 \cdot 0 \cdot e^{-i\cdot\omega\cdot 0} = L \\ \qquad\qquad \Leftrightarrow \quad c_1 = L \\ v_{c,0} = (v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta) \\ \quad \Leftrightarrow \quad \dfrac{d}{dt}\left(c_1 \cdot e^{-i\cdot\omega\cdot t} + c_2 \cdot t \cdot e^{-i\cdot\omega\cdot t}\right)\Big|_{t=0} = (v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta) \\ \quad \Leftrightarrow \quad \left[-c_1 \cdot i \cdot \omega \cdot e^{-i\cdot\omega\cdot 0} + c_2 \cdot e^{-i\cdot\omega\cdot 0} - c_2 \cdot i \cdot \omega \cdot 0 \cdot e^{-i\cdot\omega\cdot 0}\right] = (v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta) \\ \quad \Leftrightarrow \quad c_2 = (v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta + L \cdot \omega) \end{cases}$$

The final general solution has then the following complex form (remember Euler's formula, whereby $e^{\beta\cdot i} = \cos\beta + i \cdot \sin\beta$):

$$\begin{aligned} r_c &= c_1 \cdot e^{-i\cdot\omega\cdot t} + c_2 \cdot t \cdot e^{-i\cdot\omega\cdot t} \\ &= [L] \cdot e^{-i\cdot\omega\cdot t} + [(v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta + L \cdot \omega)] \cdot t \cdot e^{-i\cdot\omega\cdot t} \\ &= L \cdot [\cos(\omega t) - i \cdot \sin(\omega t)] + [(v_0 \cdot \cos\theta) + i \cdot (v_0 \cdot \sin\theta + L \cdot \omega)] \cdot t \cdot [\cos(\omega t) - i \cdot \sin(\omega t)] \\ &= (\ L \cdot \cos(\omega t) + [v_0 \cdot \cos\theta \cdot \cos(\omega t) + (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \sin(\omega t)] \cdot t\ ) \ - \\ &\qquad i \cdot (\ L \cdot \sin(\omega t) + [v_0 \cdot \cos\theta \cdot \sin(\omega t) - (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \cos(\omega t)] \cdot t\ ) \end{aligned}$$

Since r_c has the form $r_c = x + i \cdot y$, the x- and y-coordinate of the acorn's position vector $\vec{r}$ are the following:

$$\begin{cases} x(t) = L \cdot \cos(\omega t) + [v_0 \cdot \cos\theta \cdot \cos(\omega t) + (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \sin(\omega t)] \cdot t \\ y(t) = -L \cdot \sin(\omega t) - [v_0 \cdot \cos\theta \cdot \sin(\omega t) - (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \cos(\omega t)] \cdot t \end{cases}$$

The time the acorn stays on the merry-go-round is equal to the time during which the distance of the acorn from the origin has become equal to the radius R of the merry-go-round. In a next moment, the acorn will fall off of the merry-go-round, as it will be at a distance from the origin that is greater than R. This condition is mathematically translated in the following way:

$$x^2 + y^2 = R^2$$

$$\Leftrightarrow \quad \left(L \cdot \cos(\omega t) + [v_0 \cdot \cos\theta \cdot \cos(\omega t) + (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \sin(\omega t)] \cdot t \right)^2 +$$

$$\left(-L \cdot \sin(\omega t) - [v_0 \cdot \cos\theta \cdot \sin(\omega t) - (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \cos(\omega t)] \cdot t \right)^2 = R^2$$

$$\Leftrightarrow \quad \left[v_0^2 + 2 \cdot L \cdot v_0 \cdot \omega \cdot \sin\theta + L^2 \cdot \omega^2\right] \cdot t^2 + [2 \cdot L \cdot v_0 \cdot \cos\theta] \cdot t + \left[L^2 - R^2\right] = 0$$

$$\Leftrightarrow \quad \left[1.24^2 + 2 \cdot 2.00 \cdot 1.24 \cdot 0.455 \cdot \sin(156°) + 2.00^2 \cdot 0.455^2\right] \cdot t^2 +$$

$$[2 \cdot 2.00 \cdot 1.24 \cdot \cos(156°)] \cdot t + \left[2.00^2 - 2.55^2\right] = 0$$

The physically sensible (i.e., $t > 0$) solution to the above quadratic equation is equal to $t = 1.80$ s. From the moment that the acorn fell onto the merry-go-round until right before it falls off of it due to the inertial forces $\vec{F}_{CF}$ and $\vec{F}_C$, the acorn has spent a total time of $t = 1.80$ s on the merry-go-round.

(2) After having traveled along the path marked by the white dashed line in Fig. 11 for a total of $t = 1.80$ s, the acorn finds itself at the following coordinates (with respect to the rotating reference frame):

$$\begin{cases} x(t) = L \cdot \cos(\omega t) + [v_0 \cdot \cos\theta \cdot \cos(\omega t) + (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \sin(\omega t)] \cdot t \\ \quad = 2.00 \cdot \cos\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right) + \left[1.24 \cdot \cos(156°) \cdot \cos\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right) + \right. \\ \quad\quad \left. [1.24 \cdot \sin(156°) + 2.00 \cdot 0.455] \cdot \sin\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right)\right] \cdot 1.80 \\ \quad = 1.84 \text{ m} \\ y(t) = -L \cdot \sin(\omega t) - [v_0 \cdot \cos\theta \cdot \sin(\omega t) - (v_0 \cdot \sin\theta + L \cdot \omega) \cdot \cos(\omega t)] \cdot t \\ \quad = -2.00 \cdot \sin\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right) - \left[1.24 \cdot \cos(156°) \cdot \sin\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right) - \right. \\ \quad\quad \left. [1.24 \cdot \sin(156°) + 2.00 \cdot 0.455] \cdot \cos\left(0.455 \cdot 1.80 \cdot \dfrac{180°}{\pi}\right)\right] \cdot 1.80 \\ \quad = 1.77 \text{ m} \end{cases}$$

Exercise 12

Problem Statement

Two large asteroids (with mass $m_1 = 2.62 \times 10^{12}$ kg and m_2) are tumbling through the vast emptiness of space with a speed of $v_1 = 88,394$ km/h and $v_2 = 52,872$ km/h, respectively, whereby asteroid 2 is following a straight path at an angle of $\theta = 76.5°$ with respect to the straight path of asteroid 1 and both move in the same plane. The shape of both asteroids can be approximated by a cuboid, i.e., a rectangular prism, with length $l_1 = 38.1$ km, width $w_1 = 20.6$ km, and height $h_1 = 90.3$ km for asteroid 1 and length $l_2 = 155$ km, width $w_2 = 53.8$ km, and height $h_2 = 72.4$ km for asteroid 2. Asteroid 1 rotates every 6.50 hours clockwise around the z-axis through its center of mass, while asteroid 2, which spins counterclockwise around an axis that runs parallel to the z-axis and along one of the four edges, requires 22.3 hours to complete one revolution. Moreover, asteroid 2 has a hole of cuboidal shape at its center that stretches across the entire height h_2 and whereby its length and width are about one fifth of that of the asteroid. At one point, the two asteroids collide and merge their mass into one spherical-like object, which has a radius of $R = 53,750$ m and rotates about an axis running through its center of mass. If you know that the consolidated asteroid is headed into the direction that makes an angle of $\alpha = 58.9°$ relative to the original path of the first asteroid, (1) what was the mass m_2 of asteroid 2? (2) At which speed is the newly assembled asteroid hurtling through space? (3) How long does it take the spherical asteroid to spin just once around its rotation axis? Assume that no mass is lost during the collision and transformation of the asteroids.

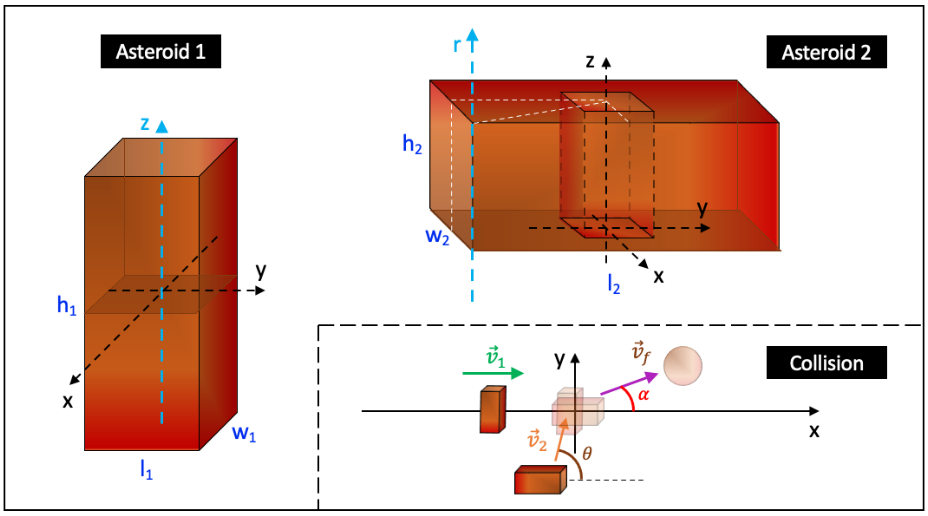

Figure 12

Solution

(1) In the vast emptiness of space, the only force acting upon each asteroid is the gravitational force $\vec{F}_G$ manifested by the presence of the other asteroid. However, in the isolated system "asteroid 1 plus asteroid 2", these two forces cancel each other out, so that the total linear momentum $\vec{p}_{tot}$ is conserved. We can therefore write:

$$\begin{cases} x : (m_1 \cdot v_1) + (m_2 \cdot v_2 \cdot \cos\theta) = (m_1 + m_2) \cdot v_f \cdot \cos\alpha \\ \\ y : m_2 \cdot v_2 \cdot \sin\theta = (m_1 + m_2) \cdot v_f \cdot \sin\alpha \end{cases}$$

If we formulate an expression for the final speed v_f based on the equation of the y-direction and plug it into the equation of the x-direction, we find the mass m_2:

$$m_2 \cdot v_2 \cdot \sin\theta = (m_1 + m_2) \cdot v_f \cdot \sin\alpha \quad \Leftrightarrow \quad v_f = \frac{m_2 \cdot v_2 \cdot \sin\theta}{(m_1 + m_2) \cdot \sin\alpha}$$

$$\Rightarrow \quad (m_1 \cdot v_1) + (m_2 \cdot v_2 \cdot \cos\theta) = (m_1 + m_2) \cdot \left[\frac{m_2 \cdot v_2 \cdot \sin\theta}{(m_1 + m_2) \cdot \sin\alpha}\right] \cdot \cos\alpha$$

$$\Leftrightarrow \quad m_2 = \frac{m_1 \cdot v_1}{v_2 \cdot (\cot\alpha \cdot \sin\theta - \cos\theta)} = \frac{2.62 \times 10^{12} \cdot 2.46 \times 10^4}{1.47 \times 10^4 \cdot [\cot(58.9^\circ) \cdot \sin(76.5^\circ) - \cos(76.5^\circ)]}$$

$$= 1.24 \times 10^{13} \text{ kg}$$

(2) The final speed v_f of the spherical-like asteroid post-collision is then calculated with the assistance of the expression obtained in part (1):

$$v_f = \frac{m_2 \cdot v_2 \cdot \sin\theta}{(m_1 + m_2) \cdot \sin\alpha} = \frac{1.24 \times 10^{13} \cdot 1.47 \times 10^4 \cdot \sin(76.5^\circ)}{(2.62 \times 10^{12} + 1.24 \times 10^{13}) \cdot \sin(58.9^\circ)} = 1.38 \times 10^4 \text{ m/s or } 49,600 \text{ km/h}$$

(3) Only at the moment when the two asteroids come together and start their transformation process is the net torque of the system "asteroid 1 plus asteroid 2" zero—before that, there is a non-zero net torque due to the gravitational force $\vec{F}_G$ exerted upon asteroid 2 by asteroid 1, because the rotation axis does not run through its center of mass. As a result, the total angular momentum $\vec{L}_{tot}$ of the system is conserved only at the start of their transformation. Given that asteroid 1 spins clockwise and asteroid 2 counterclockwise, we write the following equation:

$$\vec{L}_{tot,i} = \vec{L}_{tot,f} \quad \Leftrightarrow \quad (-I_1 \cdot \omega_1) + (I_2 \cdot \omega_2) = I_f \cdot \omega_f$$

Let us first determine the different moments of inertia. With regard to asteroid 1, the origin of the coordinate system sits at its center of mass and the asteroid rotates around the z-axis. Therefore, asteroid 1 rotates within the xy-plane, so that only the width w_1 and the length l_1 impact its moment of inertia I_1, which can be calculated as follows:

$$I_1 = \frac{m_1}{12} \cdot \left(w_1^2 + l_1^2\right) = \frac{2.62 \times 10^{12}}{12} \cdot \left[\left(20.6 \times 10^3\right)^2 + \left(38.1 \times 10^3\right)^2\right] = 4.10 \times 10^{20} \text{ kg·m}^2$$

With respect to asteroid 2, it is also rotating within the xy-plane, so, similarly, only its length l_2 and width w_2 are relevant to its moment of inertia I_2. However, the rotating axis r runs parallel to the z-axis at a distance $d = \sqrt{\left(\frac{w_2}{2}\right)^2 + \left(\frac{l_2}{2}\right)^2}$. With the help of the parallel-axis theorem, the moment of inertia $I_{2,full}$ of the asteroid, disregarding the hole in the middle, is therefore equal to:

$$I_{2,full} = \frac{m_2}{12} \cdot \left(w_2^2 + l_2^2\right) + m_2 \cdot d^2 = \frac{m_2}{12} \cdot \left(w_2^2 + l_2^2\right) + m_2 \cdot \left[\left(\frac{w_2}{2}\right)^2 + \left(\frac{l_2}{2}\right)^2\right] = \frac{m_2}{3} \cdot \left(w_2^2 + l_2^2\right)$$

To find I_2 we have to subtract the moment of inertia I_h of the hole from $I_{2,full}$. Given that the width and length of the cuboidal-shaped hole are one fifth of that of the asteroid, the missing mass m_h of the hole relative to m_2 can be determined by the ratio of the respective volumes in the following way:

$$\frac{m_h}{m_2} = \frac{V_h}{V_2} = \frac{\frac{w_2}{5} \cdot \frac{l_2}{5} \cdot h_2}{w_2 \cdot l_2 \cdot h_2} \quad \Leftrightarrow \quad m_h = \frac{m_2}{25}$$

The moment of inertia I_h is then expressed as:

$$I_h = \frac{m_2}{25} \cdot \frac{1}{12} \cdot \left[\left(\frac{w_2}{5}\right)^2 + \left(\frac{l_2}{5}\right)^2\right] = \frac{m_2}{7,500} \cdot \left(w_2^2 + l_2^2\right)$$

The value of the moment of inertia I_2 of asteroid 2 can now be calculated as follows:

$$I_2 = I_{2,full} - I_h = \frac{m_2}{3} \cdot \left(w_2^2 + l_2^2\right) - \frac{m_2}{7,500} \cdot \left(w_2^2 + l_2^2\right)$$

$$= \frac{2,499 \cdot m_2}{7,500} \cdot \left(w_2^2 + l_2^2\right) = \frac{2,499 \cdot 1.24 \times 10^{13}}{7,500} \cdot \left[\left(53.8 \times 10^3\right)^2 + \left(155 \times 10^3\right)^2\right]$$

$$= 1.11 \times 10^{23} \text{ kg}\cdot\text{m}^2$$

Based on the above equation related to the conservation of angular momentum, we can determine the period of rotation P of the spherical-shaped merged asteroid:

$$(-I_1 \cdot \omega_1) + (I_2 \cdot \omega_2) = I_f \cdot \omega_f = \left[\frac{2}{5} \cdot (m_1 + m_2) \cdot R^2\right] \cdot \omega_f$$

$$\Leftrightarrow \quad \omega_f = \frac{5}{2} \cdot \left[\frac{(-I_1 \cdot \omega_1) + (I_2 \cdot \omega_2)}{(m_1 + m_2) \cdot R^2}\right] = \frac{5}{2} \cdot \left[\frac{\left(-4.10 \times 10^{20} \cdot \frac{2\cdot\pi}{6.50\cdot 3,600}\right) + \left(1.11 \times 10^{23} \cdot \frac{2\cdot\pi}{22.3\cdot 3,600}\right)}{(2.62 \times 10^{12} + 1.24 \times 10^{13}) \cdot 53,750^2}\right]$$

$$= 4.95 \times 10^{-4} \text{ rad/s}$$

$$\Leftrightarrow \quad P = \frac{2 \cdot \pi}{\omega_f \cdot 3,600} = \frac{2 \cdot \pi}{4.95 \times 10^{-4} \cdot 3,600} = 3.52 \text{ hours}$$

Since the magnitude of the angular velocity $\vec{\omega}_f$ of the newly assembled asteroid is positive, the magnitude of its angular momentum $\vec{L}_f$ is also positive, so that the asteroid is spinning anticlockwise.

Exercise 13

Problem Statement

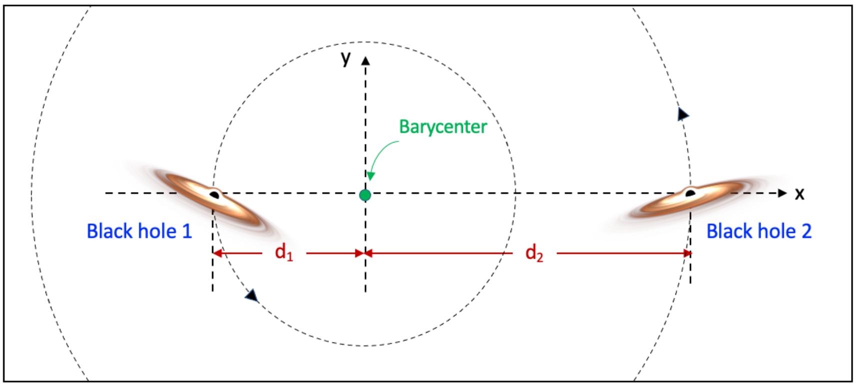

Figure 13

Nina is pursuing her PhD in Theoretical Astrophysics at the Lorentz Institute in Leiden, The Netherlands, and she is specifically interested in studying binary systems of spinning black holes, a.k.a. Kerr black holes. A binary system consists of two massive bodies orbiting each other around their common center of mass called the barycenter. When a spinning object is orbiting around a certain point in space, its total angular moment $\vec{L}_{tot}$ is composed of two terms, i.e., the orbital angular momentum $\vec{L}_{orb} = \vec{r} \times \vec{p}$ and the spin angular momentum $\vec{J} = I \cdot \vec{\Omega}$. For her doctoral thesis, Nina is currently investigating the galaxy NGC 7674, which is located within the Pegasus constellation about 400 million light years away and houses a binary system of supermassive black holes. Nina finds that the magnitude of the orbital angular momentum $\vec{L}_{orb}$ is equal to $L_{orb} = M_1 \cdot \sqrt{G \cdot d_1 \cdot M_2}$, with d_1 the distance of black hole 1 from the barycenter and M_1 and M_2 the mass of the black hole 1 and 2, respectively. How did Nina obtain this result? Make the assumption that the orbits are circular in nature.

Solution

Since no net torque is present in the system "black hole 1 plus black hole 2", the orbital angular momentum $\vec{L}_{orb}$ is conserved, which means that $\vec{L}_{orb}$ is a constant vector and thus maintains its direction and magnitude at all times. As a result, both the position vector $\vec{r}$ and the velocity vector $\vec{v}$ always remain perpendicular to L_{orb}. Put another way, the orbital motion of the two black holes occur in a plane, i.e., the xy-plane, as $\vec{L}_{orb}$ points in the z-direction (out of your screen).

Given that the black holes are orbiting the barycenter in an anticlockwise fashion, their respective velocities within our (co-rotating) coordinate system are equal to $\vec{v}_1 = -v_1 \cdot \vec{i}_y$ and $\vec{v}_2 = v_2 \cdot \vec{i}_y$ (in the specific position of Fig. 13). The orbital angular momentum $\vec{L}_{orb}$ of the system is then formulated as follows:

$$
\begin{aligned}
\vec{L}_{orb} = \vec{L}_{orb,1} + \vec{L}_{orb,2} &= (\vec{r}_1 \times \vec{p}_1) + (\vec{r}_2 \times \vec{p}_2) \\
&= \left[\left(-d_1 \cdot \vec{i}_x\right) \times \left(-M_1 \cdot v_1 \cdot \vec{i}_y\right)\right] + \left[\left(d_2 \cdot \vec{i}_x\right) \times \left(M_2 \cdot v_2 \cdot \vec{i}_y\right)\right] \\
&= \left(M_1 \cdot d_1 \cdot v_1 \cdot \vec{i}_z\right) + \left(M_2 \cdot d_2 \cdot v_2 \cdot \vec{i}_z\right)
\end{aligned}
$$

In the co-rotating frame of reference, we know that the centrifugal force $\vec{F}_{CF}$ must be balanced by the gravitational force $\vec{F}_G$, if we want the black holes to remain in their respective orbit. For the two supermassive black holes, we therefore obtain the following expression for their orbital speed v_1 and v_2, respectively:

$$F_{CF} = F_G \quad \Leftrightarrow \quad \begin{cases} \frac{M_1 \cdot v_1^2}{d_1} = \frac{G \cdot M_1 \cdot M_2}{(d_1+d_2)^2} \quad \Leftrightarrow \quad v_1 = \frac{\sqrt{G \cdot M_2 \cdot d_1}}{(d_1+d_2)} \\ \\ \frac{M_2 \cdot v_2^2}{d_2} = \frac{G \cdot M_1 \cdot M_2}{(d_1+d_2)^2} \quad \Leftrightarrow \quad v_2 = \frac{\sqrt{G \cdot M_1 \cdot d_2}}{(d_1+d_2)} \end{cases}$$

If we insert these two expressions into the magnitude of the orbital angular momentum $\vec{L}_{orb}$, we obtain the following expression for L_{orb}:

$$L_{orb} = (M_1 \cdot d_1 \cdot v_1) + (M_2 \cdot d_2 \cdot v_2) = \left(M_1 \cdot d_1 \cdot \left[\frac{\sqrt{G \cdot M_2 \cdot d_1}}{(d_1+d_2)}\right]\right) + \left(M_2 \cdot d_2 \cdot \left[\frac{\sqrt{G \cdot M_1 \cdot d_2}}{(d_1+d_2)}\right]\right)$$

Based on the definition of the center of mass, we can write d_1 and d_2 as follows:

$$d_1 = \frac{(d_1+d_2) \cdot M_2}{(M_1+M_2)} \qquad\qquad d_2 = \frac{(d_1+d_2) \cdot M_1}{(M_1+M_2)}$$

We now insert these two expressions into the above expression for L_{orb} and we find Nina's result:

$$\begin{aligned} L_{orb} &= \left(M_1 \cdot d_1 \cdot \left[\frac{\sqrt{G \cdot M_2 \cdot d_1}}{(d_1+d_2)}\right]\right) + \left(M_2 \cdot d_2 \cdot \left[\frac{\sqrt{G \cdot M_1 \cdot d_2}}{(d_1+d_2)}\right]\right) \\ &= \left(M_1 \cdot \left[\frac{(d_1+d_2) \cdot M_2}{(M_1+M_2)}\right] \cdot \left[\frac{\sqrt{G \cdot M_2 \cdot \left[\frac{(d_1+d_2) \cdot M_2}{(M_1+M_2)}\right]}}{(d_1+d_2)}\right]\right) + \left(M_2 \cdot \left[\frac{(d_1+d_2) \cdot M_1}{(M_1+M_2)}\right] \cdot \left[\frac{\sqrt{G \cdot M_1 \cdot \left[\frac{(d_1+d_2) \cdot M_1}{(M_1+M_2)}\right]}}{(d_1+d_2)}\right]\right) \\ &= \left[\frac{M_1 \cdot M_2^2}{(M_1+M_2)} \cdot \sqrt{\frac{G \cdot (d_1+d_2)}{(M_1+M_2)}}\right] + \left[\frac{M_2 \cdot M_1^2}{(M_1+M_2)} \cdot \sqrt{\frac{G \cdot (d_1+d_2)}{(M_1+M_2)}}\right] \\ &= M_1 \cdot M_2 \cdot \sqrt{\frac{G \cdot (d_1+d_2)}{(M_1+M_2)}} \\ &= M_1 \cdot M_2 \cdot \sqrt{G \cdot \left[\frac{d_1}{M_2}\right]} \\ &= M_1 \cdot \sqrt{G \cdot d_1 \cdot M_2} \end{aligned}$$

Exercise 14

Problem Statement

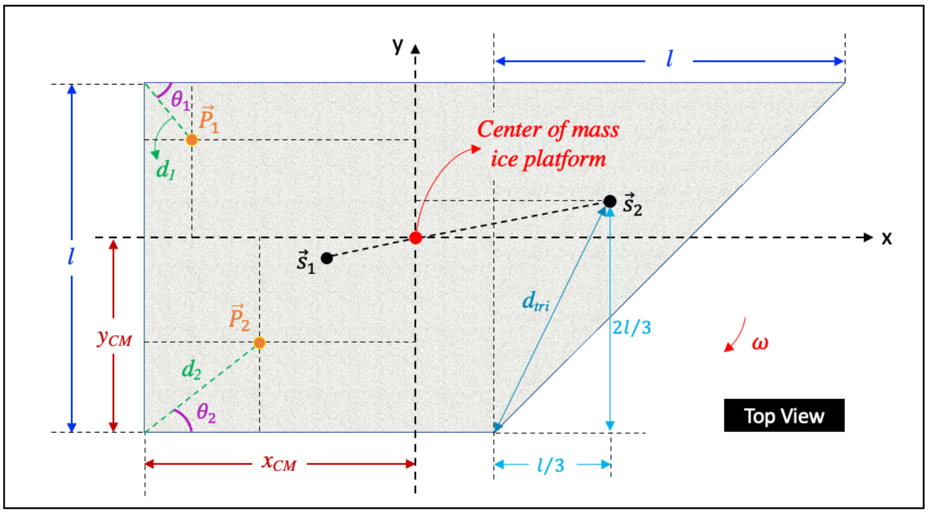

Figure 14

The sub-Antarctic island of South Georgia, which belongs to the British Overseas Territories, is home to the world's largest colony of King Penguins. During the winter, many are often found more southwards along the coastal regions of the Antarctic continent. On one of the Antarctic islands called Spert Island, three King Penguins feel particularly playful today and they suddenly notice a floating piece of ice near the shore. Two of them ($m_{p1} = 12.3$ kg and $m_{p2} = 17.6$ kg) are quick to react, make their way towards the ice platform and jump onto it. Right before the third penguin ($m_{p3} = 23.2$ kg) also jumps onto the platform, the ice shelf is rotating slowly in the clockwise direction around its center of mass at a rate of 1 revolution every 1.22 minutes. The shape of the platform is a square prism (with length $l = 2.00$ m and height $h = 5.00$ cm) onto which an isosceles right-angled triangular prism of corresponding dimensions is attached to one of its sides. When visualizing the triangular prism at the right-hand side of the square prism, then the two penguins are standing $d_1 = 35.5$ cm and $d_2 = 82.7$ cm from the top left and bottom left corner under an angle of $\theta_1 = 50.6°$ and $\theta_2 = 38.2°$ with the horizontal, respectively. If you know that the density of ice is equal to $\rho = 917$ kg/m^3 and that the third penguin lands right at the center of mass, at what rate is the ice platform now rotating? Treat the ice shelf as a thin plate and the penguins as solid cylinders with an internal radius equal to $r_1 = 16.9$ cm, $r_2 = 19.1$ cm, and $r_3 = 22.3$ cm, respectively.

Solution

For this problem, we consider the isolated system "ice platform plus the three penguins", so that we can disregard any potential wobbling of the platform due to the water. In a first step, we determine the total mass m_{tot} of the ice shelf. The total volume V_{tot} of the platform is equal to the sum of the volume V_{sq} of the square prism and the volume V_{tri} of the right triangular prism:

$$V_{tot} = V_{sq} + V_{tri} = \left(l^2 \cdot h\right) + \left(\frac{l^2}{2} \cdot h\right) = \frac{3}{2} \cdot l^2 \cdot h = \frac{3}{2} \cdot 2.00^2 \cdot 5.00 \times 10^{-2} = 0.300 \text{ m}^3$$

The total mass is then equal to $m_{tot} = \rho \cdot V_{tot} = 917 \cdot 0.300 = 275$ kg. The mass m_{sq} and m_{tri} of the square and triangular prism, respectively, are equal to $m_{sq} = \rho \cdot V_{sq} = \rho \cdot (l^2 \cdot h) =$

$917 \cdot (2.00^2 \cdot 5.00 \times 10^{-2}) = 183$ kg and $m_{tri} = m_{tot} - m_{sq} = 275 - 183 = 91.7$ kg. Now that we know the mass of the ice shelf, we will treat the platform as a thin plate from now onwards.

In a next step, we calculate the distances x_{CM} and y_{CM}, which are the distances from which the platform's center of mass—this includes the mass of penguin 1 and 2—is positioned from its bottom left corner in the x- and y-direction, respectively. If we now relocate for a moment the origin of our coordinate system to the bottom left corner of the ice shelf (at the level of the surface), then we know that the point $\vec{s}_1$, which corresponds to the center of mass of the square prism, has the coordinates $\vec{s}_1 = (1.00, 1.00)$. The center of mass of the triangular prism is positioned at $\vec{s}_2 = \left(\frac{8}{3}, \frac{4}{3}\right)$. The center of mass $\vec{r}_c$ of the ice platform *without the two penguins* is therefore located at:

$$\vec{r}_c = (x_c, y_c) = \left(\frac{s_{1,x} \cdot m_{sq} + s_{2,x} \cdot m_{tri}}{m_{tot}}, \frac{s_{1,y} \cdot m_{sq} + s_{2,y} \cdot m_{tri}}{m_{tot}}\right) = \left(\frac{1.00 \cdot 183 + \frac{8}{3} \cdot 91.7}{275}, \frac{1.00 \cdot 183 + \frac{4}{3} \cdot 91.7}{275}\right)$$

$$= (1.56, 1.11)$$

The center of mass $\vec{r}_{CM}$ of the ice platform *including the two penguins* (this corresponds to the red dot in Fig. 14) is then calculated in a similar fashion:

$$\vec{r}_{CM} = \left(\frac{x_c \cdot m_{tot} + (d_1 \cdot \cos\theta_1) \cdot m_{p1} + (d_2 \cdot \cos\theta_2) \cdot m_{p2}}{m_{tot} + m_{p1} + m_{p2}}, \frac{y_c \cdot m_{tot} + (l - d_1 \cdot \sin\theta_1) \cdot m_{p1} + (d_2 \cdot \sin\theta_2) \cdot m_{p2}}{m_{tot} + m_{p1} + m_{p2}}\right)$$

$$= \left(\frac{1.56 \cdot 275 + [0.355 \cdot \cos(50.6°)] \cdot 12.3 + [0.827 \cdot \cos(38.2°)] \cdot 17.6}{275 + 12.3 + 17.6},\right.$$

$$\left.\frac{1.11 \cdot 275 + [2.00 - 0.355 \cdot \sin(50.6°)] \cdot 12.3 + [0.827 \cdot \sin(38.2°)] \cdot 17.6}{275 + 12.3 + 17.6}\right) = (1.45, 1.10)$$

The distances x_{CM} and y_{CM} are thus equal to $x_{CM} = 1.45$ m and $y_{CM} = 1.10$ m, respectively. For the remainder of this exercise, we place the origin of the coordinate system at the platform's center of mass.

Since in our isolated system "ice platform plus the three penguins" the net torque is equal to the null vector—the gravitational force $\vec{F}_G$ acts on the platform's center of mass which is also the location of the rotation axis—we know that the total angular momentum $\vec{L}_{tot}$ remains constant, so that the rotational motion of the ice shelf occurs in a plane, i.e., the xy-plane. If the moment of inertia I_{ice} represents the moment of inertia of the ice platform containing two penguins, we can then write (with $\vec{\omega}_f$ the final angular velocity after the third penguin has jumped onto the ice shelf):

$$\vec{L}_{tot,i} = \vec{L}_{tot,f} \quad \Leftrightarrow \quad I_{ice} \cdot \vec{\omega}_i = \left(I_{ice} + \frac{m_{p3} \cdot r_3^2}{2}\right) \cdot \vec{\omega}_f$$

The moment of inertia I_{ice} consists of four sub-moments of inertia, i.e., those related to the two penguins, one for the square prism (L_{sq}), and one for the triangular prism (L_{tri}). Regarding L_{sq}, we can write the moment of inertia for a rectangular prism (with equal length and width) and apply the parallel-axis theorem across the distance between the platform's center of mass and the point $\vec{s}_1$:

$$I_{sq} = \frac{m_{sq}}{12} \cdot \left(l^2 + l^2\right) + m_{sq} \cdot \left[\sqrt{(x_{CM} - 1)^2 + (y_{CM} - 1)^2}\right]^2$$

$$= \frac{183}{12} \cdot \left(2.00^2 + 2.00^2\right) + 183 \cdot \left[\sqrt{(1.45 - 1)^2 + (1.10 - 1)^2}\right]^2 = 161 \text{ kg·m}^2$$

With respect to I_{tri}, we derived in Exercise 8 the formula for the moment of inertia of a right-angled triangle rotating around one of its two vertices adjacent to the hypotenuse, i.e. $I_v = \frac{m_{tri}}{2} \cdot \left(l^2 + \frac{l^2}{3}\right)$. Before we can obtain I_{tri}, we need to find the moment of inertia $I_{t,CM}$ of the triangular prism at its center of mass $\vec{s}_2$. If we consider the bottom left vertex of the triangle as the point of rotation and if we refer to d_{tri} as the distance between that point and $\vec{s}_2$, we can then write:

$$I_v = I_{t,CM} + m_{tri} \cdot d_{tri}^2 \quad \Leftrightarrow \quad I_{t,CM} = I_v - m_{tri} \cdot d_{tri}^2$$

$$= \frac{m_{tri}}{2} \cdot \left(l^2 + \frac{l^2}{3}\right) - m_{tri} \cdot \left[\sqrt{\left(\frac{l}{3}\right)^2 + \left(\frac{2 \cdot l}{3}\right)^2}\right]^2$$

$$= \frac{m_{tri} \cdot l^2}{9}$$

The moment of inertia I_{tri} of the triangular prism rotating about the platform's center of mass, which is located at a distance $|\vec{s}_2|$ away from the prism's center of mass, is then calculated as follows:

$$I_{tri} = I_{t,CM} + m_{tri} \cdot |\vec{s}_2|^2$$

$$= \frac{m_{tri} \cdot l^2}{9} + m_{tri} \cdot \left[\sqrt{\left(\frac{l}{3} + (l - x_{CM})\right)^2 + \left(\frac{2 \cdot l}{3} - y_{CM}\right)^2}\right]^2$$

$$= \frac{91.7 \cdot 2.00^2}{9} + 91.7 \cdot \left[\sqrt{\left(\frac{2.00}{3} + (2.00 - 1.45)\right)^2 + \left(\frac{2 \cdot 2.00}{3} - 1.10\right)^2}\right]^2 = 182 \text{ kg·m}^2$$

The moment of inertia I_{ice} of the platform containing the two penguins is then equal to:

$$I_{ice} = I_{sq} + I_{tri} + \left(\frac{m_{p1} \cdot r_1^2}{2}\right) + \left(\frac{m_{p2} \cdot r_2^2}{2}\right) = 161 + 182 + \left(\frac{12.3 \cdot 0.169^2}{2}\right) + \left(\frac{17.6 \cdot 0.191^2}{2}\right)$$

$$= 343 \text{ kg·m}^2$$

In a final step, based on the above equation for the conservation of the angular momentum, we determine the magnitude of the angular velocity $\vec{\omega}_f$ after the third penguin has made its way onto the ice platform (given the choice of our coordinate system, note that the clockwise direction is the negative direction of rotation):

$$I_{ice} \cdot \vec{\omega}_i = \left(I_{ice} + \frac{m_{p3} \cdot r_3^2}{2}\right) \cdot \vec{\omega}_f \quad \Leftrightarrow \quad I_{ice} \cdot [-\omega_i]\,\vec{i}_z = \left(I_{ice} + \frac{m_{p3} \cdot r_3^2}{2}\right) \cdot \omega_f \cdot \vec{i}_z$$

$$\Leftrightarrow \quad \omega_f = -\frac{I_{ice} \cdot \omega_i}{\left(I_{ice} + \frac{m_{p3} \cdot r_3^2}{2}\right)}$$

$$= -\frac{343 \cdot \left[\frac{2 \cdot \pi}{1.22 \cdot 60}\right]}{\left(343 + \frac{23.2 \cdot 0.223^2}{2}\right)}$$

$$= -8.58 \times 10^{-2} \text{ rad/s}$$

The ice shelf barely slows down after the third penguin jumped onto the platform; it is still taking $t = \frac{2 \cdot \pi}{|\omega_f| \cdot 60} = \frac{2 \cdot \pi}{8.58 \times 10^{-2} \cdot 60} = 1.22$ minutes to complete one revolution. The difference in rotational motion is only noticeable on the millisecond scale. Whereas the rotational period of the ice shelf is initially equal to 1 min 13.2 s, the final period is measured as 1 min 13.2274 s. In other words, the ice shelf has slowed down its rate of rotation by 27.4 ms.

Exercise 15

Problem Statement

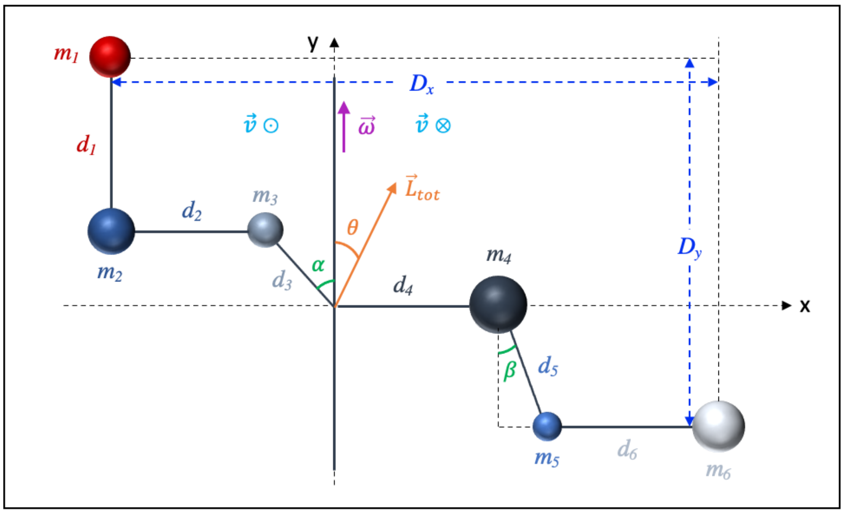

Figure 15

Laniyan is one of the Cameroonian artists who are invited to exhibit their work at the temporary exposition "l'Asymétrie et la Rotation" at the contemporary art center doul'art at Douala, Cameroon. Laniyan designed his own creative version of a newly discovered, young planetary system of six planets. Instead of orbiting within a fixed plane, the planets are spatially arranged in a stepwise fashion whereby their total angular momentum $\vec{L}_{tot}$ is tilted by an angle of $\theta = 23.2°$ relative to the axis of rotation, i.e., the y-axis. Planet 1 ($m_1 = 6.55$ kg), which is the planet in the highest orbit, is located at a distance of $D_x = 92.2$ cm horizontally and $D_y = 66.6$ cm vertically from the planet in the lowest orbit, i.e., Planet 6 ($m_6 = 8.21$ kg), at the other end of the planetary system. Planet 2 ($m_2 = 7.60$ kg) is positioned $d_1 = 42.5$ cm to the south of Planet 1, and Planet 3 ($m_3 = 4.35$ kg), which finds itself $d_2 = 24.4$ cm to the east of Planet 2, is the planet closest to the origin of the coordinate system at a distance of $d_3 = 15.0$ cm, making thereby an angle α with the rotation axis. Planet 4 (m_4) is orbiting at $d_4 = 26.4$ cm to the east of the origin, and $d_5 = 13.9$ cm away from Planet 4 in the direction east of south at an angle β is the location of the orbit of Planet 5 ($m_5 = 3.67$ kg). Finally, Planet 6 is positioned $d_6 = 26.7$ cm further to the east relative to Planet 5. If you know that in Laniyan's planetary system the planets are not spinning and rotate counterclockwise, what is the mass m_4 of Planet 4? Neglect the mass of the connecting rods between the planets.

Solution

To find the mass m_4 of Planet 4 through the definition of the orbital angular momentum $\vec{L}_{tot} = \vec{r} \times \vec{p}$, we will need to know the coordinates of every position vector $\vec{r}$ of the six planets (at the moment shown in Fig. 15). Therefore, let us in a first instance determine the value of the angles α and β. Based on the two constraints $D_x = 92.2$ cm and $D_y = 66.6$ cm, we obtain the following two equations:

$$x: \; D_x = d_2 + d_3 \cdot \sin\alpha + d_4 + d_5 \cdot \sin\beta + d_6 \qquad y: \; D_y = d_1 + d_3 \cdot \cos\alpha + d_5 \cdot \cos\beta$$

$$\Leftrightarrow \quad (D_x - d_2 - d_4 - d_6) = d_3 \cdot \sin\alpha + d_5 \cdot \sin\beta \qquad \Leftrightarrow \quad (D_y - d_1) = d_3 \cdot \cos\alpha + d_5 \cdot \cos\beta$$

$$\Leftrightarrow \quad c_1 = d_3 \cdot \sin\alpha + d_5 \cdot \sin\beta \qquad \Leftrightarrow \quad c_2 = d_3 \cdot \cos\alpha + d_5 \cdot \cos\beta$$

whereby $c_1 = (D_x - d_2 - d_4 - d_6) = (92.2 - 24.4 - 26.4 - 26.7) = 14.7$ cm and $c_2 = (D_y - d_1) = (66.6 - 42.5) = 24.1$ cm. Rearranging and subsequently squaring the two equations and then adding them together as well as bearing in mind that the linear combination of a cosine and a sine function, i.e., "$a \cdot \cos\gamma + b \cdot \sin\gamma$", can be replaced by a single cosine function "$z \cdot \cos(\gamma + \delta)$", whereby $z = sgn(a)\sqrt{a^2+b^2}$ and $\delta = \tan^{-1}(-\frac{b}{a})$, allows us to find the value for the angle β:

$$\begin{cases} [c_1 - d_3 \cdot \sin\alpha]^2 = [d_5 \cdot \sin\beta]^2 \\ \\ [c_2 - d_3 \cdot \cos\alpha]^2 = [d_5 \cdot \cos\beta]^2 \end{cases} \Rightarrow \quad c_1^2 + c_2^2 - 2 \cdot d_3 \cdot (c_1 \cdot \sin\alpha + c_2 \cdot \cos\alpha) + d_3^2 = d_5^2$$

$$\Leftrightarrow \quad c_1^2 + c_2^2 - 2 \cdot d_3 \cdot \left(\sqrt{c_1^2 + c_2^2} \cdot \cos\left[\alpha + \tan^{-1}\left(-\frac{c_1}{c_2}\right)\right]\right) + d_3^2 = d_5^2$$

$$\Leftrightarrow \quad \cos\left[\alpha + \tan^{-1}\left(-\frac{c_1}{c_2}\right)\right] = \frac{(c_1^2 + c_2^2 + d_3^2 - d_5^2)}{2 \cdot d_3 \cdot \sqrt{c_1^2 + c_2^2}}$$

$$\Leftrightarrow \quad \alpha = \cos^{-1}\left[\frac{(c_1^2 + c_2^2 + d_3^2 - d_5^2)}{2 \cdot d_3 \cdot \sqrt{c_1^2 + c_2^2}}\right] - \tan^{-1}\left(-\frac{c_1}{c_2}\right)$$

$$= \cos^{-1}\left[\frac{(0.147^2 + 0.241^2 + 0.150^2 - 0.139^2)}{2 \cdot 0.150 \cdot \sqrt{0.147^2 + 0.241^2}}\right] - \tan^{-1}\left(-\frac{0.147}{0.241}\right) = 43.3°$$

Based on the constraint related to, for instance, the x-direction, we then obtain the value for the angle β:

$$c_1 = d_3 \cdot \sin\alpha + d_5 \cdot \sin\beta \quad \Leftrightarrow \quad \beta = \sin^{-1}\left[\frac{c_1 - d_3 \cdot \sin\alpha}{d_5}\right] = \sin^{-1}\left[\frac{0.147 - 0.150 \cdot \sin(43.3°)}{0.139}\right] = 18.5°$$

We can now calculate the coordinates of all the position vectors (at the moment shown in Fig. 15):

$$\begin{cases} \vec{r}_1 = (x_3 - d_2, y_3 + d_1, 0) = (-10.3 - 24.4, 10.9 + 42.5, 0) & = (-34.7, 53.4, 0) \text{ cm} \\ \vec{r}_2 = (x_1, y_3, 0) & = (-34.7, 10.9, 0) \text{ cm} \\ \vec{r}_3 = (-d_3 \cdot \sin\alpha, d_3 \cdot \cos\alpha, 0) = (-15.0 \cdot \sin(43.3°), 15.0 \cdot \cos(43.3°, 0) & = (-10.3, 10.9, 0) \text{ cm} \\ \vec{r}_4 = (d_4, 0, 0) & = (26.4, 0, 0) \text{ cm} \\ \vec{r}_5 = (d_4 + d_5 \cdot \sin\beta, -d_5 \cdot \cos\beta, 0) = (26.4 + 13.9 \cdot \sin(18.5°), -13.9 \cdot \cos(18.5°), 0) & = (30.8, -13.2, 0) \text{ cm} \\ \vec{r}_6 = (x_5 + d_6, y_5, 0) = (30.8 + 26.7, -13.2, 0) & = (57.5, -13.2, 0) \text{ cm} \end{cases}$$

Given that the total orbital angular momentum vector $\vec{L}_{tot}$ is the sum of the angular momenta of the six planets, we can write:

$$\vec{L}_{tot} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \vec{L}_4 + \vec{L}_5 + \vec{L}_6$$

Given a counterclockwise rotation, the orbital velocity vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ of the Planets 1, 2, and 3, respectively, are pointing, at the moment shown in Fig. 15, out of your screen (represented by the symbol $\odot$), whereas the orbital velocity vectors $\vec{v}_4$, $\vec{v}_5$, and $\vec{v}_6$ of the Planets 4, 5, and 6, respectively, are pointing into your screen (represented by the symbol $\otimes$). Therefore, the orbital angular momentum of Planet 1 is equal to $\vec{L}_1 = \vec{r}_1 \times \vec{p}_1 = (-x_1, y_1, 0) \times [m_1 \cdot (0, 0, v_1)] = (m_1 \cdot y_1 \cdot v_1) \cdot \vec{i}_x + (m_1 \cdot x_1 \cdot v_1) \cdot \vec{i}_y$. Note that here x_1, y_1, and v_1 represent *only the magnitude* of the respective vector (e.g., $x_1 = 0.347$ and not $x_1 = -0.347$). For the remaining five planets, we find:

$$\begin{cases} \vec{L}_2 = \vec{r}_2 \times \vec{p}_2 = (-x_2, y_2, 0) \times [m_2 \cdot (0, 0, v_2)] = (m_2 \cdot y_2 \cdot v_2) \cdot \vec{i}_x + (m_2 \cdot x_2 \cdot v_2) \cdot \vec{i}_y \\ \vec{L}_3 = \vec{r}_3 \times \vec{p}_3 = (-x_3, y_3, 0) \times [m_3 \cdot (0, 0, v_3)] = (m_3 \cdot y_3 \cdot v_3) \cdot \vec{i}_x + (m_3 \cdot x_3 \cdot v_3) \cdot \vec{i}_y \\ \vec{L}_4 = \vec{r}_4 \times \vec{p}_4 = (x_4, 0, 0) \times [m_4 \cdot (0, 0, -v_4)] = (m_4 \cdot x_4 \cdot v_4) \cdot \vec{i}_y \\ \vec{L}_5 = \vec{r}_5 \times \vec{p}_5 = (x_5, -y_5, 0) \times [m_5 \cdot (0, 0, -v_5)] = (m_5 \cdot y_5 \cdot v_5) \cdot \vec{i}_x + (m_5 \cdot x_5 \cdot v_5) \cdot \vec{i}_y \\ \vec{L}_6 = \vec{r}_6 \times \vec{p}_6 = (x_6, -y_6, 0) \times [m_6 \cdot (0, 0, -v_6)] = (m_6 \cdot y_6 \cdot v_6) \cdot \vec{i}_x + (m_6 \cdot x_6 \cdot v_6) \cdot \vec{i}_y \end{cases}$$

Since the magnitude of the orbital velocity $\vec{v}_i$ is equal to $v_i = x_i \cdot \omega$, the total orbital angular momentum vector $\vec{L}_{tot}$ then becomes:

$$\begin{aligned} \vec{L}_{tot} = &[m_1 \cdot y_1 \cdot x_1 + m_2 \cdot y_2 \cdot x_2 + m_3 \cdot y_3 \cdot x_3 + m_5 \cdot y_5 \cdot x_5 + m_6 \cdot y_6 \cdot x_6] \cdot \omega \cdot \vec{i}_x \; + \\ &\left[m_1 \cdot x_1^2 + m_2 \cdot x_2^2 + m_3 \cdot x_3^2 + m_4 \cdot x_4^2 + m_5 \cdot x_5^2 + m_6 \cdot x_6^2\right] \cdot \omega \cdot \vec{i}_y \end{aligned}$$

To simplify our calculations let us first calculate the value of the x-component of $\vec{L}_{tot}$:

$$\begin{aligned} L_x =& [m_1 \cdot y_1 \cdot x_1 + m_2 \cdot y_2 \cdot x_2 + m_3 \cdot y_3 \cdot x_3 + m_5 \cdot y_5 \cdot x_5 + m_6 \cdot y_6 \cdot x_6] \cdot \omega \\ =& [6.55 \cdot 0.534 \cdot 0.347 + 7.60 \cdot 0.109 \cdot 0.347 + 4.35 \cdot 0.109 \cdot 0.103 + 3.67 \cdot 0.132 \cdot 0.308 \; + \\ & 8.21 \cdot 0.132 \cdot 0.575] \cdot \omega \\ =& 2.32 \cdot \omega \quad (\text{kg}\cdot\text{m}^2/\text{s}) \end{aligned}$$

Given that the total orbital angular momentum vector $\vec{L}_{tot}$ is tilted by an angle $\theta = 23.2°$ with respect to the rotation axis, we know that $\tan\theta = \frac{L_x}{L_y}$, so that we can calculate the mass m_4 of Planet 4 as follows:

$$\tan\theta \cdot L_y = L_x$$

$$\Leftrightarrow \quad \tan\theta \cdot \left[m_1 \cdot x_1^2 + m_2 \cdot x_2^2 + m_3 \cdot x_3^2 + m_4 \cdot x_4^2 + m_5 \cdot x_5^2 + m_6 \cdot x_6^2\right] \cdot \omega = 2.32 \cdot \omega$$

$$\Leftrightarrow \quad m_4 = \frac{2.32}{x_4^2 \cdot \tan\theta} - \frac{[m_1 \cdot x_1^2 + m_2 \cdot x_2^2 + m_3 \cdot x_3^2 + m_5 \cdot x_5^2 + m_6 \cdot x_6^2]}{x_4^2}$$

$$= \frac{2.32}{0.264^2 \cdot \tan(23.2^\circ)} - \frac{[6.55 \cdot 0.347^2 + 7.60 \cdot 0.347^2 + 4.35 \cdot 0.103^2 + 3.67 \cdot 0.308^2 + 8.21 \cdot 0.575^2]}{0.264^2}$$

$$= 8.67 \text{ kg}$$

Exercise 16

Problem Statement

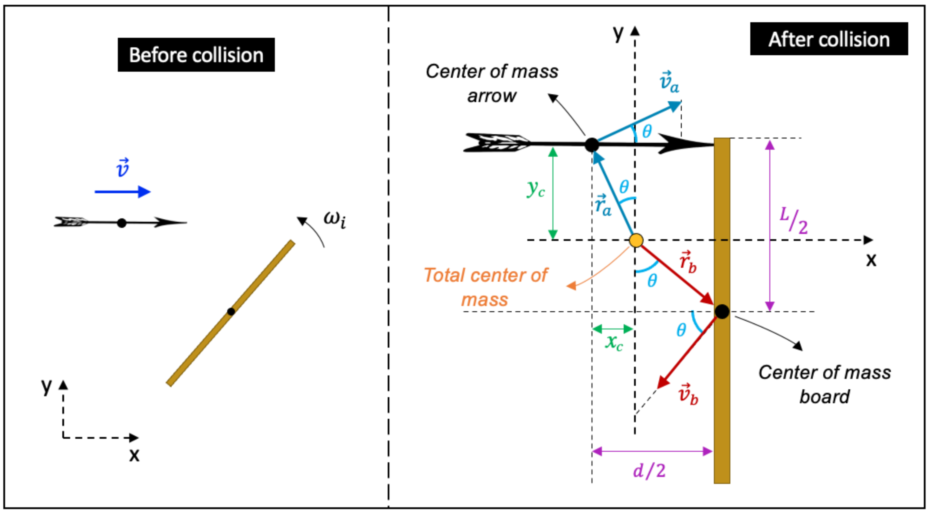

Figure 16

Neylan is trekking with her friend Eldar through the national park Bozdağ Milli Parkı, which is located east of Konya, Turkey, and they just made a stop since Neylan wishes to practice her Robin Hood archery skills. Eldar finds a thin wooden board of length $L = 82.5$ cm and mass $m_b = 855$ g, balances it upright on two fingers, and throws it up in the air. The board spins in a counterclockwise direction around the axis perpendicular to its length and parallel to its width at 210 rpm, whereby the angular velocity vector points southwards. From a distance of $\Delta x = 55.0$ m away, Neylan shoots an arrow of length $d = 61.6$ cm and mass $m_a = 40.6$ g with an initial speed of $v = 89.3$ m/s eastwards towards the spinning board. When the arrow is at its highest point during its trajectory, it hits the uppermost end of the board, which is at that precise moment vertically oriented, right in the middle. If you know that the arrow remains stuck after hitting the wooden board, at what rate does the combined object now spin and in which direction? Treat the arrow as a long rod.

Solution

When disregarding drag forces, objects in free fall are subject to only one force, i.e., the gravitational force $\vec{F}_G$, which acts at their center of mass in a downward direction. Therefore, $\vec{F}_G$ is unable to manifest any torque, so that the total angular momentum $\vec{L}_{tot}$ remains constant. Also, note that the z-axis is the axis of rotation and that, since the arrow hits the wooden board right in the middle, the board will not spin around the y-axis.

Before we determine the initial and final angular momentum of the collision between the arrow and the wooden board, let us calculate the distances x_c and y_c, which represent the magnitude of the x- and y-component of the position vector $\vec{r}_a$, respectively, between the total center of mass of the two objects combined, i.e., the location of the origin of our coordinate system, and the arrow's center of mass. Based on the definition of the center of mass, we can write:

$$x: \quad 0 = \frac{-m_a \cdot x_c + m_b \cdot \left(\frac{d}{2} - x_c\right)}{m_a + m_b} \quad \Leftrightarrow \quad x_c = \frac{m_b \cdot d}{2 \cdot (m_a + m_b)}$$

$$y: \quad 0 = \frac{m_a \cdot y_c - m_b \cdot \left(\frac{L}{2} - y_c\right)}{m_a + m_b} \quad \Leftrightarrow \quad y_c = \frac{m_b \cdot L}{2 \cdot (m_a + m_b)}$$

The magnitude of the position vectors $\vec{r}_a$ and $\vec{r}_b$, respectively, is then equal to:

$$\begin{cases} r_a = \sqrt{\left[-\dfrac{m_b \cdot d}{2 \cdot (m_a + m_b)}\right]^2 + \left[\dfrac{m_b \cdot L}{2 \cdot (m_a + m_b)}\right]^2} = \dfrac{m_b \cdot \sqrt{d^2 + L^2}}{2 \cdot (m_a + m_b)} \\ \\ r_b = \sqrt{\left[\dfrac{d}{2} - x_c\right]^2 + \left[y_c - \dfrac{L}{2}\right]^2} = \sqrt{\left[\dfrac{d}{2} - \dfrac{m_b \cdot d}{2 \cdot (m_a + m_b)}\right]^2 + \left[\dfrac{m_b \cdot L}{2 \cdot (m_a + m_b)} - \dfrac{L}{2}\right]^2} = \dfrac{m_a \cdot \sqrt{d^2 + L^2}}{2 \cdot (m_a + m_b)} \end{cases}$$

The initial total angular momentum $\vec{L}_{tot,i}$ is the sum of the spin angular momentum $\vec{L}_{b,i}$ of the rotating wooden board and the orbital angular momentum $\vec{L}_{a,i}$ of the arrow with respect to the perpendicular distance to the total center of mass (keep in mind that, as the board does not rotate around the y-axis, its moment of inertia $I_{CM,b}$ is equal to that of a long, uniform rod, and, furthermore, that the arrow's velocity at the highest point in its trajectory has a zero y-component):

$$\begin{aligned} \vec{L}_{tot,i} = \vec{L}_{b,i} + \vec{L}_{a,i} &= [I_{CM,b} \cdot \vec{\omega}_i] + [\vec{r}_a \times \vec{p}_i] \\ &= \left[\left(\frac{m_b \cdot L^2}{12} \cdot \omega_i\right) \cdot \vec{i}_z\right] + \left[\left(-\frac{m_b \cdot d}{2 \cdot (m_a + m_b)}, \frac{m_b \cdot L}{2 \cdot (m_a + m_b)}, 0\right) \times m_a \cdot (v, 0, 0)\right] \\ &= \left[\left(\frac{m_b \cdot L^2}{12} \cdot \omega_i\right) \cdot \vec{i}_z\right] + \left[-\frac{m_a \cdot m_b \cdot v \cdot L}{2 \cdot (m_a + m_b)} \cdot \vec{i}_z\right] \end{aligned}$$

The final total angular momentum $\vec{L}_{tot,f}$ of the combined object is the sum of the total angular momenta $\vec{L}_{tot,a}$ and $\vec{L}_{tot,b}$ of the arrow and the board, respectively, which in turn each consist of a spin angular momentum around their own center of mass and an orbital angular momentum of their individual center of mass rotating around the total center of mass (we will make the assumption that the combined system rotates clockwise):

$$\begin{aligned} \vec{L}_{tot,f} &= \vec{L}_{tot,a} + \vec{L}_{tot,b} \\ &= [I_{CM,a} \cdot \vec{\omega}_f + (\vec{r}_a \times \vec{p}_a)] + [I_{CM,b} \cdot \vec{\omega}_f + (\vec{r}_b \times \vec{p}_b)] \\ &= \left[\left(-\frac{m_a \cdot d^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z + m_a \cdot (-r_a \cdot \sin\theta, r_a \cdot \cos\theta, 0) \times (v_a \cdot \cos\theta, v_a \cdot \sin\theta, 0)\right] + \\ &\quad \left[\left(-\frac{m_b \cdot L^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z + m_b \cdot (r_b \cdot \sin\theta, -r_b \cdot \cos\theta, 0) \times (-v_b \cdot \cos\theta, -v_b \cdot \sin\theta, 0)\right] \\ &= \left[\left(-\frac{m_a \cdot d^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z - (m_a \cdot r_a \cdot v_a) \cdot \vec{i}_z\right] + \left[\left(-\frac{m_b \cdot L^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z - (m_b \cdot r_b \cdot v_b) \cdot \vec{i}_z\right] \end{aligned}$$

Given that $\vec{v}_a$ and $\vec{v}_b$ represent the respective orbital velocity of the arrow and the board around the total center of mass, their magnitude is equal to $v_a = \omega_f \cdot r_a$ and $v_b = \omega_f \cdot r_b$, respectively. The final total angular momentum $\vec{L}_{tot,f}$ then becomes:

$$\vec{L}_{tot,f} = \left[\left(-\frac{m_a \cdot d^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z - \left(m_a \cdot r_a^2 \cdot \omega_f\right) \cdot \vec{i}_z\right] + \left[\left(-\frac{m_b \cdot L^2}{12} \cdot \omega_f\right) \cdot \vec{i}_z - \left(m_b \cdot r_b^2 \cdot \omega_f\right) \cdot \vec{i}_z\right]$$

$$= -\left[\frac{m_a \cdot d^2}{12} + \frac{m_b \cdot L^2}{12} + m_a \cdot \left[\frac{m_b \cdot \sqrt{d^2 + L^2}}{2 \cdot (m_a + m_b)}\right]^2 + m_b \cdot \left[\frac{m_a \cdot \sqrt{d^2 + L^2}}{2 \cdot (m_a + m_b)}\right]^2\right] \cdot \omega_f \cdot \vec{i}_z$$

$$= -\left[\frac{m_a \cdot d^2}{12} + \frac{m_b \cdot L^2}{12} + \frac{m_a \cdot m_b \cdot (d^2 + L^2)}{4 \cdot (m_a + m_b)}\right] \cdot \omega_f \cdot \vec{i}_z$$

The same result would have been obtained if, instead of contemplating the orbital angular momentum of the individual centers of mass, the parallel-axis theorem was directly applied. The conservation of total angular momentum then allows us to calculate the final angular velocity ω_f (as we assumed that the system rotates clockwise, which means that $\vec{\omega}_f$ points into the negative z-direction, a positive value for ω_f would confirm our assumption; a negative value would mean that the system rotates counterclockwise):

$$\vec{L}_{tot,i} = \vec{L}_{tot,f}$$

$$\Leftrightarrow \quad \left[\frac{m_b \cdot L^2}{12} \cdot \omega_i - \frac{m_a \cdot m_b \cdot v \cdot L}{2 \cdot (m_a + m_b)}\right] \cdot \vec{i}_z = -\left[\frac{m_a \cdot d^2}{12} + \frac{m_b \cdot L^2}{12} + \frac{m_a \cdot m_b \cdot (d^2 + L^2)}{4 \cdot (m_a + m_b)}\right] \cdot \omega_f \cdot \vec{i}_z$$

$$\Leftrightarrow \quad \frac{m_b \cdot L^2}{12} \cdot \omega_i - \frac{m_a \cdot m_b \cdot v \cdot L}{2 \cdot (m_a + m_b)} = -\left[\frac{m_a \cdot d^2}{12} + \frac{m_b \cdot L^2}{12} + \frac{m_a \cdot m_b \cdot (d^2 + L^2)}{4 \cdot (m_a + m_b)}\right] \cdot \omega_f$$

$$\Leftrightarrow \quad \omega_f = \frac{m_b \cdot L \cdot [6 \cdot m_a \cdot v - L \cdot \omega_i \cdot (m_a + m_b)]}{(m_a \cdot d^2 + m_b \cdot L^2) \cdot (m_a + m_b) + 3 \cdot m_a \cdot m_b \cdot (d^2 + L^2)}$$

$$= \frac{0.855 \cdot 0.825 \cdot \left[6 \cdot 0.0406 \cdot 89.3 - 0.825 \cdot \frac{2 \cdot \pi \cdot 210}{60} \cdot (0.0406 + 0.855)\right]}{(0.0406 \cdot 0.616^2 + 0.855 \cdot 0.825^2) \cdot (0.0406 + 0.855) + 3 \cdot 0.0406 \cdot 0.855 \cdot (0.616^2 + 0.825^2)}$$

$$= 6.02 \text{ rad/s or } 57.5 \text{ rpm}$$

As we obtained a positive value for the magnitude of the final angular velocity $\vec{\omega}_f$, we know that the combined object of the arrow stuck within the wooden board indeed rotates in the clockwise direction, even though the board started out rotating counterclockwise.

Exercise 17

Problem Statement

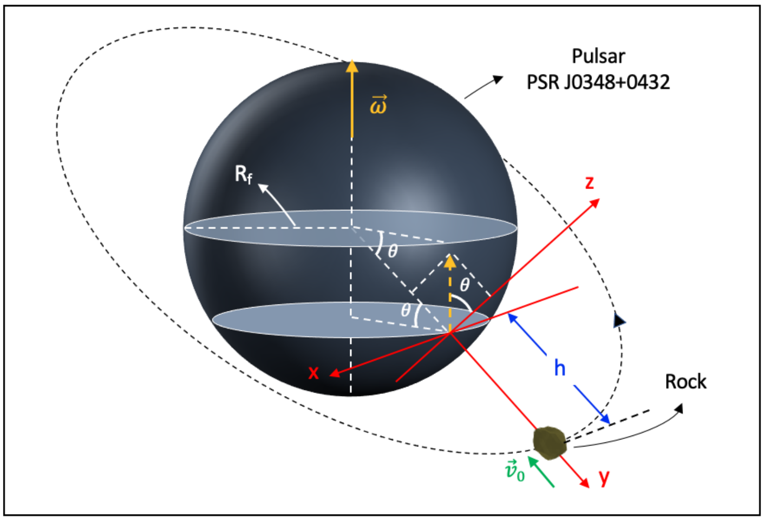

Figure 17

When stars exhaust their nuclear fuel at the end of their lifetime, they shed off their outer layers, often accompanied by a supernova explosion, and the stellar core remnant converts, broadly speaking, into a white dwarf, a neutron star, or a black hole. About 2.6 billion years ago, this process created a rapidly spinning neutron star—called a pulsar—which goes by the name PSR J0348+0432. (1) If we suppose that the original star had a mass, diameter, and rotational period of $M_i = 4.68 \cdot M_s$, $d_i = 2.56 \times 10^4$ km, and $T_i = 1.05$ days, respectively, that during the formation of the pulsar a total of 57% of its mass was lost (without dissipating any angular momentum) and that the star's diameter shrunk by 99.9%, at what rate is the pulsar now spinning? (2) Suppose that a rock ($m_r = 1.87 \times 10^4$ kg) is following a synchronous, circular orbit around the pulsar PSR J0348+0432 and is suddenly hit by an asteroid from outer space along the radial direction of the rock's orbit. As a result of the collision, the rock is sent straight down towards the pulsar's surface at a velocity of $\vec{v}_0 = -15.5 \cdot \vec{i}_y$ km/s. If you know that the rock at the moment of impact is positioned above the pulsar's southern hemisphere at a latitude of $51°18'4.04''$S, by how much is the rock deflected due to the Coriolis effect when it hits the pulsar's surface? Use the average value of the gravitational field strength g between the orbital height and the pulsar's surface, and remember that the universal gravitational constant G is equal to $G = 6.67 \times 10^{-11}$ m^3/(kg$\cdot$ s^2) and that one solar mass measures $M_s = 1.99 \times 10^{30}$ kg.

Solution

(1) Apart from the internal forces at play during the transformation of the star into a pulsar, there are no net external forces giving rise to any net torque within the isolated system "the original star converting into a pulsar", so that the total angular momentum $\vec{L}_{tot}$ is constant under the assumption that the system is not dissipating any angular momentum while ejecting mass. Given that the mass and the radius of the pulsar are equal to $M_f = (1 - 0.57) \cdot M_i = (1 - 0.57) \cdot 4.68 \cdot M_s = 2.01 \cdot M_s$ and $R_f = (1 - 0.999) \cdot \frac{d_i}{2} = (1 - 0.999) \cdot \frac{2.56 \times 10^4}{2} = 12.8$ km, respectively, we can therefore write (suppose that $\vec{\omega}$ is pointing in the positive direction of the upwards oriented rotation axis r, which is not drawn in Fig. 17):

$$\vec{L}_{tot,i} = \vec{L}_{tot,f} \quad \Leftrightarrow \quad I_i \cdot \vec{\omega}_i = I_f \cdot \vec{\omega}_f \quad \Leftrightarrow \quad \left[\frac{2}{5} \cdot M_i \cdot \left(\frac{d_i}{2}\right)^2\right] \cdot \left(\frac{2 \cdot \pi}{T_i}\right) \cdot \vec{i}_r = \left[\frac{2}{5} \cdot M_f \cdot R_f^2\right] \cdot \omega_f \cdot \vec{i}_r$$

$$\Leftrightarrow \quad \omega_f = \frac{M_i \cdot \left(\frac{d_i}{2}\right)^2 \cdot \left(\frac{2\cdot\pi}{T_i}\right)}{M_f \cdot R_f^2}$$

$$= \frac{4.68 \cdot M_s \cdot \left(\frac{2.56\times10^7}{2}\right)^2 \cdot \left(\frac{2\cdot\pi}{1.05\cdot 86{,}400}\right)}{2.01 \cdot M_s \cdot 12{,}800^2}$$

$$= 161 \text{ rad/s}$$

In other words, whereas the original star rotated every 1.05 days once around its axis, it is taking the pulsar only $T_f = \frac{2\cdot\pi}{\omega_f} = \frac{2\cdot\pi}{161} = 39.0$ ms. Put another way, the pulsar is able to complete 25.6 revolutions in just one second.

(2) Let us in a first instance determine some facts about the rock's orbit. Given that the rock follows a synchronous orbit, which exhibits the same period as that of the pulsar, we find the rock's orbital height h with the help of Kepler's third law:

$$T_f^2 = \frac{4\cdot\pi^2}{G\cdot M_f} \cdot (R_f + h)^3 \quad \Leftrightarrow \quad h = \sqrt[3]{\frac{T_f^2 \cdot G \cdot M_f}{4\cdot\pi^2}} - R_f$$

$$= \sqrt[3]{\frac{0.0390^2 \cdot 6.67\times10^{-11} \cdot 2.01 \cdot 1.99\times10^{30}}{4\cdot\pi^2}} - 12{,}800$$

$$= 205 \text{ km}$$

Viewed from an inertial reference frame, the rock's orbital speed is then equal to $v_{orb} = (R_f + h)\cdot\omega_f = (12{,}800 + 2.05\times10^5)\cdot 161 = 3.50\times10^4$ km/s. Also, as the asteroid only provides the rock with a velocity $\vec{v}_0$ in the radial direction (along the y-axis), the rock's motion is left undisturbed in the two other spatial directions, so that the rock is still moving within its orbital plane after the impact.

In the co-rotating coordinate system (x,y,z), the orbital speed, represented by the x-component of the rock's velocity, is equal to zero. Note furthermore that the co-rotating coordinate system is fixed with respect to the pulsar's surface—in other words, the latitude angle θ remains constant—and does therefore not rotate within the inclined orbital plane of the rock. This means that, from the perspective of someone standing at the origin of the co-rotating coordinate system, the rock is moving up and down along an arched path with the same curvature as an imaginary sphere of radius $R_f + h$. The vertical velocity $\vec{v}_v$ from the rock's lowest to the highest point in its orbit (and vice versa) has therefore both a y- and a z-component which oscillate as the rock travels along its inclined orbit. In fact, the y-component of the velocity $\vec{v}_v$ is negatively (positively) oriented when the rock is ascending (descending), whereas the z-component is pointing in the positive (negative) direction as the rock is ascending (descending). Both the components are zero when the rock is located at its lowest and highest point in orbit.

Given that the angle θ is equal to $\theta = 51 + \frac{18}{60} + \frac{4.04}{3600} = 51.3°$, that the vertical distance d_v between the orbit's lowest and highest point measures $d_v = 2 \cdot (R_f + h) \cdot \sin\theta = 2 \cdot (12,800 + 2.05 \times 10^5) \cdot \sin(51.3°) = 340$ km, that the time passed between these two points is equal to half the period T_f, and that the oscillating behaviour as described above is reflected by the sine function "$\sin(\omega_f t)$", the vertical velocity $\vec{v}_v$ can then be expressed as follows:

$$\vec{v}_v = \left[-\left(\frac{2 \cdot d_v}{T_f} \cdot \sin(\omega_f t)\right)\sin\theta\right] \cdot \vec{i}_y + \left[\left(\frac{2 \cdot d_v}{T_f} \cdot \sin(\omega_f t)\right)\cos\theta\right] \cdot \vec{i}_z$$

Since the asteroid collides with the rock in the radial direction, the rock's velocity in the tangential direction (viewed from an inertial frame) is left undisturbed and therefore also the harmonic motion of the vertical velocity $\vec{v}_v$ in the co-rotating reference frame. The rock's inwards, radial motion in the negative y-direction as a result of the collision with the asteroid can be described as a free fall towards the pulsar's surface, whereby the final velocity $\vec{v}_a$ at time t when the rock hits the pulsar's surface is expressed as $\vec{v}_a = -(v_0 + g \cdot t) \cdot \vec{i}_y$. Given the angular velocity $\vec{\omega} = (-\omega_f \cdot \sin\theta) \cdot \vec{i}_y + (\omega_f \cdot \cos\theta) \cdot \vec{i}_z$, we find the following expression for the acceleration in the x-direction due to the Coriolis effect:

$$\vec{F}_{cor} = 2 \cdot m_r \cdot (\vec{v} \times \vec{\omega}) = 2 \cdot m_r \cdot [\ (0, v_a + v_{vy}, v_{vz}) \times (0, -\omega_f \cdot \sin\theta, \omega_f \cdot \cos\theta)\ ]$$

$$\Leftrightarrow \quad \vec{a}_{cor} = 2 \cdot [(v_a + v_{vy}) \cdot \omega_f \cdot \cos\theta + v_{vz} \cdot \omega_f \cdot \sin\theta] \cdot \vec{i}_x$$

$$= 2 \cdot \left[\left[-(v_0 + g \cdot t) - \left(\frac{2 \cdot d_v}{T_f} \cdot \sin(\omega_f t)\right)\sin\theta\right] \cdot \omega_f \cdot \cos\theta + \left[\left(\frac{2 \cdot d_v}{T_f} \cdot \sin(\omega_f t)\right)\cos\theta\right] \cdot \omega_f \cdot \sin\theta\right] \cdot \vec{i}_x$$

$$= -2 \cdot \omega_f \cdot (v_0 + g \cdot t) \cdot \cos\theta \cdot \vec{i}_x$$

As the two components of the vertical velocity $\vec{v}_v$ each manifest a Coriolis effect equal in magnitude but opposite in direction, they cancel each other out so that only the radial velocity $\vec{v}_a$ is responsible for causing a net Coriolis acceleration. Integrating the above expression twice and taking into account the fact that the initial position and speed of the rock in the x-direction of the co-rotating frame are equal to $x_0 = 0$ m and $v_{x,0} = 0$ m/s, respectively, at $t = 0$ s, we find the following expression for the deflection x_{cor} due to the Coriolis effect:

$$a_{cor} = \frac{dv}{dt} \quad \Leftrightarrow \quad \int_{v_{x,0}}^{v_{cor}} dv' = -2 \cdot \omega_f \cdot \cos\theta \left[\int_0^t (v_0 + g \cdot t') \cdot dt'\right]$$

$$\Leftrightarrow \quad \left(v\Big|_{v_{x,0}=0}^{v_{cor}}\right) = -2 \cdot \omega_f \cdot \cos\theta \cdot \left[\left(v_0 \cdot t + \frac{g}{2} \cdot t^2\right)\Big|_{t=0}^{t}\right]$$

$$\Leftrightarrow \quad v_{cor} = -2 \cdot \omega_f \cdot \cos\theta \cdot \left(v_0 \cdot t + \frac{g}{2} \cdot t^2\right)$$

$$v_{cor} = \frac{dx}{dt} \quad \Leftrightarrow \quad \int_{x_0}^{x_{cor}} dx' = -2 \cdot \omega_f \cdot \cos\theta \cdot \left[\int_0^t \left(v_0 \cdot t' + \frac{g}{2} \cdot t'^2 \right) \cdot dt' \right]$$

$$\Leftrightarrow \quad \left(x|_{x_0=0}^{x_{cor}} \right) = -2 \cdot \omega_f \cdot \cos\theta \cdot \left[\left(\frac{v_0}{2} \cdot t^2 + \frac{g}{6} \cdot t^3 \right) \Big|_{t=0}^{t} \right]$$

$$\Leftrightarrow \quad x_{cor} = -\omega_f \cdot \cos\theta \cdot \left(v_0 \cdot t^2 + \frac{g}{3} \cdot t^3 \right)$$

At this point, we still have to determine the value of the average gravitational field strength g and the time t it takes the rock to reach the pulsar's surface before we can calculate the value of the deflection x_{cor}. Regarding the former, since the field strength g_h and g_s at the orbital height h and the pulsar's surface are equal to $g_h = \frac{G \cdot M_f}{\left(R_f + h\right)^2}$ and $g_s = \frac{G \cdot M_f}{R_f^2}$, respectively, we find the average field strength g as follows:

$$g = \frac{(g_h + g_s)}{2} = \frac{1}{2} \cdot \left[\frac{G \cdot M_f}{(R_f + h)^2} + \frac{G \cdot M_f}{R_f^2} \right]$$

$$= \frac{G \cdot M_f}{2} \cdot \left[\frac{1}{(R_f + h)^2} + \frac{1}{R_f^2} \right]$$

$$= \frac{6.67 \times 10^{-11} \cdot 2.01 \cdot 1.99 \times 10^{30}}{2} \cdot \left[\frac{1}{(12,800 + 2.05 \times 10^5)^2} + \frac{1}{12,800^2} \right]$$

$$= 8.18 \times 10^{11} \text{ m/s}^2$$

With respect to the time t, we have to bear in mind that in the co-rotating frame there are two components to the velocity in the y-direction, i.e., $\vec{v}_a$ and $\vec{v}_{vy}$. Adding these two components together and integrating the sum, we obtain an expression for the position of the rock in function of the time t:

$$\frac{dy}{dt} = v_y = v_a + v_{vy} = -v_0 - g \cdot t - \frac{2 \cdot d_v}{T_f} \cdot \sin\theta \cdot \sin(\omega_f t)$$

$$\Leftrightarrow \quad \int_h^0 dy' = \int_0^t \left(-v_0 - g \cdot t' - \frac{2 \cdot d_v}{T_f} \cdot \sin\theta \cdot \sin(\omega_f t') \right) \cdot dt'$$

$$\Leftrightarrow \quad \left(y|_h^0 \right) = -v_0 \cdot \left(t|_0^t \right) - g \cdot \left(\frac{t^2}{2} \Big|_0^t \right) - \frac{2 \cdot d_v}{T_f} \cdot \sin\theta \cdot \left(-\frac{1}{\omega_f} \cos(\omega_f t) \Big|_0^t \right)$$

$$\Leftrightarrow \quad 0 = h - v_0 \cdot t - \frac{g}{2} \cdot t^2 + \frac{2 \cdot d_v}{T_f \cdot \omega_f} \cdot \sin\theta \cdot (\cos(\omega_f t) - 1)$$

If we want to extract t from the above expression, we first have to rewrite the cosine function "$\cos(\omega_f t)$" with the assistance of the Maclaurin series expansion, which is a Taylor expansion around the point $t = 0$. If we expand to the second order, the cosine function can be written as follows:

$$\cos(\omega_f t) \approx \sum_{k=0}^{1} \frac{(-1)^k \cdot (\omega_f t)^{2k}}{(2k)!} = \frac{(-1)^0 \cdot (\omega_f t)^{2\cdot 0}}{(2\cdot 0)!} + \frac{(-1)^1 \cdot (\omega_f t)^{2\cdot 1}}{(2\cdot 1)!} = 1 - \frac{\omega_f^2 \cdot t^2}{2}$$

Inserting this expression for "$\cos(\omega_f t)$" back into the above equation of motion (in the y-direction), we find:

$$0 = h - v_0 \cdot t - \frac{g}{2} \cdot t^2 + \frac{2 \cdot d_v}{T_f \cdot \omega_f} \cdot \sin\theta \cdot \left(\left[1 - \frac{\omega_f^2 \cdot t^2}{2}\right] - 1\right) = h - v_0 \cdot t - \left[\frac{g}{2} + \frac{d_v \cdot \omega_f}{T_f} \cdot \sin\theta\right] \cdot t^2$$

$$\Leftrightarrow \quad 0 = 2.05 \times 10^5 - 1.55 \times 10^4 \cdot t - \left[\frac{8.18 \times 10^{11}}{2} + \frac{3.40 \times 10^5 \cdot 161}{\frac{2\cdot\pi}{161}} \cdot \sin(51.3°)\right] \cdot t^2$$

The physically sensible ($t > 0$) solution to the above quadratic equation is equal to $t = 7.07 \times 10^{-4}$ s. In a final step, we calculate the value of the deflection x_{cor} of the rock in the x-direction as a result of the Coriolis effect:

$$x_{cor} = -\omega_f \cdot \cos\theta \cdot \left(v_0 \cdot t^2 + \frac{g}{3} \cdot t^3\right)$$

$$= -161 \cdot \cos(51.3°) \cdot \left[1.55 \times 10^4 \cdot \left(7.07 \times 10^{-4}\right)^2 + \frac{8.18 \times 10^{11}}{3} \cdot \left(7.07 \times 10^{-4}\right)^3\right]$$

$$= -9.69 \text{ km}$$

The Coriolis effect causes the rock to deflect by 9.69 km into the same direction as the pulsar's spin, i.e., eastwards (or counterclockwise). This distance corresponds to $\frac{|x_{cor}|}{2\cdot\pi\cdot R_f} = \frac{9{,}690}{2\cdot\pi\cdot 12{,}800} = 12.0\%$ of the pulsar's circumference. Within an inertial frame of reference and disregarding the Coriolis effect, the rock has traveled a distance of $d = v_{orb} \cdot t = 3.50 \times 10^7 \cdot 7.07 \times 10^{-4} = 24.8$ km along its orbit during time t. Adding the Coriolis effect, the total distance covered by the rock is then equal to $x_{tot} = d + |x_{cor}| = 24.8 + 9.69 = 34.4$ km.

We can also calculate the height h_z of the rock within the co-rotating coordinate system at the moment when the rock hits the pulsar's surface (which we implicitly assumed as being equal to $y = 0$):

$$h_z = v_{vz} \cdot t = \left[\frac{2 \cdot d_v}{T_f} \cdot \sin(\omega_f t) \cos\theta\right] \cdot t$$

$$= \left[\frac{2 \cdot 3.40 \times 10^5}{\frac{2\cdot\pi}{161}} \cdot \sin\left(161 \cdot 7.07 \times 10^{-4} \cdot \frac{180°}{\pi}\right) \cos(51.3°)\right] \cdot 7.07 \times 10^{-4}$$

$$= 873 \text{ m}$$

Exercise 18

Problem Statement

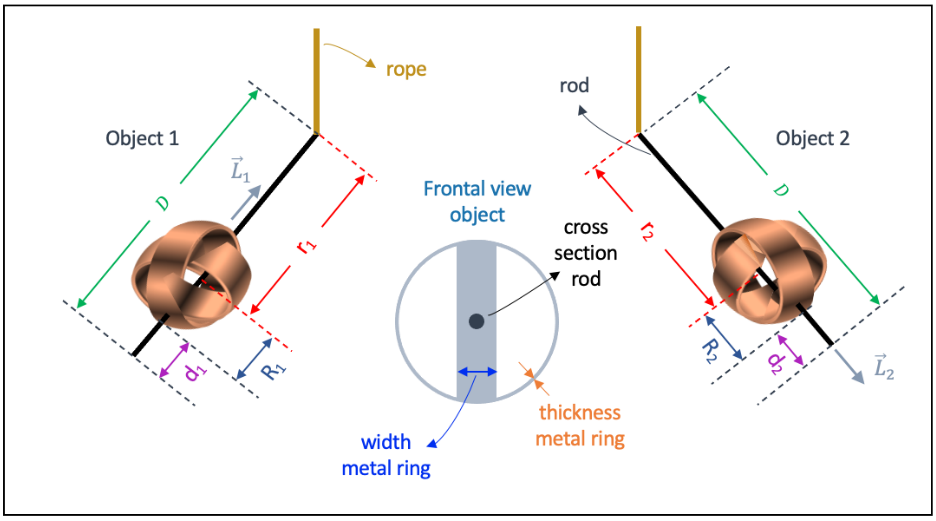

Figure 18

Tarif is living in the one house that stands somewhat isolated from the rest of the houses in his village Al-Mani'a in Algeria. Everywhere in and around his house Tarif has spinning objects hanging from the ceiling or from the roof of his veranda, since he believes that rotating objects produce powerful vortices of spiritual energy and attract benevolent spirits. In the Saharan cypress tree in his backyard, Tarif has tied two ropes to a branch, whereby two rods of length $D = 62.0$ cm are each attached to the free end of a rope. Towards the free end of each rod a spherical-like object is mounted that is able to spin around the rod. The object consists of two copper rings welded together in such a way so that the rod passes through the center of one ring and makes up the central diameter of the other ring. The rings of object 1 are $w_1 = 9.30$ cm wide and the left side of the object is located at a distance of $d_1 = 7.50$ cm from the rod's free end. Object 2 is $d_2 = 15.2$ cm away from its rod's free end and its rings have a width of $w_2 = 8.60$ cm. When holding the free end of the rod, Tarif gives object 1 a clockwise initial spin of 12 rps, whereas object 2 receives an anti-clockwise initial spin of 960 rpm. When he subsequently lets go of the respective rod, both objects start to precess. As Tarif is fascinated by periodic relations, he wants the first object to precess at half the rate of the second object. He achieves this configuration by gently touching a metal ring with a stick for about $t = 3.50$ s, producing thereby an angular deceleration of $\alpha = 16.7$ rad/s^2. (1) When looking from above, what is the direction of precession of both objects? (2) If you know that object 1 precesses initially at a higher rate, to which object does Tarif has to apply his angular deceleration technique? (3) If you know that the radius R_1 of the two metal rings of object 1 measures $R_1 = 14.2$ cm, what is the radius R_2 of the two metal rings of object 2?

Solution

(1) The point of origin around which the objects precess is located at the intersection between the rope and the rod. The position vectors $\vec{r}_1$ and $\vec{r}_2$ (not drawn in Fig. 18) point from the origin towards the center of mass of the object and have a magnitude of $|\vec{r}_1| = r_1$ and $|\vec{r}_2| = r_2$, respectively. The torque $\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_G$ of object 1 points out of your screen and since Tarif gives object 1 a clockwise spin (as seen from below) the angular momentum $\vec{L}_1 = I_1 \cdot \vec{\omega}_1$ points towards the point of origin. As the change in angular momentum $d\vec{L}_1$ has the same direction as the torque $\vec{\tau}_1$, it then follows that the angular momentum $\vec{L}_1$ will change its direction in a clockwise fashion (as seen from above),

which marks the direction of the precession of object 1.

By the same reasoning, given that the torque $\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_G$ of object 2 points into your screen and that the angular momentum $\vec{L}_2 = I_2 \cdot \vec{\omega}_2$ is directed away from the point of origin, it follows that the direction of precession of object 2 is counterclockwise (as seen from above).

(2) Since Tarif wants object 1 to precess at half the rate of object 2 and given that, initially, object 1 precesses at a higher rate ($\Omega_1 > \Omega_2$), it means that Tarif must increase the precession rate Ω_2 of object 2 by slowing down its spinning rate ω_2.

(3) To determine the radius R_2 of the metal rings of object 2, we use the constraint that the rate of precession Ω_2 of object 2 will be twice that of object 1 ($\Omega_2 = 2 \cdot \Omega_1$), once Tarif applies his angular deceleration technique to object 2. We now need to calculate the distances r_1 and r_2, the moments of inertia I_1 and I_2, and the angular velocities ω_1 and ω_2 of object 1 and 2, respectively. Note that the precession rate does not depend on the mass of the object, so we do not need to calculate their masses.

The angular velocity ω_1 of object 1 is equal to $\omega_1 = 2 \cdot \pi \cdot 12 = 75.4$ rad/s, and that of object 2 under the condition that $\Omega_2 = 2 \cdot \Omega_1$ is calculated to be $\omega_2 = \omega_i - \alpha \cdot t = \frac{2 \cdot \pi \cdot 960}{60} - 16.7 \cdot 3.5 = 42.1$ rad/s. From Fig. 18 we can furthermore see that the distance r_1 is equal to $r_1 = D - d_1 - R_1 = 62.0 - 7.50 - 14.2 = 40.3$ cm.

With respect to their mass moment of inertia, we know that the moment of inertia of a thin ring whereby the rotation axis runs through its center point is equal to $I_{cp} = M \cdot R^2$ and that the moment of inertia of a thin ring with the central diameter as its rotation axis is equal to $I_{cd} = \frac{M \cdot R^2}{2} + \frac{M \cdot w^2}{12}$ (with w the width of the ring), so that the moment of inertia I of the object as described in our problem can be expressed as follows:

$$I = I_{cp} + I_{cd} = M \cdot R^2 + \frac{M \cdot R^2}{2} + \frac{M \cdot w^2}{12} = \frac{M}{12} \cdot \left(18 \cdot R^2 + w^2\right)$$

Based on the constraint $\Omega_2 = 2 \cdot \Omega_1$ we find the following quadratic expression for the radius R_2 of the metal rings of object 2 (keep in mind that the objects consist of two metal rings, so we need to multiply the mass of the object in the numerator by a factor of 2):

$$\Omega_2 = 2 \cdot \Omega_1 \quad \Leftrightarrow \quad \frac{2 \cdot M_2 \cdot r_2 \cdot g}{I_2 \cdot \omega_2} = 2 \cdot \frac{2 \cdot M_1 \cdot r_1 \cdot g}{I_1 \cdot \omega_1}$$

$$\Leftrightarrow \quad \frac{2 \cdot M_2 \cdot [D - d_2 - R_2] \cdot g}{\left[\frac{M_2}{12} \cdot (18 \cdot R_2^2 + w_2^2)\right] \cdot \omega_2} = 2 \cdot \frac{2 \cdot M_1 \cdot r_1 \cdot g}{\left[\frac{M_1}{12} \cdot (18 \cdot R_1^2 + w_1^2)\right] \cdot \omega_1}$$

$$\Leftrightarrow \quad [36 \cdot r_1 \cdot \omega_2] \cdot R_2^2 + \left[\omega_1 \cdot \left(18 \cdot R_1^2 + w_1^2\right)\right] \cdot R_2 + \left[2 \cdot r_1 \cdot \omega_2 \cdot w_2^2 - (D - d_2) \cdot \omega_1 \cdot \left(18 \cdot R_1^2 + w_1^2\right)\right] = 0$$

$$\Leftrightarrow \quad [36 \cdot 0.403 \cdot 42.1] \cdot R_2^2 + \left[75.4 \cdot \left(18 \cdot 0.142^2 + 0.0930^2\right)\right] \cdot R_2 +$$

$$\left[2 \cdot 0.403 \cdot 42.1 \cdot 0.0860^2 - (0.620 - 0.152) \cdot 75.4 \cdot \left(18 \cdot 0.142^2 + 0.0930^2\right)\right] = 0$$

The physically sensible ($R_2 > 0$) solution to the above quadratic equation is equal to $R_2 = 12.4$ cm.

Exercise 19

Problem Statement

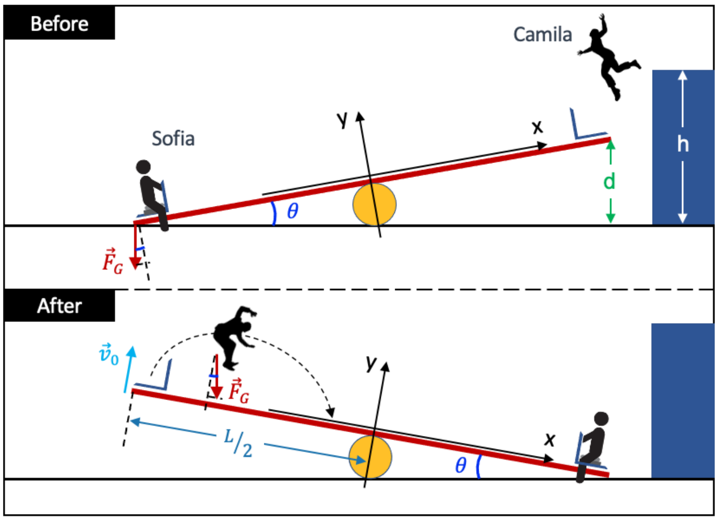

Figure 19

Yulissa decided to take her five-year-old nieces Camila ($m_1 = 20.6$ kg) and Sofia ($m_2 = 18.3$ kg) to the playground in the Parque Central Juan Pablo Duarte in Nagua, The Dominican Republic, so that their parents could celebrate their eighth wedding anniversary during a lunch at the seafood restaurant Junior Natura. The first attraction to which Camila and Sofia run off when they arrive at the Parque Central is the seesaw, which has a length and mass of $L = 4.50$ m and $M = 12.5$ kg, respectively, and makes an angle of $\theta = 9.50°$ with the horizontal when one side touches the ground. After only 10 minutes, Camila, who is the cheekiest of the two sisters, gets bored and climbs on the nearby Disney tower. From up there, she sees that Sofia is still sitting on the seesaw and without hesitation Camila jumps from a height of $h = 2.25$ m onto her empty seat. To Camila's great delight (and also Sofia's), Sofia is being ejected out of her seat for just a brief moment in time. (1) Where exactly does Sofia land? (2) How high did she go? Assume that the average force of impact that Camila exerts upon her empty seat is about twice her kinetic energy (per unit length) right before landing on the seesaw.

Solution

(1) As Camila jumps down from the Disney tower, the conservative gravitational force $\vec{F}_G$ is the only force acting upon Camila, so that her total mechanical energy remains constant. Right before landing in her empty seat, Camila's kinetic energy is equal to:

$$\begin{aligned} E_{tot,i} = E_{tot,f} \quad &\Leftrightarrow \quad m_1 \cdot g \cdot h = E_k + m_2 \cdot g \cdot d \\ &\Leftrightarrow \quad E_k = m_1 \cdot g \cdot (h - d) = m_1 \cdot g \cdot (h - L \cdot \sin\theta) \\ &\qquad\qquad = 20.6 \cdot 9.81 \cdot [2.25 - 4.50 \cdot \sin(9.50°)] \\ &\qquad\qquad = 305 \text{ J} \end{aligned}$$

Given that the force of impact $\vec{F}_C$ of Camila landing in her seat is twice her kinetic energy (per unit length $l = 1$ m), the magnitude of $\vec{F}_C$ is then equal to $F_C = 2 \cdot \frac{E_k}{l} = 2 \cdot \frac{305}{1} = 609$ N.

In a next step, let us determine the net torque $\vec{\tau}_n$ of the seesaw, which is the sum of the torques $\vec{\tau}_S$ and $\vec{\tau}_C$ manifested by the weight of Sofia and the impact force of Camila, respectively. With the origin of our coordinate system located at the center of the seesaw, we write:

$$
\begin{aligned}
\vec{\tau}_n = \vec{\tau}_S + \vec{\tau}_C &= \left[\vec{r}_S \times \vec{F}_{G,S}\right] + \left[\vec{r}_C \times \vec{F}_C\right] \\
&= \left[\left(-\frac{L}{2}, 0, 0\right) \times (-m_2 \cdot g \cdot \sin\theta, -m_2 \cdot g \cdot \cos\theta, 0)\right] + \\
&\qquad \left[\left(\frac{L}{2}, 0, 0\right) \times (-F_C \cdot \sin\theta, -F_C \cdot \cos\theta, 0)\right] \\
&= \frac{L}{2} \cdot (m_2 \cdot g - F_C) \cdot \cos\theta \cdot \vec{i}_z \\
&= \frac{4.50}{2} \cdot (18.3 \cdot 9.81 - 609) \cdot \cos(9.50°) \cdot \vec{i}_z \\
&= -954 \cdot \vec{i}_z \text{ N·m}
\end{aligned}
$$

In other words, the seesaw will turn clockwise. From the moment Camila lands on her seat, the seesaw's angular velocity $\vec{\omega}$ starts increasing and points into your screen, and so does the angular momentum $\vec{L}$. The vector $\vec{L}$ does not change direction—it remains directed into your screen in the negative z-direction—but constantly changes its magnitude, i.e., it increases, so that the vector related to the change in angular momentum $d\vec{L}$ runs parallel, as it should, to the net torque $\vec{\tau}_n$ pointing into your screen.

Before we calculate the angular acceleration $\vec{\alpha}$ of the seesaw, let us determine its mass moment of inertia I. We find:

$$
\begin{aligned}
I = \frac{M \cdot L^2}{12} + m_1 \cdot \left(\frac{L}{2}\right)^2 + m_2 \cdot \left(\frac{L}{2}\right)^2 &= \frac{L^2}{12} \cdot [M + 3 \cdot (m_1 + m_2)] \\
&= \frac{4.50^2}{12} \cdot [12.5 + 3 \cdot (20.6 + 18.3)] \\
&= 218 \text{ kg·m}^2
\end{aligned}
$$

The angular acceleration $\vec{\alpha}$ then becomes:

$$
\vec{\tau}_n = I \cdot \vec{\alpha} \quad \Leftrightarrow \quad \vec{\alpha} = \frac{\vec{\tau}_n}{I} = \frac{-954}{218} \cdot \vec{i}_z = -4.37 \cdot \vec{i}_z \text{ rad/s}^2
$$

What we need to find is the initial velocity $\vec{v}_0$ of Sofia when Camila's end of the seesaw touches the ground. Therefore, let us first calculate the magnitude of the angular velocity $\vec{\omega}$ with the help of the following rotational equation of motion (bear in mind that Sofia has rotated over an angle 2θ right before being sent flying into the air):

$$\omega^2 - \omega_0^2 = 2 \cdot \alpha \cdot \theta \quad \Leftrightarrow \; \omega = \sqrt{\omega_0^2 + 2 \cdot \alpha \cdot (2\theta)} = \sqrt{0^2 + 2 \cdot 4.37 \cdot \left(2 \cdot 9.50° \cdot \frac{\pi}{180°}\right)} = 1.70 \text{ rad/s}$$

Since the angular velocity vector $\vec{\omega}$ points into the negative z-direction, we find the initial velocity $\vec{v}_0$ as follows:

$$\vec{v}_0 = \vec{\omega} \times \vec{r}_S = (0, 0, -\omega) \times \left(-\frac{L}{2}, 0, 0\right) = \omega \cdot \frac{L}{2} \cdot \vec{i}_y = 1.70 \cdot \frac{4.50}{2} \cdot \vec{i}_y = 3.83 \cdot \vec{i}_y \text{ m/s}$$

The time t Sofia spend in the air can be calculated with the assistance of the (linear) equation of motion in the y-direction:

$$y = y_0 + v_0 \cdot t + \frac{a_y}{2} \cdot t^2 \quad \Leftrightarrow \quad 0 = 0 + v_0 \cdot t - \frac{g \cdot \cos\theta}{2} \cdot t^2 \quad \Leftrightarrow \quad t = \frac{2 \cdot v_0}{g \cdot \cos\theta} = \frac{2 \cdot 3.83}{9.81 \cdot \cos(9.50°)} = 0.792 \text{ s}$$

The (linear) equation in the x-direction then gives us the location on the seesaw where Sofia lands after her brief aerial adventure:

$$x = x_0 + v_0 \cdot t + \frac{a_x}{2} \cdot t^2 \quad \Leftrightarrow \quad x = -\frac{L}{2} + 0 \cdot t + \frac{g \cdot \sin\theta}{2} \cdot t^2 = -\frac{4.50}{2} + \frac{9.81 \cdot \sin(9.50°)}{2} \cdot 0.792^2 = -1.74 \text{ m}$$

In other words, Sofia landed $s = \frac{L}{2} + x = \frac{4.50}{2} - 1.74 = 50.8$ cm to the right of her seat.

(2) During this short moment of free fall, Sofia attained a maximum height h_{max} of (as viewed from the perspective of our coordinate system):

$$v^2 - v_0^2 = 2 \cdot a_y \cdot h_{max} \quad \Leftrightarrow \quad h_{max} = \frac{v^2 - v_0^2}{2 \cdot (-g \cdot \cos\theta)} = \frac{0^2 - 3.83^2}{2 \cdot [-9.81 \cdot \cos(9.50°)]} = 75.9 \text{ cm}$$

Viewed from a coordinate system fixed to the ground, at her highest point Sofia was a vertical distance D removed from the seesaw, which can be calculated as follows with the help of some trigonometry:

$$D = \left[\frac{-(v_0 \cdot \cos\theta)^2}{2 \cdot (-g)}\right] + \left[\frac{s}{2} \cdot \sin\theta\right] = \left[\frac{-[3.83 \cdot \cos(9.50°)]^2}{2 \cdot (-9.81)}\right] + \left[\frac{0.508}{2} \cdot \sin(9.50°)\right] = 77.0 \text{ cm}$$

Exercise 20

Problem Statement

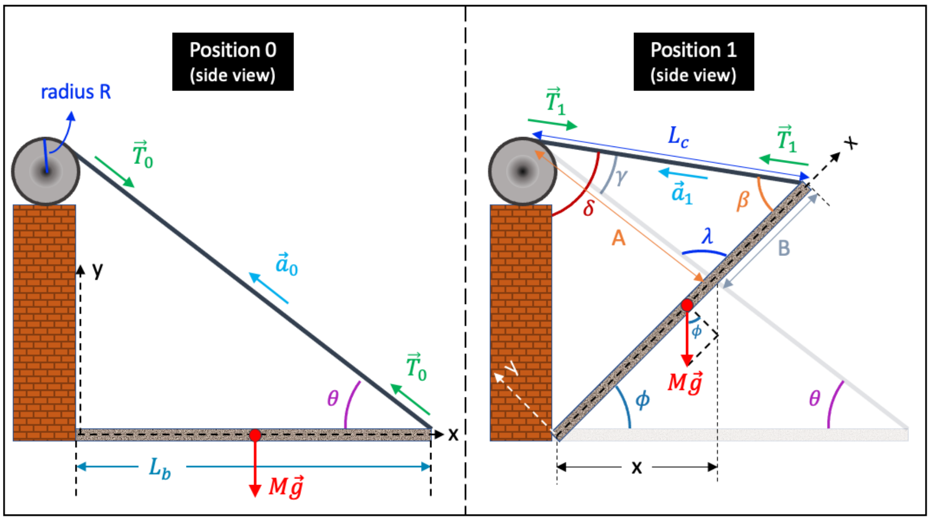

Figure 20

The local council of the city of Bruges in Belgium is calling tenders for the construction of a new pedestrian drawbridge over one of the city's many canals. The bridge has to be $L_b = 12.0$ m long and two galvanized steel cables will have to do the important job of safely drawing up the bridge. Two large iron disk-shaped pulleys with a radius and mass of $R = 57.5$ cm and $M_p = 165$ kg, respectively, are rolling up the cables as the bridge is being lifted. Bart is one of the engineers responsible for listing the technical details for one of the tenders. According to his conservative calculations, the cables combined could lift a bridge of no more than $M = 4,250$ kg. When the bridge is down the cables make an angle of $\theta = 42.0°$ with the horizontal, and Bart estimates that the small boats could easily pass under the bridge when it is drawn up at an angle of $\phi_{up} = 65.0°$ with the horizontal. Bart furthermore assesses that the motor that rotates the pulleys can comfortably provide a torque of $\vec{\tau}_0 = 8,985 \cdot \vec{i}_z$ N·m at the initial moment when the bridge is being lifted (position 0), whereas at an angle of $\phi = 45.0°$ (position 1)—that is, when the process of drawing up the bridge is slowing down—a minimum torque of 54.5% of its initial value is required. (1) What angle do the cables make with the bridge when it is in position 1? (2) What is the length of the visible part of the cables at that moment? (3) What is the magnitude of the tension forces $\vec{T}_0$ and $\vec{T}_1$ in the cables when the bridge is in position 0 and 1, respectively? (4) What is the average amount of time needed for the bridge to reach position 1?

Solution

(1) The angle that we need to find is the angle β. When applying the sine rule on the triangle $\mathrm{ABL_C}$, we can write the following equations:

$$\frac{A}{\sin\beta} = \frac{B}{\sin\gamma} = \frac{L_C}{\sin\lambda}$$

To determine angle β, let us focus on the left-hand side of the above equation and find an expression for the angle γ in terms of the angle β. If we consider the large triangle made up of the brick wall, the cables, and the bridge (in position 1), we can write:

$$\delta = 180° - (90.0° - \phi) - \beta = 180° - (90.0° - 45.0°) - \beta = 135° - \beta$$

If we consider the triangle when the bridge is in position 0, we then find:

$$\delta - \gamma = 90.0° - \theta = 90.0° - 42.0° = 48.0° \quad \Leftrightarrow \quad \delta = \gamma + 48.0°$$

Combining the above two expression for the angle δ, we find an expression for the angle γ in terms of the angle β:

$$\delta = 135° - \beta = \gamma + 48.0° \quad \Leftrightarrow \quad \gamma = 87° - \beta$$

We find an expression for the angle β by solving the left-hand side equation of the sine rule (whereby we make use of the angle subtraction theorem "$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cdot \cos\theta_2 - \cos\theta_1 \cdot \sin\theta_2$") :

$$\begin{aligned}
\frac{A}{\sin\beta} = \frac{B}{\sin\gamma} \quad \Leftrightarrow \quad A \cdot \sin\gamma = B \cdot \sin\beta \quad &\Leftrightarrow \quad A \cdot \sin(87° - \beta) = B \cdot \sin\beta \\
&\Leftrightarrow \quad A \cdot [\sin(87°) \cdot \cos\beta - \cos(87°) \cdot \sin\beta] = B \cdot \sin\beta \\
&\Leftrightarrow \quad \tan\beta = \frac{A \cdot \sin(87°)}{A \cdot \cos(87°) + B} \\
&\Leftrightarrow \quad \beta = \tan^{-1}\left[\frac{A \cdot \sin(87°)}{A \cdot \cos(87°) + B}\right]
\end{aligned}$$

To find the value of the angle β, we need to determine the lengths A and B. Let us first calculate the x-coordinate (relative to the coordinate system related to position 0) of the intersection between the two straight lines represented, on the one hand, by the cables when the bridge is in position 0, and, on the other hand, by the bridge in position 1. We find:

$$\begin{cases} \text{cables: } y = L_b \cdot \tan\theta - \tan\theta \cdot x \\ \\ \text{bridge: } y = x \end{cases} \quad \Leftrightarrow \quad x = \frac{L_b}{1 + \cot\theta} = \frac{12.0}{1 + \cot(42.0°)} = 5.69 \text{ m}$$

The lengths A and B are then equal to:

$$\begin{cases} A = \dfrac{x}{\cos\theta} = \dfrac{5.69}{\cos(42.0°)} = 7.65 \text{ m} \\ \\ B = L_b - \dfrac{x}{\cos\phi} = 12.0 - \dfrac{5.69}{\cos(45.0°)} = 3.96 \text{ m} \end{cases}$$

The value of the angle β then becomes:

$$\beta = \tan^{-1}\left[\frac{A\cdot\sin(87°)}{A\cdot\cos(87°)+B}\right] = \tan^{-1}\left[\frac{7.65\cdot\sin(87°)}{7.65\cdot\cos(87°)+3.96}\right] = 60.3°$$

(2) The length of the cables when the bridge is in position 1 is represented by the length L_c. Given that the angle λ is equal to $\lambda = 180° - \phi - \theta = 180° - 45.0° - 42.0° = 93.0°$, we can use the sine rule equation established in part (1) to determine the value of L_c:

$$\frac{A}{\sin\beta} = \frac{L_C}{\sin\lambda} \quad\Leftrightarrow\quad L_C = \sin\lambda\cdot\frac{A}{\sin\beta} = \sin(93.0°)\cdot\frac{7.65}{\sin(60.3°)} = 8.80 \text{ m}$$

(3) As we have to determine the magnitude of the tension forces $\vec{T}_0$ and $\vec{T}_1$ per cable and given that the drawbridge is supported by two cables, the mass $M = 4,250$ kg of the bridge will be equally divided between the two cables. We will therefore work with the mass $M_b = \frac{M}{2} = \frac{4,250}{2} = 2,125$ kg to find the tension in one cable.

The tension forces can be determined using Newton's second law for the rotational motion of the bridge, but to find the angular acceleration of the bridge, we will in a first instance have to write an expression for the acceleration of the cables. With respect to the cables when the bridge is in position 0, the magnitude of the acceleration $\vec{a}_0$, which runs tangential to the rotating pulley, can be expressed as follows by applying Newton's second law for rotation to the pulley:

$$\vec{\tau}_{net} = \vec{\tau}_0 + \left(\vec{R}\times\vec{T}_0\right) = I\cdot\vec{\alpha}_0 \quad\Leftrightarrow\quad \tau_0 - R\cdot T_0 = \left(\frac{M_p\cdot R^2}{2}\right)\cdot\left(\frac{a_0}{R}\right) \quad\Leftrightarrow\quad a_0 = \frac{2}{M_p\cdot R}\cdot(\tau_0 - R\cdot T_0)$$

In position 1, the motor is providing 54.5% of its initial torque to the pulley, so that the torque $\vec{\tau}_1$ related to the motor is equal to $\vec{\tau}_1 = 0.545\cdot\vec{\tau}_0 = 0.545\cdot 8,985\cdot\vec{i}_z = 4,897\cdot\vec{i}_z$ N·m. The magnitude of the acceleration $\vec{a}_1$ of the cables is then expressed as:

$$\vec{\tau}_{net} = \vec{\tau}_1 + \left(\vec{R}\times\vec{T}_1\right) = I\cdot\vec{\alpha}_1 \quad\Leftrightarrow\quad \tau_1 - R\cdot T_1 = \left(\frac{M_p\cdot R^2}{2}\right)\cdot\left(\frac{a_1}{R}\right) \quad\Leftrightarrow\quad a_1 = \frac{2}{M_p\cdot R}\cdot(\tau_1 - R\cdot T_1)$$

In a next step, we apply Newton's second law for rotation to the bridge in position 0. As we placed the origin of our coordinate system at the pivot point, the forces acting on the bridge due to the bearings at the foot of the brick wall will not contribute to the net torque. Note furthermore that the magnitude of the tangential acceleration $\vec{a}_{0,t}$ of the bridge is equal to $a_{0,t} = a_0\cdot\sin\theta$. We can therefore write (with $\vec{r}_1 = (L_b, 0, 0)$ and $\vec{r}_2 = \left(\frac{L_b}{2}, 0, 0\right)$):

$$\vec{\tau}_{net} = \left(\vec{r}_1\times\vec{T}_0\right) + \left(\vec{r}_2\times\vec{F}_G\right) = I\cdot\vec{\alpha}_0$$

$$\Leftrightarrow\quad (T_0\cdot\sin\theta)\cdot L_b - (M_b\cdot g)\cdot\frac{L_b}{2} = I\cdot\left(\frac{a_{0,t}}{R}\right) = \left(\frac{M_b\cdot L_b^2}{3}\right)\cdot\left(\frac{[a_0\cdot\sin\theta]}{L_b}\right)$$

$$= \left(\frac{M_b\cdot L_b^2}{3}\right)\cdot\left(\left[\frac{2}{M_p\cdot R}\cdot(\tau_0 - R\cdot T_0)\right]\cdot\frac{\sin\theta}{L_b}\right)$$

$$\Leftrightarrow \quad T_0 = \frac{M_b}{2 \cdot R \cdot \sin\theta} \cdot \left[\frac{3 \cdot g \cdot M_p \cdot R + 4 \cdot \sin\theta \cdot \tau_0}{3 \cdot M_p + 2 \cdot M_b}\right]$$

$$= \frac{2,125}{2 \cdot 0.575 \cdot \sin(42.0°)} \cdot \left[\frac{3 \cdot 9.81 \cdot 165 \cdot 0.575 + 4 \cdot \sin(42.0°) \cdot 8,985}{3 \cdot 165 + 2 \cdot 2,125}\right]$$

$$= 15,600 \text{ N}$$

Similarly, the magnitude of the tangential acceleration $\vec{a}_{1,t}$ of the bridge in position 1 is equal to $a_{1,t} = a_1 \cdot \sin\beta$, so that we calculate the magnitude of the tension force $\vec{T}_1$ as follows:

$$\vec{\tau}_{net} = \left(\vec{r}_1 \times \vec{T}_1\right) + \left(\vec{r}_2 \times \vec{F}_G\right) = I \cdot \vec{\alpha}_1$$

$$\Leftrightarrow \quad (T_1 \cdot \sin\beta) \cdot L_b - (M_b \cdot g \cdot \cos\phi) \cdot \frac{L_b}{2} = I \cdot \left(\frac{a_{1,t}}{R}\right) = \left(\frac{M_b \cdot L_b^2}{3}\right) \cdot \left(\frac{[a_1 \cdot \sin\beta]}{L_b}\right)$$

$$= \left(\frac{M_b \cdot L_b^2}{3}\right) \cdot \left(\left[\frac{2}{M_p \cdot R} \cdot (\tau_1 - R \cdot T_1)\right] \cdot \frac{\sin\beta}{L_b}\right)$$

$$\Leftrightarrow \quad T_1 = \frac{M_b}{2 \cdot R \cdot \sin\beta} \cdot \left[\frac{3 \cdot g \cdot M_p \cdot R \cdot \cos\phi + 4 \cdot \sin\beta \cdot \tau_1}{3 \cdot M_p + 2 \cdot M_b}\right]$$

$$= \frac{2,125}{2 \cdot 0.575 \cdot \sin(60.3°)} \cdot \left[\frac{3 \cdot 9.81 \cdot 165 \cdot 0.575 \cdot \cos(45.0°) + 4 \cdot \sin(60.3°) \cdot 4,897}{3 \cdot 165 + 2 \cdot 2,125}\right]$$

$$= 8,510 \text{ N}$$

(4) The distance over which the bridge accelerates to reach position 1 is equal to the arc length $s = L_b \cdot \phi = 12.0 \cdot \left(45.0° \cdot \frac{\pi}{180°}\right) = 9.42$ m. As we are interested in finding the average amount of time t, we use the following magnitude of the average acceleration $\vec{a}_{av}$ for our calculations:

$$a_{av} = \frac{a_{0,t} + a_{1,t}}{2} = \frac{1}{2} \cdot \left[\frac{2 \cdot \sin\theta}{M_p \cdot R} \cdot (\tau_0 - R \cdot T_0) + \frac{2 \cdot \sin\beta}{M_p \cdot R} \cdot (\tau_1 - R \cdot T_1)\right]$$

$$= \frac{1}{2} \cdot \left[\frac{2 \cdot \sin(42.0°)}{165 \cdot 0.575} \cdot (8,985 - 0.575 \cdot 15,600) + \frac{2 \cdot \sin(60.3°)}{165 \cdot 0.575} \cdot (4,897 - 0.575 \cdot 8,510)\right]$$

$$= 1.97 \times 10^{-2} \text{ m/s}^2$$

The following equation of motion then allows us to calculate the average amount of time t the bridge needs to reach position 1:

$$s = \frac{a_{av}}{2} \cdot t^2 \quad \Leftrightarrow \quad t = \sqrt{\frac{2 \cdot s}{a_{av}}} = \sqrt{\frac{2 \cdot 9.42}{1.97 \times 10^{-2}}} = 30.9 \text{ s}$$